Network Design via Core Detouring for Problems Without a Core

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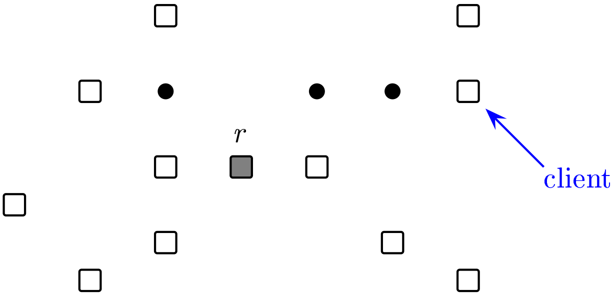
Single-Sink Rent-or-Buy

Input:

- Undirected graph $G = (V, E)$, edge cost $c : E \to \mathbb{Q}_+$
- clients $D \subseteq V$, root r, parameter $M \geq 1$

Goal: Find Steiner Tree T (incl. r) minimizing

$$M \cdot c(T) + \sum_{v \in D} \text{dist}(v, T)$$



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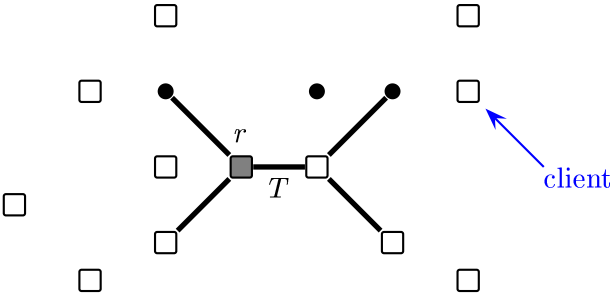

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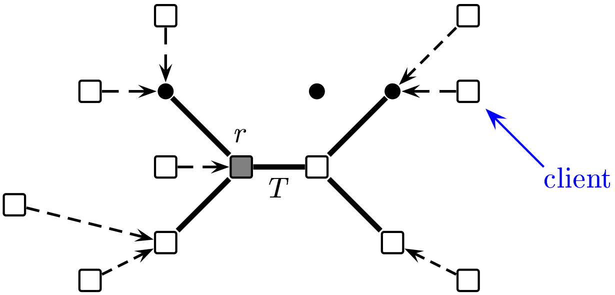

More network design settings

- Single-Sink Rent-or-Buy
- Connected Facility Location
- Single-Sink Buy-at-Bulk
- Unsplittable Single-Sink Buy-at-Bulk
- Virtual Private Network (VPN)

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- Single-Sink Rent-or-Buy
  - 4.55-apx [Swamy, Kumar '02]
  - 3.55-apx [Gupta, Kumar, Roughgarden '03]
  - 2.92-apx [Eisenbrand, Grandoni, R., Schäfer '08]
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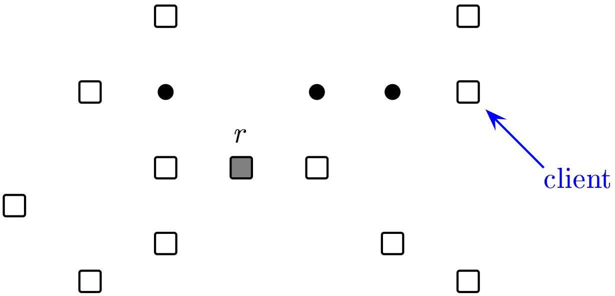
Best approximation algorithms use **random sampling (*)**.

Single-Sink Rent-or-Buy



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(1) Sample each client with prob $\frac{\Theta(1)}{M}$

(2) Output approximate Steiner tree on sampled clients

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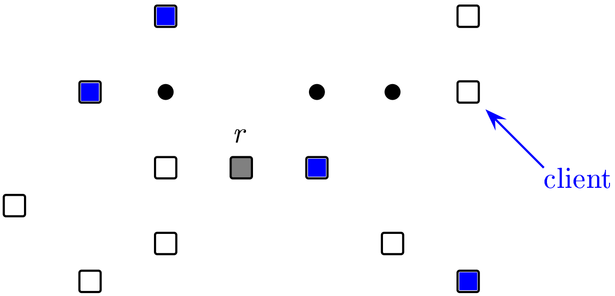

