A PTAS for the Highway Problem

Fabrizio Grandoni & Thomas Rothvoß

Institute of Mathematics
EPFL, Lausanne

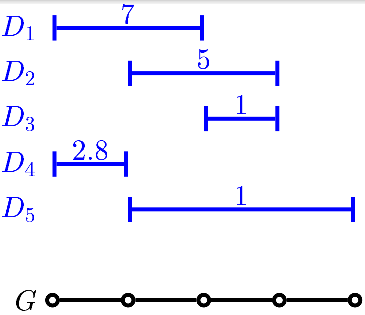
The Highway Problem

Def: Highway Problem

Given:

- a line graph $G = (e_1, \dots, e_n)$ (*highway*)
- subpaths $D_1, \dots, D_m$ (*drivers*)
- $b_j \in \mathbb{Q}_{\geq 0}$ (*budget*)

Find: *tolls* $w : E \to \mathbb{Q}_{\geq 0}$, max. the *profit* $\sum_{j : w(D_j) \leq b_j} w(D_j)$



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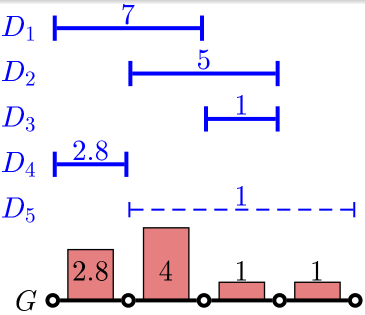

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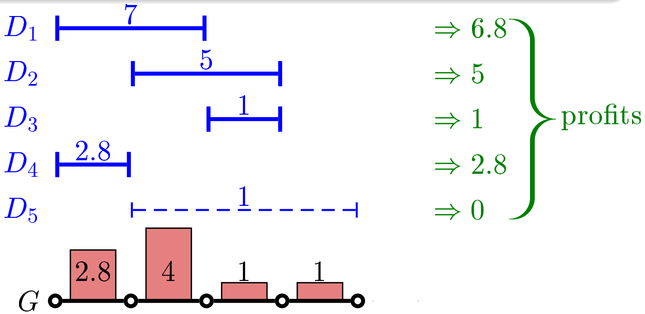

Known Results

- Is **NP**-hard? [Guruswami, Hartline, Karlin, Kempe, Kenyon, McSherry '05]
- Weakly **NP**-hard [Briest, Krysta '06]
- Strongly **NP**-hard [Elbassioni, Raman, Ray '09]

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- $O(1)$-apx for uniform-length drivers [Balcan, Blum '06]
- FPTAS for $n = O(1)$ [Hartline, Koltun '05]
- FPTAS for $|D_j| = O(1)$ [Guruswami et al. '05]
- FPTAS for $b_j = O(1)$ [Guruswami et al. '05]
- FPTAS for laminar drivers [Briest, Krysta '06]

Our Results

Theorem

There is a (deterministic) PTAS for the highway problem.

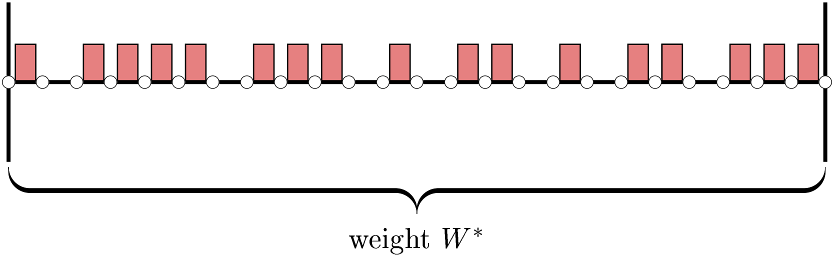
- Even $O(1)$-apx was open

Preprocessing

- By **rounding**: $b_j \in \{1, \ldots, m/\varepsilon^2\}$
- By **total unimodularity**, optimal weights $w^* : E \to \mathbb{Z}_+$
- By **edge duplication**, $w^* : E \to \{0, 1\}$
- By **dummy edges**, $W^* := \sum_{e \in E} w^*(e) = \gamma^\ell$, $\ell \in \mathbb{N}$ and $\gamma = (1/\varepsilon)^{1/\varepsilon}$.

