Steiner Tree Approximation via Iterative Randomized Rounding

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DISCRETE OPTIMIZATION

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

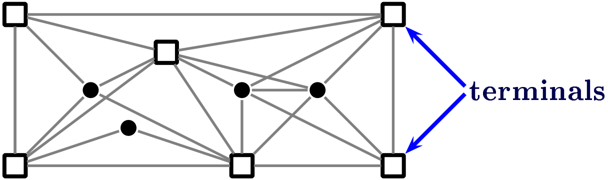
Steiner Tree

Given:

- undirected graph $G = (V, E)$
- cost $c : E \to \mathbb{Q}_+$
- terminals $R \subseteq V$

Find: Min-cost Steiner tree, spanning R.

$$OPT := \min\{c(S) \mid S \text{ spans } R\}$$



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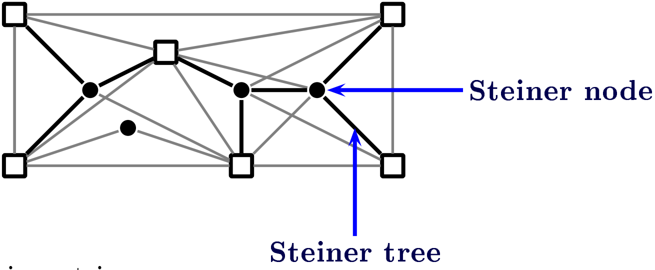

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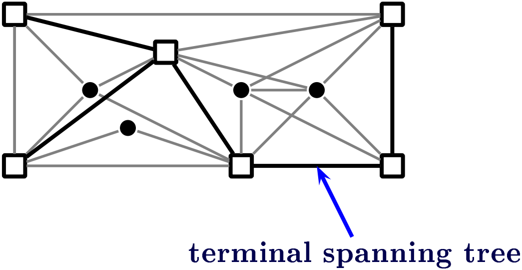
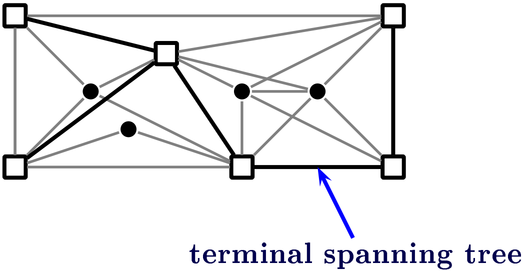
Spanning tree



Min-cost terminal spanning tree (MST):

Spanning tree



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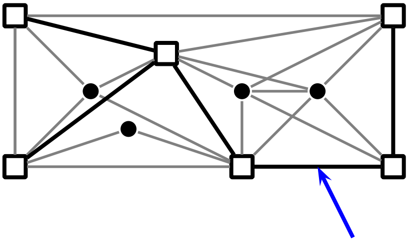

Min-cost terminal spanning tree (MST):

- Can be computed in poly-time.
- Costs $\leq 2 \cdot OPT$.

Known results for Steiner tree:

Approximations:

- 2-apx (*minimum spanning tree heuristic*)
- 1.83-apx [Zelikovsky '93]
- 1.667-apx [Prömel & Steger '97]
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Hardness:

- **NP**-hard even if edge costs $\in \{1, 2\}$ [Bern & Plassmann '89]
- no $< \frac{96}{95}$-apx unless $\mathbf{NP} = \mathbf{P}$ [Chlebik & Chlebikova '02]

Our results:

Theorem

There is a polynomial time 1.39-approximation.

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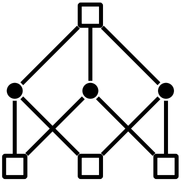
Theorem

The Directed-Component Cut Relaxation has an integrality gap of at most 1.55.

- First < 2 bound for *any* LP-relaxation.