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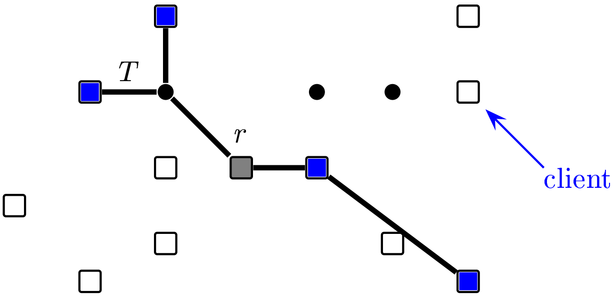

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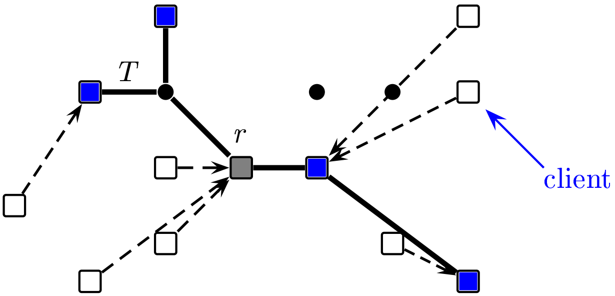

Algorithm: [Gupta, Kumar, Roughgarden '03]

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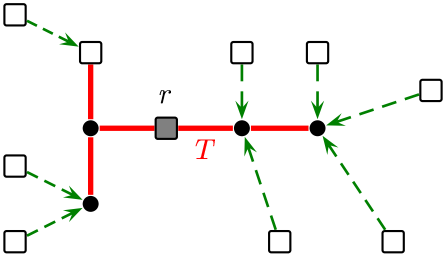
$\rightarrow$ 2.92-approximation [Eisenbrand, Grandoni, R., Schäfer '08]

Core detouring theorem

Theorem (Core Detouring Theorem - [EGSR '08])

Let $D \subseteq V$, connected subgraph T, root $r \in T$. Sample any node in D with prob. $p \in]0,1]$

$$E\left[\sum_{v \in D} dist(v, sampled\ node \cup \{r\})\right] \leq \frac{0.81}{p} \cdot c(T) + 2 \sum_{v \in D} dist(v, T)$$



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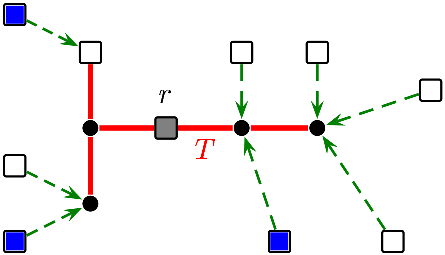

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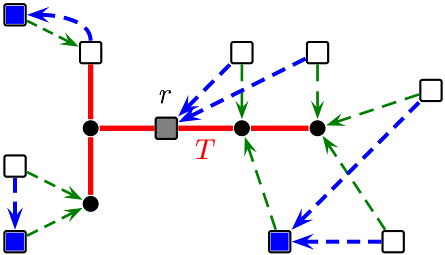

Virtual Private Network (VPN)

Given:

- Undirected graph $G = (V, E)$, costs $c : E \to \mathbb{Q}_+$
- Outgoing traffic bound $b_v^+ \in \mathbb{N}_0$, ingoing traffic bound $b_v^- \in \mathbb{N}_0$

Find: Paths P_{uv}, capacities x_e s.t. to minimize the cost

$$\sum_{e \in E} c(e) \cdot x_e$$

and every *valid* traffic scenario $(D_{u,v})_{u,v \in V}$ can be routed.
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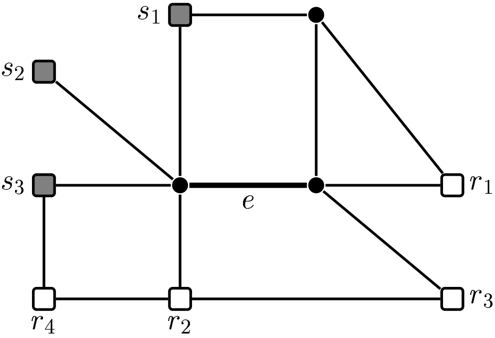
W.l.o.g.:

- senders $s \in S$: $b_s^+ = 1, b_s^- = 0$
- receivers $r \in R$: $b_r^+ = 0, b_r^- = 1$
- non-terminals v: $b_v^+ = b_v^- = 0$
- $|S| \leq |R|$

Example



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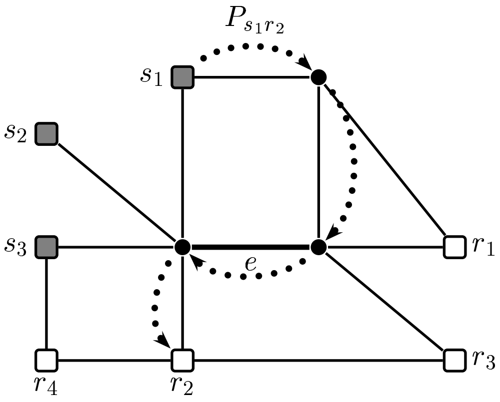

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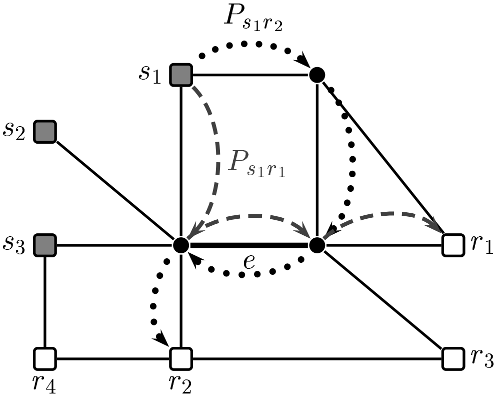

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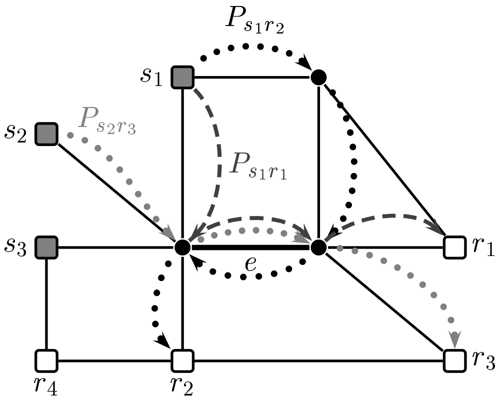

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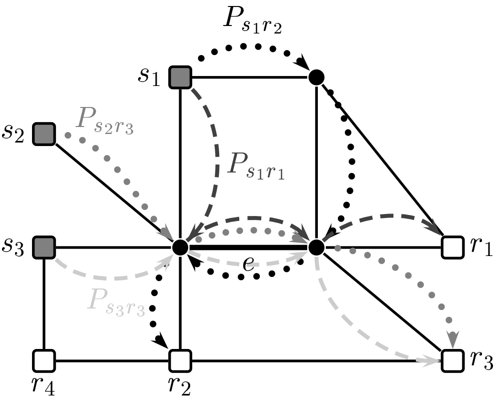

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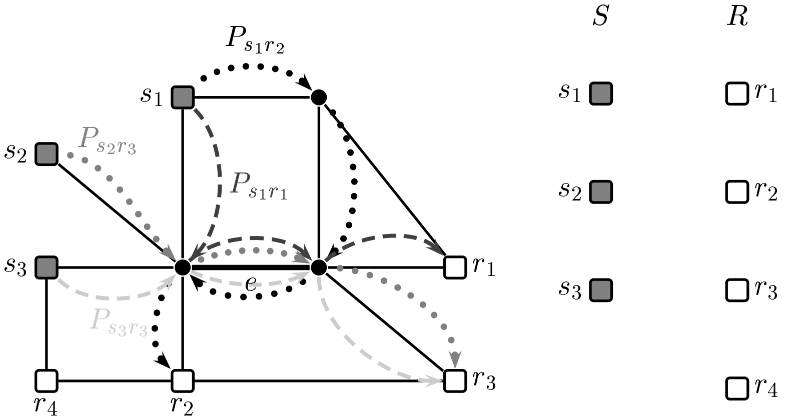