Dissection of the optimum solution

- Let $w^* : E \to \{0, 1\}$ optimum weight assignment



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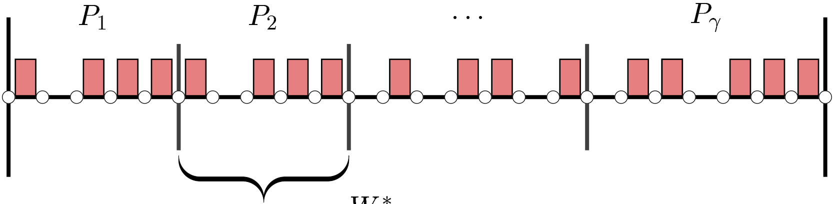

weight $w^*(P_i) = \dfrac{W^*}{\gamma}$

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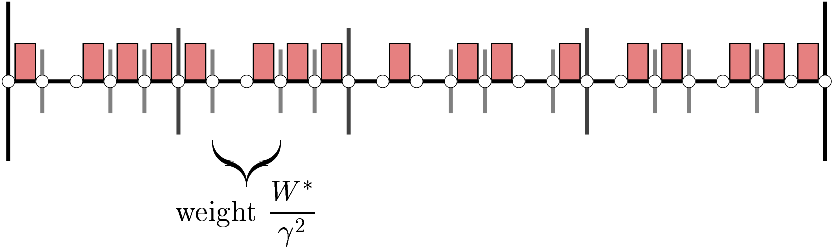

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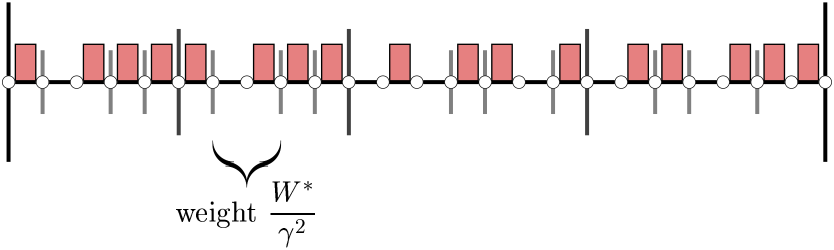
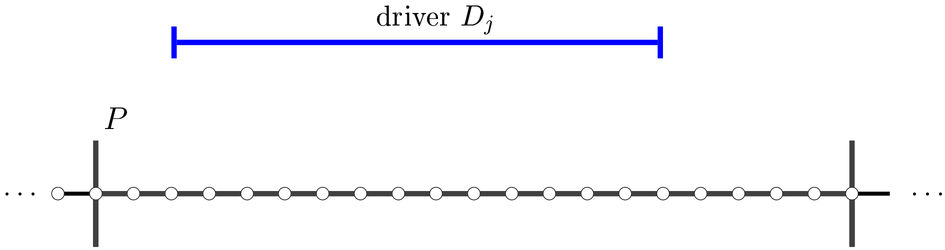


- Iterate until paths have weight $O(1)$.
- We obtain a **dissection** of degree $\gamma = (1/\varepsilon)^{1/\varepsilon}$

Why is such a dissection useful?