Bi-directed cut relaxation



Bi-directed cut relaxation

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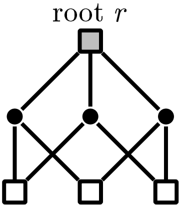

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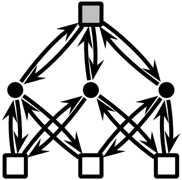

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$$
\begin{aligned}
\min \sum_{e \in E} c(e) z_e & \qquad \text{(BCR)} \\
\sum_{e \in \delta^+(U)} z_e \geq 1 & \qquad \forall U \subseteq V \setminus \{r\} : U \cap R \neq \emptyset \\
z_e \geq 0 & \qquad \forall e \in E.
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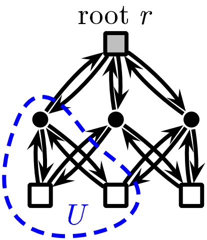

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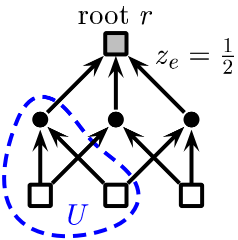

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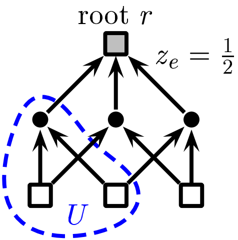


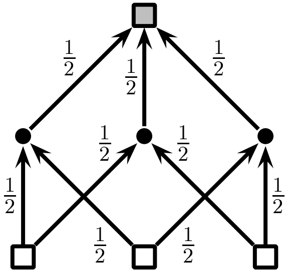
Theorem (Edmonds '67)

$R = V \Rightarrow BCR$ integral

- Integrality gap $\leq 4/3$ for quasi-bipartite graphs [Chakrabarty, Devanur, Vazirani '08]
- Integrality gap $\in [1.16, 2]$

How to exploit BCR?

- Is there always an edge e with $x_e \geq 1/2$?



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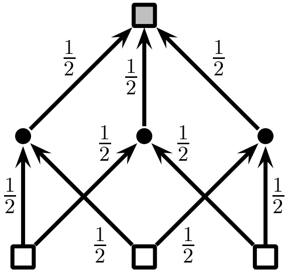

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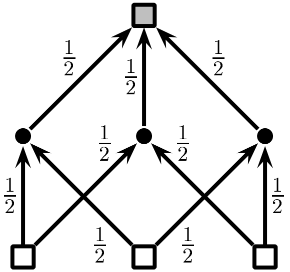

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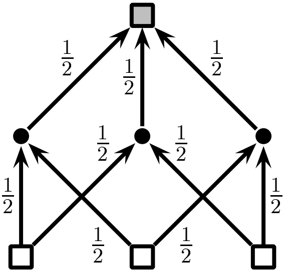

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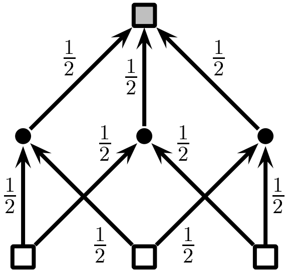

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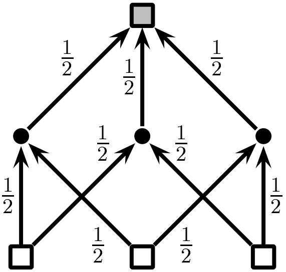

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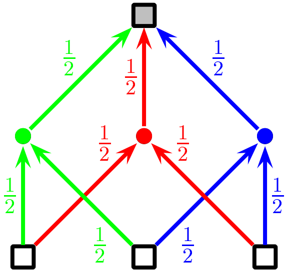

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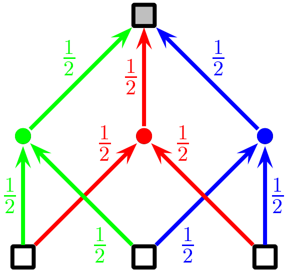
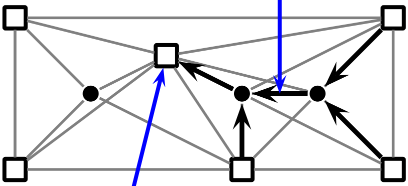


Components



- $\mathbf{C}$ = set of directed components

Directed component cut relaxation

$$\min \sum_{C \in \mathbf{C}} c(C) \cdot x_C \qquad \text{(DCR)}$$

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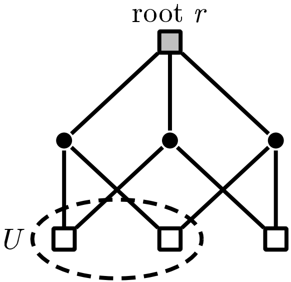