$P_{s_1r_2}$
s_1
s_2
$P_{s_2r_3}$
$P_{s_1r_1}$
s_3
e
r_1
$P_{s_3r_3}$
r_4
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S
R
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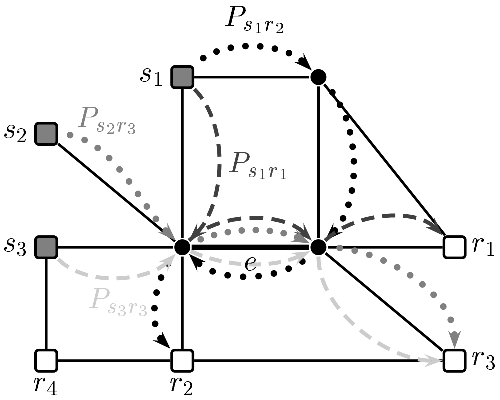

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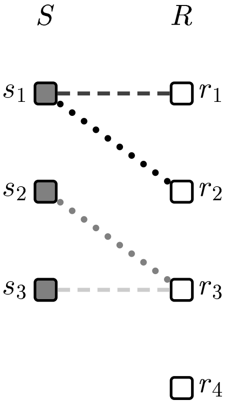

S
R
s_1
r_1
s_2
r_2
s_3
r_3
r_4

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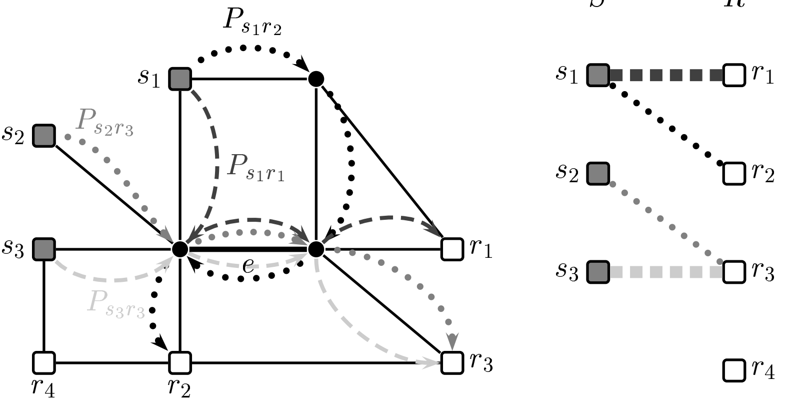

$$x_e \quad = \quad \text{maximal cardinality of a matching in } G_e = (S \cup R, E_e)$$
$$\text{with } (s,r) \in E_e \Leftrightarrow e \in P_{sr}$$

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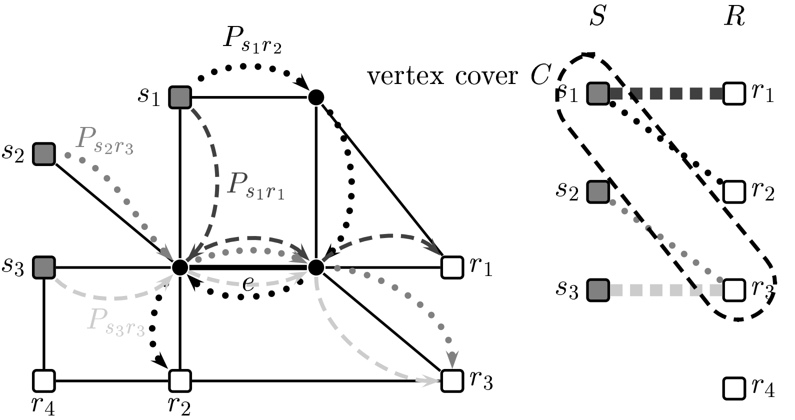

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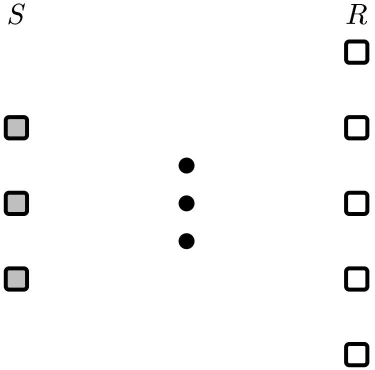
The algorithm of [EGOS '07]

(1) Sample each receiver with prob. $\frac{\alpha}{|S|} \to R'$
(plus an extra random root $r^* \to R'$)