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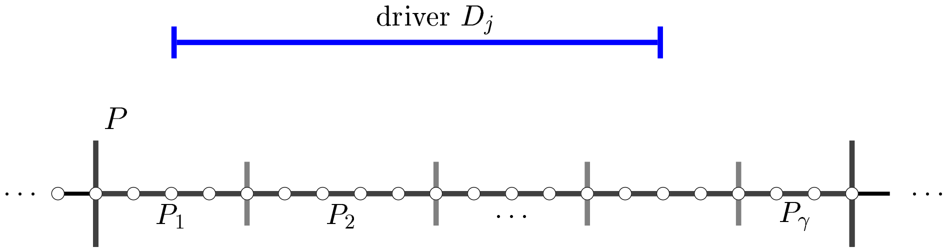

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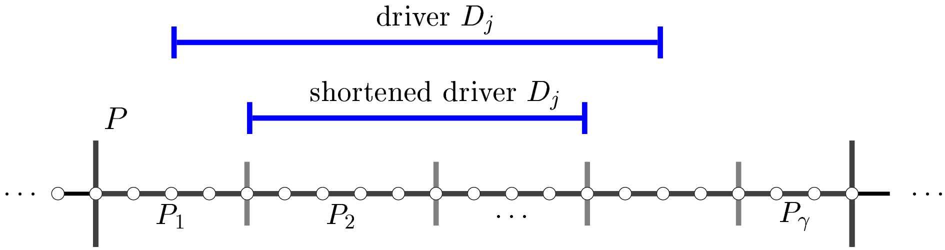

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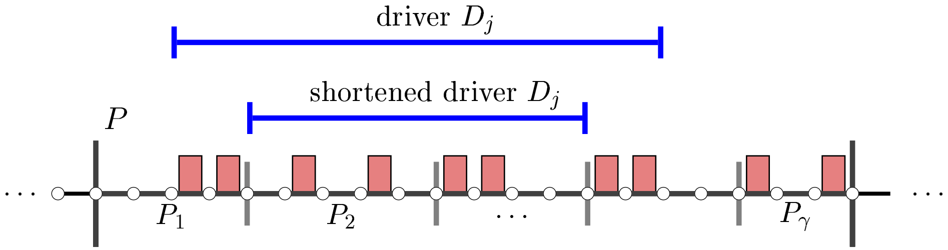

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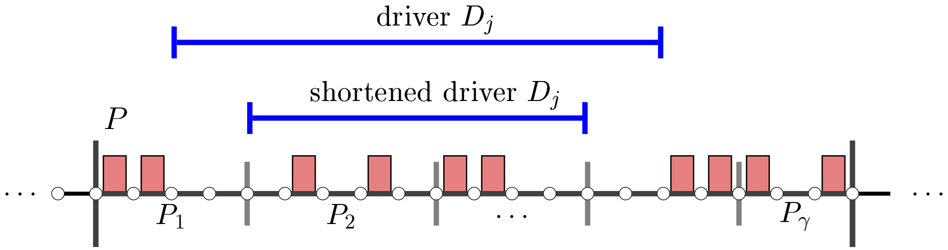

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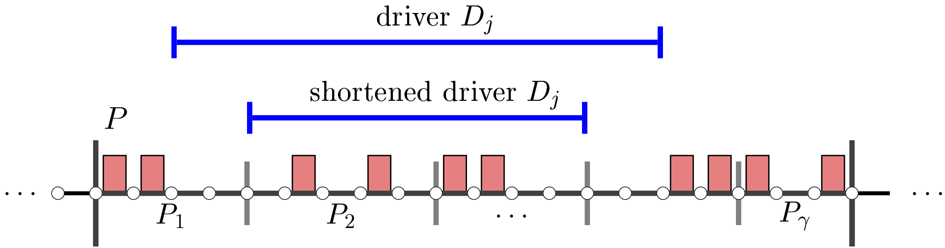

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$$\#(P_i \subseteq D_j) \geq \frac{1}{2\varepsilon}$$