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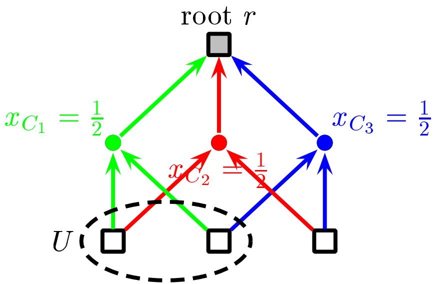

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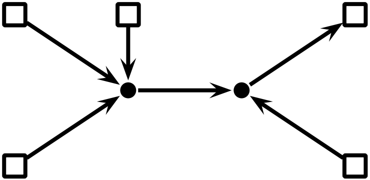
Properties:

- Number of variables: exponential
- Number of constraints: exponential
- Approximable within $1 + \varepsilon$ (we ignore the ε here).

Solvability of the LP

Lemma

For any $\varepsilon > 0$, a solution x of cost $\leq (1+\varepsilon)OPT_f$ can be computed in polynomial time.



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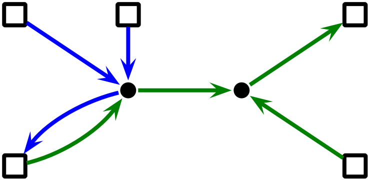

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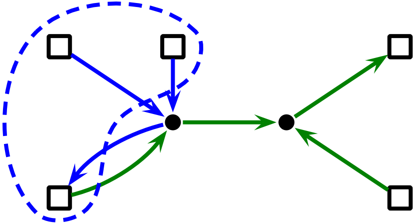

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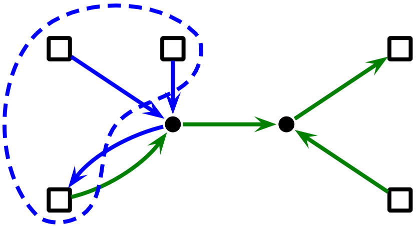

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- Compact flow formulation $\rightarrow$ # constraints polynomial
  (or solve with ellipsoid method). □

An iterative randomized rounding algo

(1) FOR $t = 1, \ldots, \infty$ DO

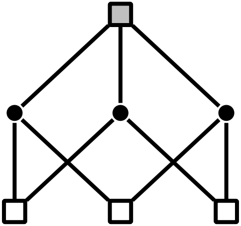
(2) Compute opt. LP solution x

(3) Sample a component:

$$\Pr[\text{sample } C] = \frac{x_C}{\mathbf{1}^T x}$$

and contract it.

(4) IF all terminals connected THEN output sampled components



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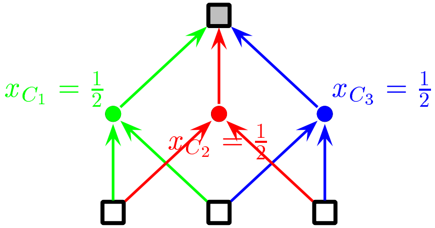

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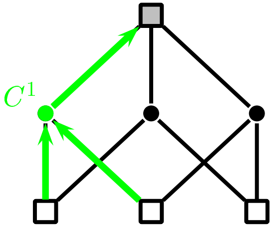

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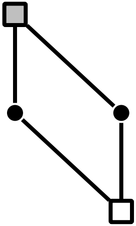

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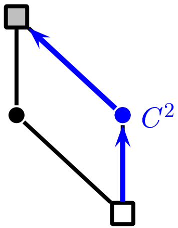

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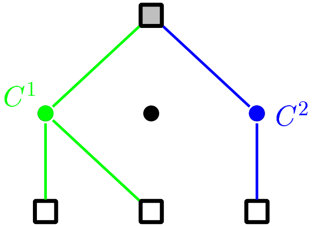