(2) FOR ANY $s \in S$ DO
install cap. 1 on approximate Steiner tree on $\{s\} \cup R'$

(3) FOR ANY $r \in R$ DO
install cap. 1 from r to closest node in R'



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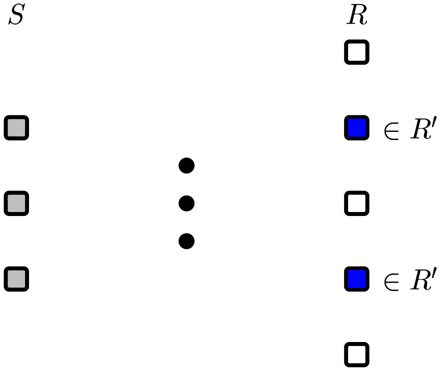

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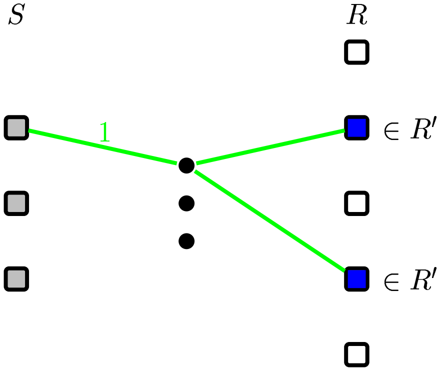

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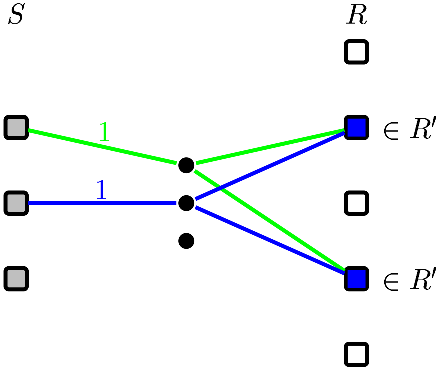

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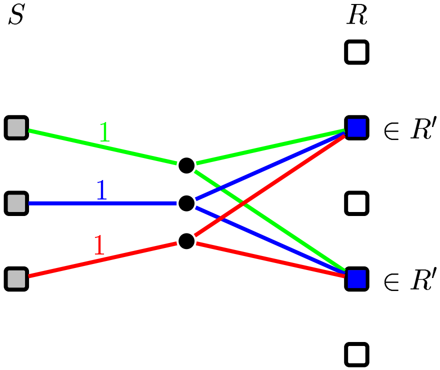

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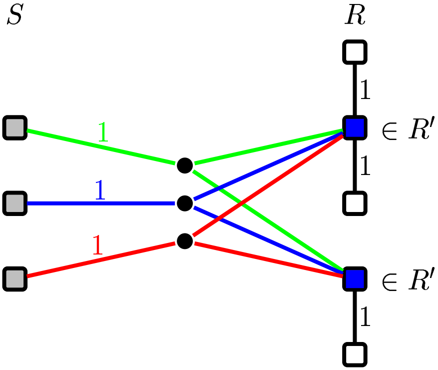
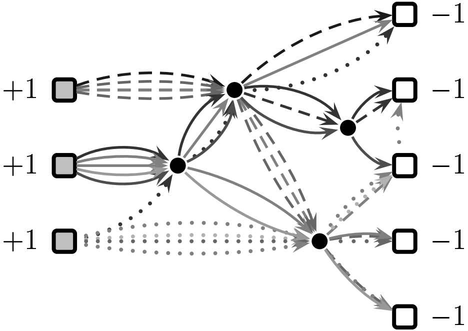
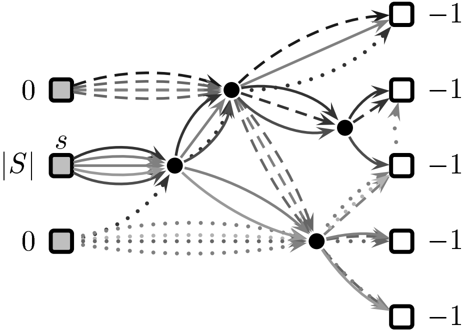


Aggregating senders (1)



Aggregating senders (1)