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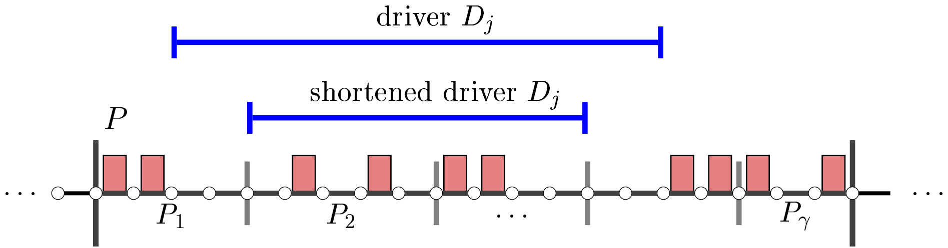

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- For a good driver

$$\frac{w(\text{driver } D_j)}{w(\text{shortened driver } D_j)} = 1 + O(\varepsilon)$$

Theorem

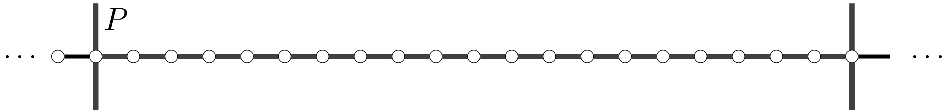
We can find a $w : E \to \mathbb{Q}_+$ in poly-time that maximizes profit from good shortened drivers.

The Dynamic program

Table entries: For any subpath $P \subseteq G$, weights $W \in \gamma^{\mathbb{N}}$

$$\phi(P, W) := \max_{\substack{\text{dissections of } P \\ \text{installing weight } W \text{ on } P}} \left\{ \begin{array}{c} \text{profits from good} \\ \text{shortened drivers } D_j \subseteq P \end{array} \right\}$$

Computing $\phi(P, W)$:



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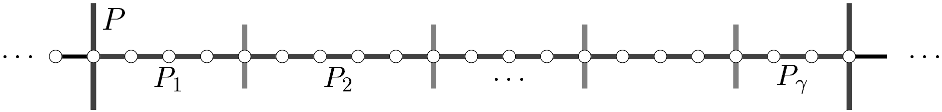

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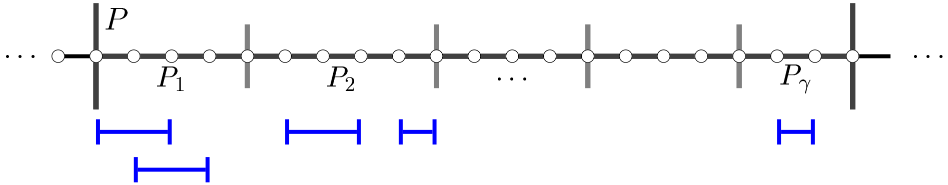

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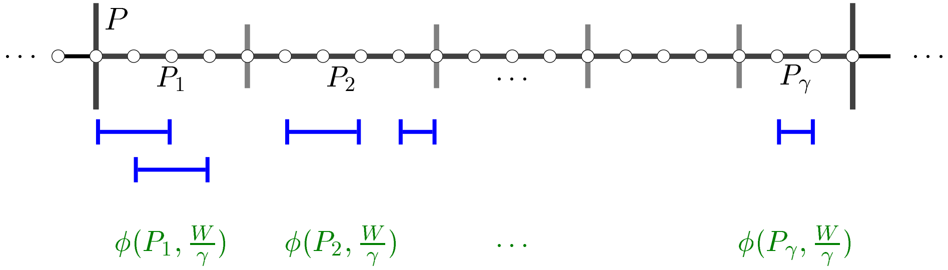