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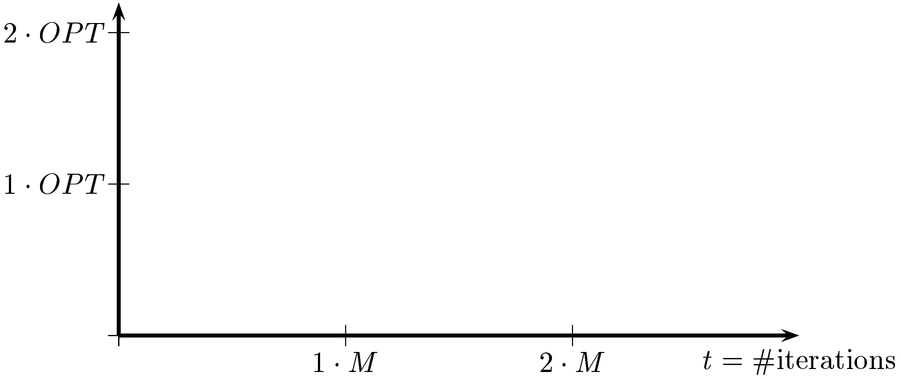
- W.l.o.g. $M := \mathbf{1}^T x$ invariant

Roadmap

- In one iteration t:

  $$E[c(\text{comp. sampled in it. } t)] = \sum_C \frac{x_C}{M} \cdot c(C) \leq \frac{1}{M} \cdot OPT \text{ in it } t$$



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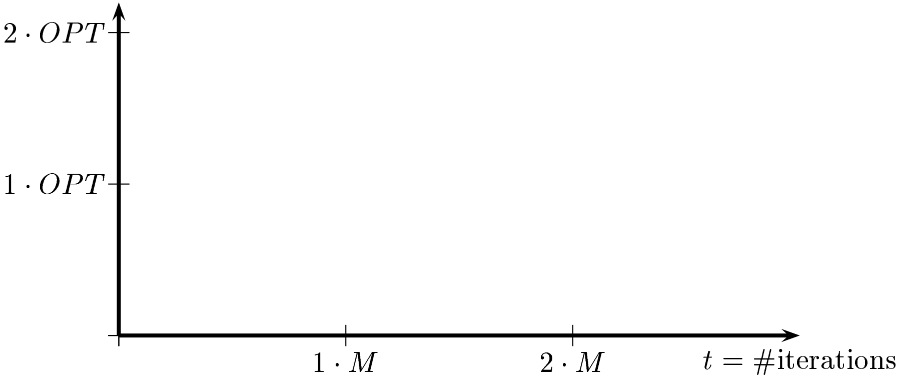

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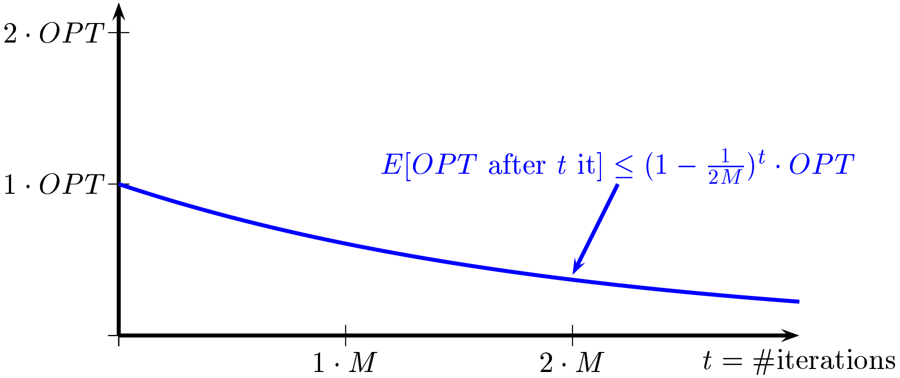