- Ideally aggregate outgoing capacity into random sender $s \in S \to OPT_s$



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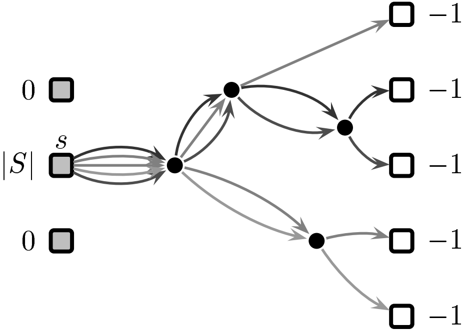

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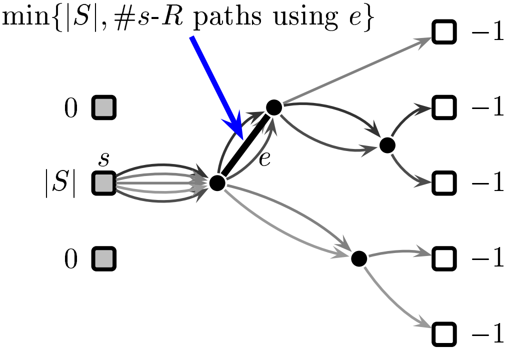

Lemma

$E[OPT_s] \leq OPT$.

- Show: $E[\text{new capacity on edge } e] \leq \text{old capacity on } e$

Aggregating senders (2)

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$$E[\text{new cap. on } e]$$

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\begin{aligned}
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\end{aligned}
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Lemma

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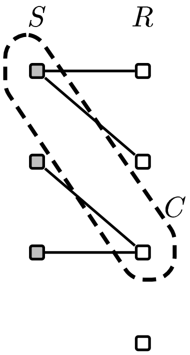
- Let C be vertex cover in capacity graph: $|C| = \text{cap. in } OPT$



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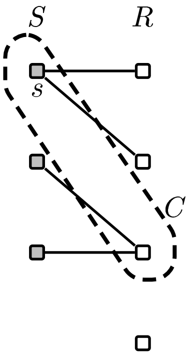

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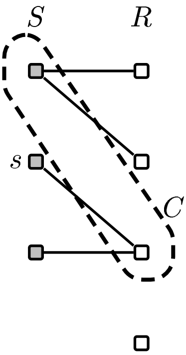

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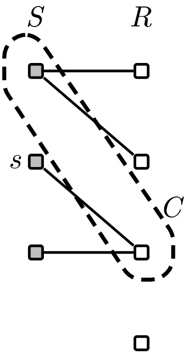

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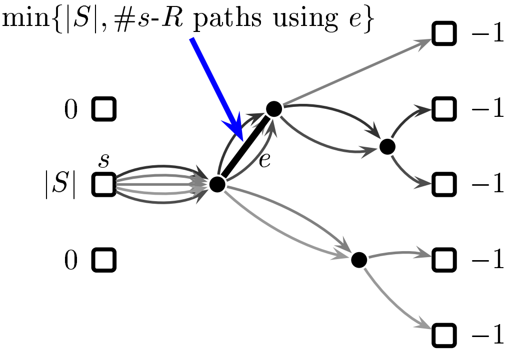
The Core



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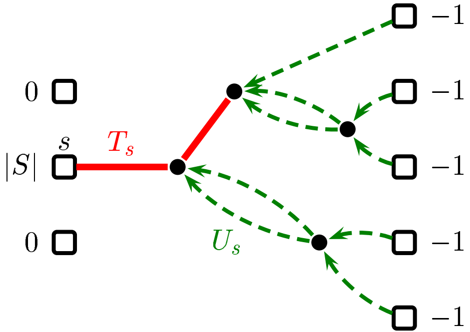
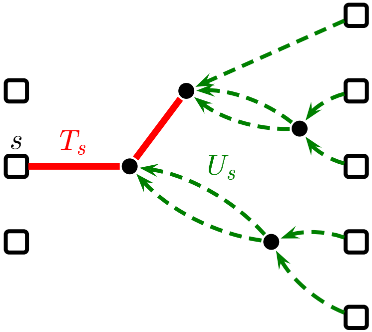


- OPT_s is tree
- $T_s := \{e \mid cap = |S|\}$, $U_s = \{e \mid cap < |S|\}$
- $|S| \cdot E[c(T_s)] + E[c(U_s)] \leq OPT$

The Analysis (1)