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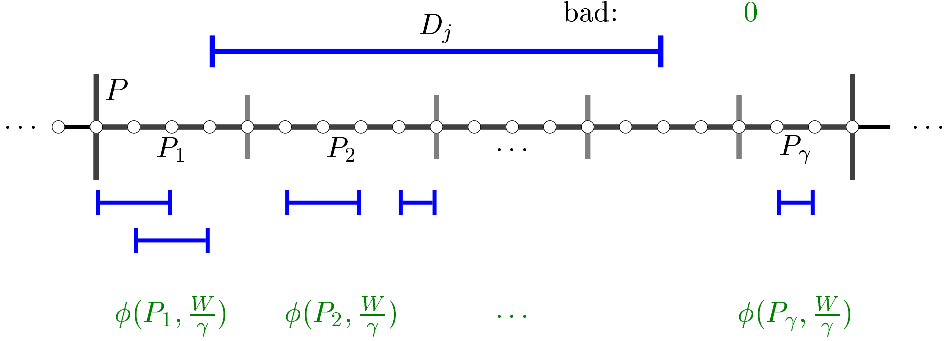
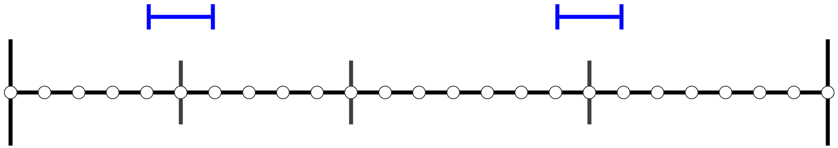
Computing $\phi(P, W)$:

good:	$\frac{W}{\gamma} \cdot \#P_i \subseteq D_j$
b_j exceeded:	0
bad:	0

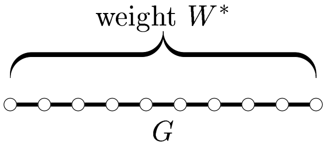


A pathological situation

- Maybe there is no dissection such that most drivers are good!



Randomization



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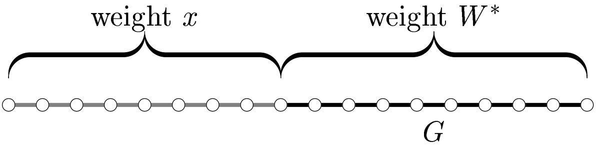

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- Choose $x \in \{1, \dots, W^*\}$ randomly
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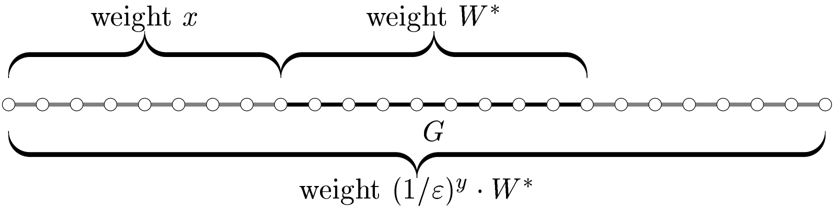

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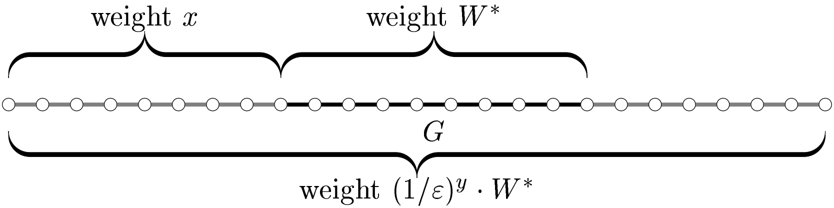
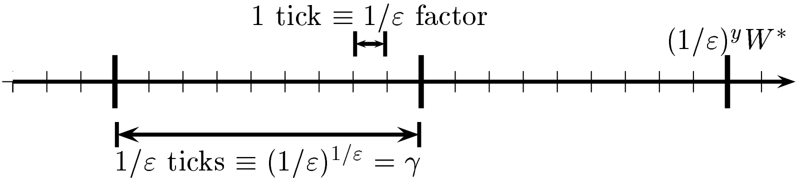


- *New randomized optimum solution!*

Lemma

For any driver D_j and the dissection induced by the randomized optimum solution: $\Pr[D_j \textit{ is good}] \geq 1 - 3\varepsilon$.