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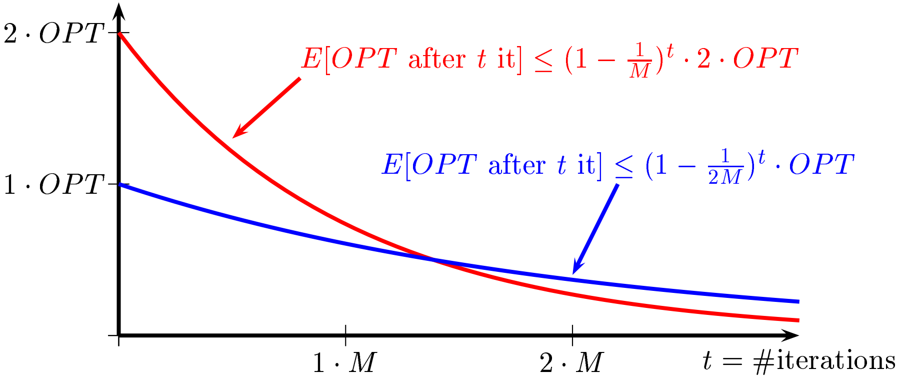
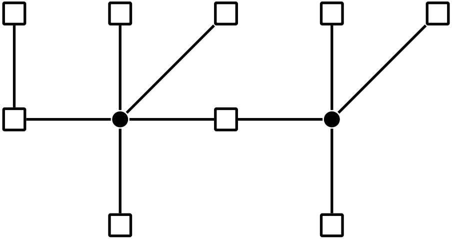


Bridges

- Let S be Steiner tree



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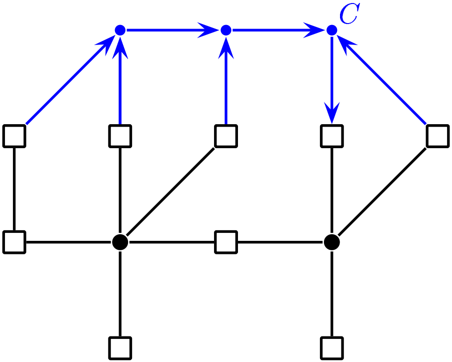

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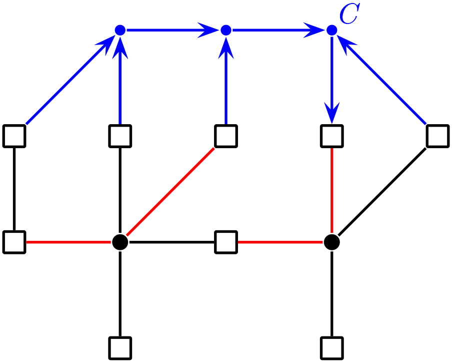
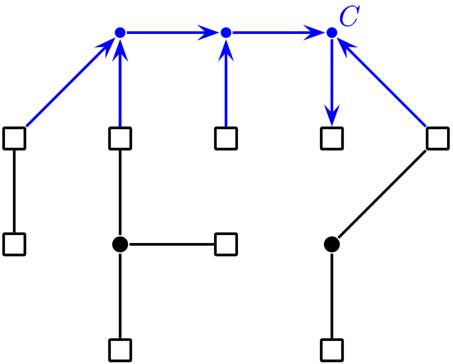


- **Bridges:**

$$Br_S(C) = \text{argmax}\{c(B) \mid B \subseteq S,\ S \backslash B \cup C \text{ is connected}\}$$

Bridges

- Let S be Steiner tree, C a component



- **Bridges:**

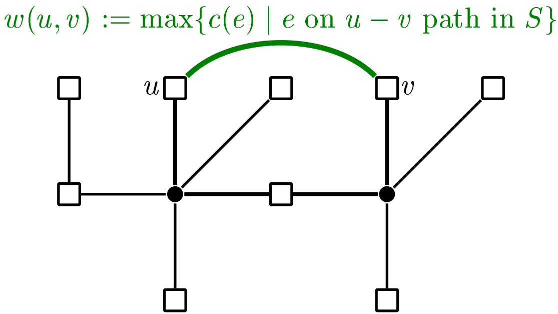
$$Br_S(C) = \text{argmax}\{c(B) \mid B \subseteq S,\ S \backslash B \cup C \text{ is connected}\}$$