Lemma

$$E[c(apx\ Steiner\ tree\ on\ \{s\} \cup R')] \leq 1.39 \cdot \left(c(T_s) + \frac{\alpha}{|S|} \cdot c(U_s)\right)$$



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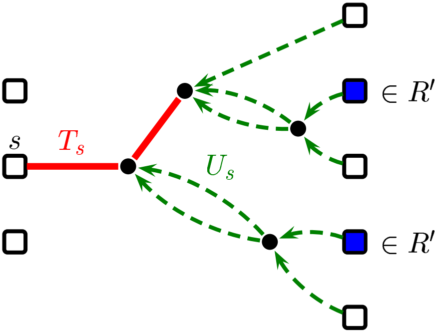
$$E[c(apx\ Steiner\ tree\ on\ \{s\} \cup R')] \leq 1.39 \cdot \left(c(T_s) + \frac{\alpha}{|S|} \cdot c(U_s) \right)$$



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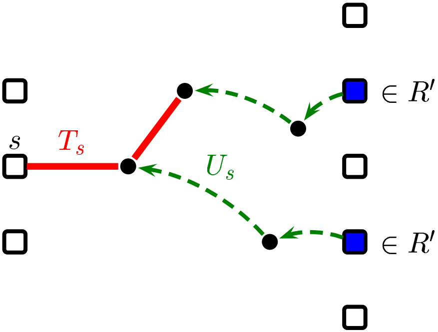

- Take Steiner tree $T_s + \bigcup_{r \in R'}$(shortest path from r to T_s).
- Use 1.39-apx for Steiner Tree [BGRS '10]

The Analysis (2)

Lemma

$$E[\sum_{r \in R} dist(r, R')] \leq 0.81 \cdot \frac{|S|}{\alpha} \cdot E[c(T_s)] + 2 \cdot E[c(U_s)]$$

- Apply Core Detouring Theorem!

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The algorithm gives a 2.8-apx.

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\begin{aligned}
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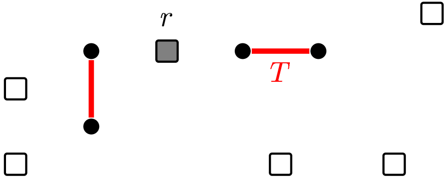
\leq \quad & 2.8 \cdot OPT \qquad \square
\end{aligned}
$$

Multi Core Detouring theorem

Theorem (Multi Core Detouring Theorem)

Let $D \subseteq V$, subgraph T, root r. Sample any node in D with prob. $p \in]0,1]$

$$E\left[\sum_{v \in D} dist(v, sampled\ nodes \cup \{r\})\right] \leq \frac{0.81}{p} c(T) + 2 \sum_{v \in D} dist_T(v, r)$$



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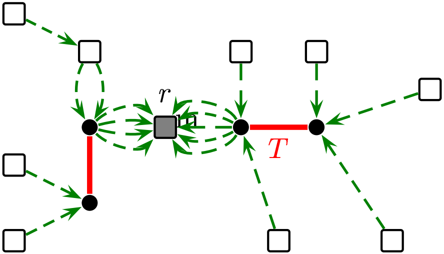

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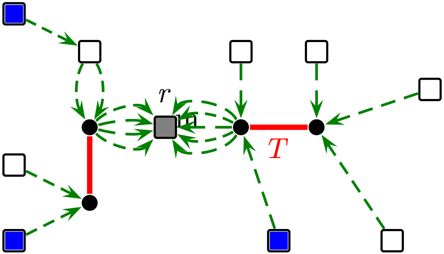

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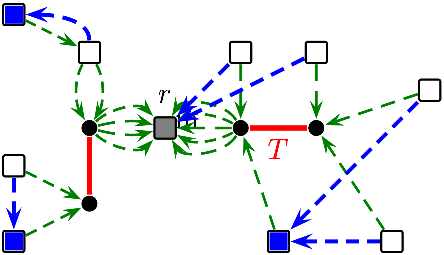

Our results

- Virtual Private Network (VPN)
  - 3.55-apx [Eisenbrand, Grandoni, Oriolo, Skutella '07]
  - **2.80-apx**
- Single-Sink Buy-at-Bulk
  - 25-apx [Grandoni, Italiano '06]
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Thanks for your attention