- Consider boundary sizes in the dissection: $(1/\varepsilon)^{y+\mathbb{Z}/\varepsilon}$



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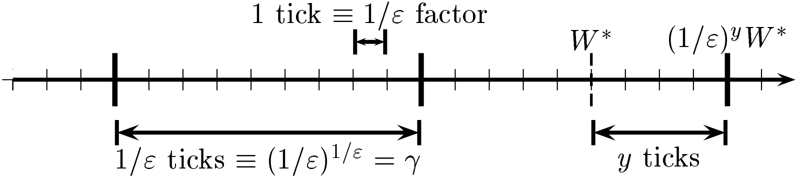

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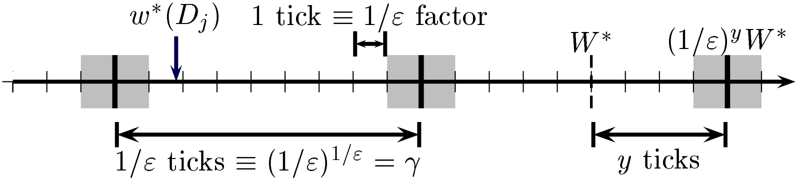

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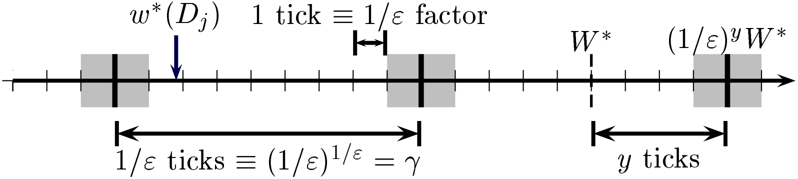

$$\Pr[w^*(D_j) \text{ is within a} \leq 1/\varepsilon \text{ factor such a boundary}] \leq 2\varepsilon$$

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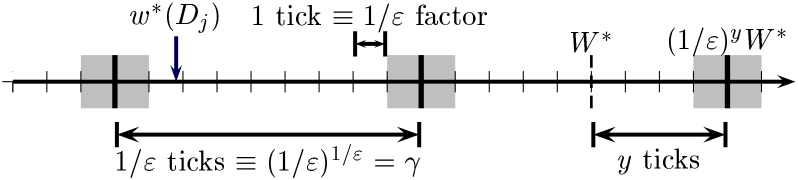

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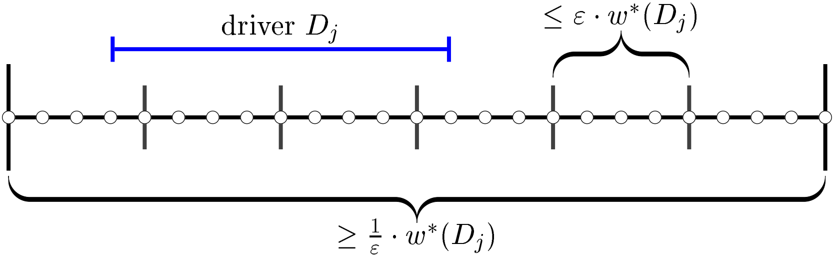

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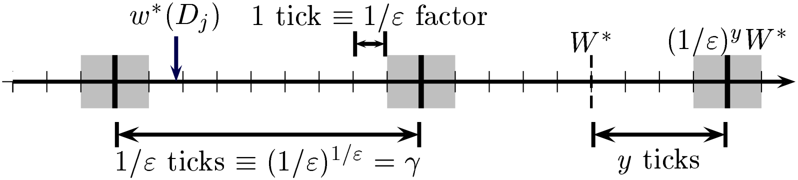

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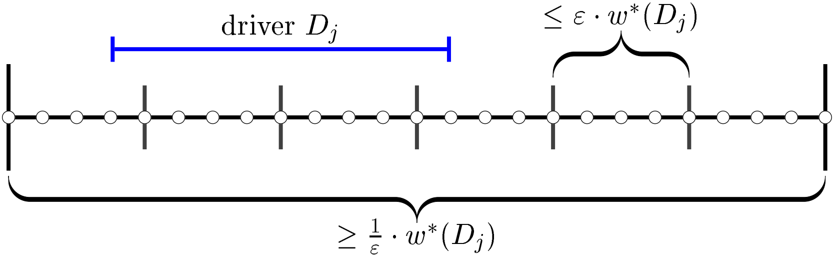

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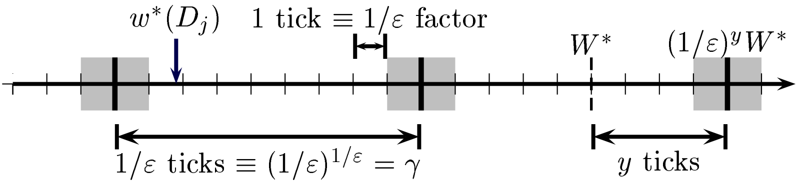

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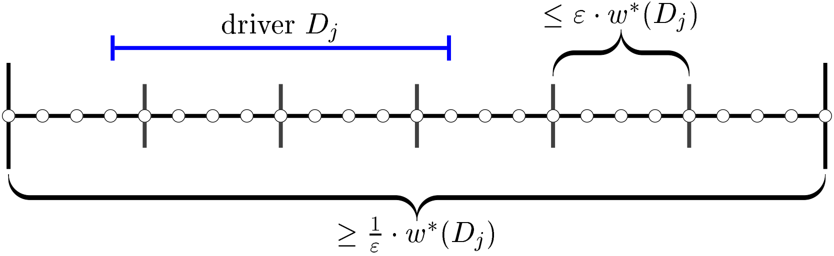
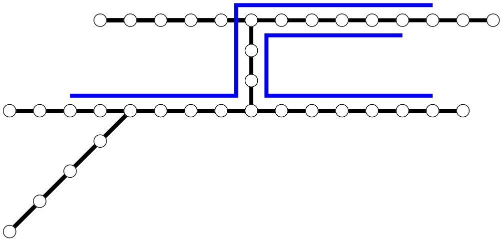


- D_j contains $\geq \frac{1}{\varepsilon} - 1 \geq \frac{1}{2\varepsilon}$ smaller subintervals
- $\Pr[D_j \text{ crosses bigger boundary}] \leq \varepsilon$ $\square$

Extensions (1)

Theorem

There is a PTAS for the generalization, where the input graph is a tree with $O(1)$ leafs.



Extensions (2)

Def: Max Feasible Subsystem Problem For Interval Matrices

Given: Interval matrix $A = \begin{pmatrix} a_1 \\ \cdots \\ a_m \end{pmatrix} \in \{0,1\}^{m \times n}$ (consecutive ones in rows a_j), bounds $\ell_j, u_j \in \mathbb{Q}_+$.
Find: Vector $w \geq \mathbf{0}$ maximizing the number of satisfied constraints

$$\ell_j \leq a_j w \leq u_j$$

Example:

$$\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \leq \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \leq \begin{pmatrix} 8 \\ 6 \\ 9 \end{pmatrix}$$

- **APX**-hard [Elbassioni, Raman, Ray, Sitters '09]

Extensions (3)

Theorem

For any $\varepsilon > 0$, one can find in time $poly(n, m, \ell_{\max})$ weights $w \geq \mathbf{0}$ with

$$\ell_j \leq a_j w \leq (1+\varepsilon) \cdot u_j$$

for at least $(1-\varepsilon)OPT$ constraints.

- Previously known: Multicriteria QPTAS, multicriteria polylog-apx [Elbassioni, Raman, Ray, Sitters '09]

Open problems

- Is there an $O(1)$-apx if G is a tree? (Tollbooth Problem) (until now $O(\log n / \log\log n)$-apx)
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Thanks for your attention