The saving function

Definition

For a Steiner tree S, the **saving function** $w : E \to \mathbb{Q}_+$ is defined as

$$w(u, v) := \max\{c(e) \mid e \text{ on } u - v \text{ path in } S\}.$$

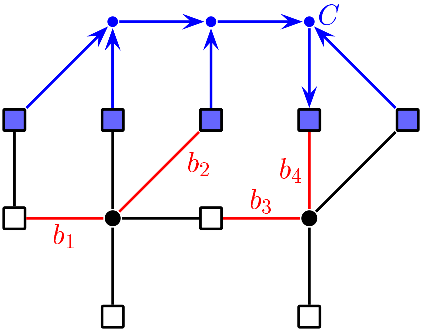


A saving lemma

Lemma

For any Steiner tree S and component C, $\exists$ saving tree spanning the terminals of C with

$$c(Br_S(C)) = w(\textit{saving tree})$$



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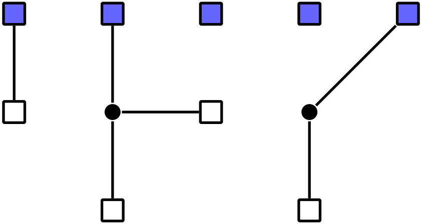

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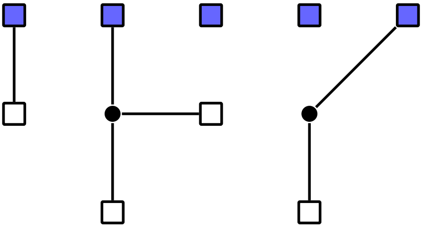

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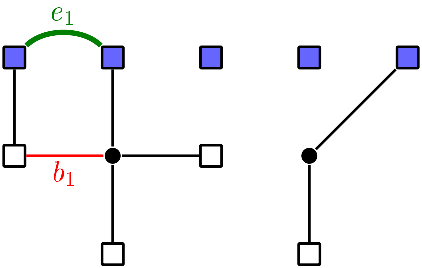

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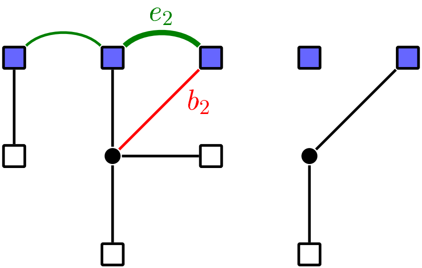

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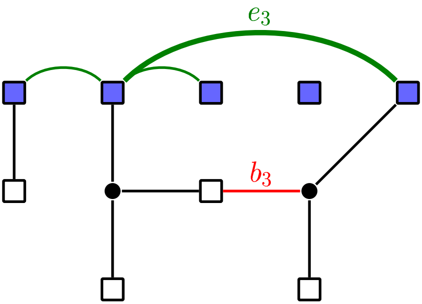

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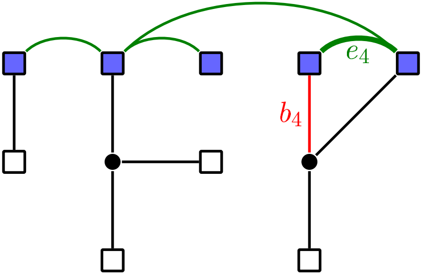

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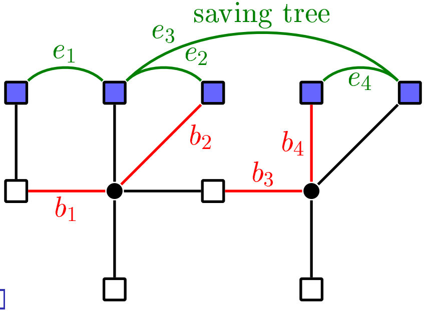

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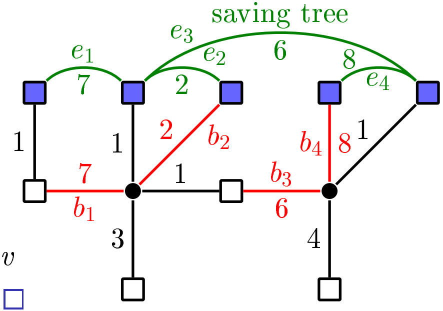
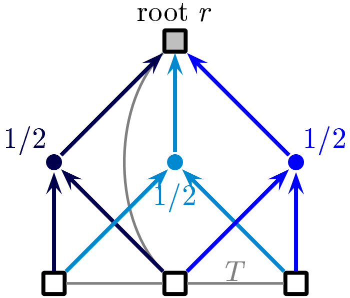


The Bridge Lemma (1)

Lemma (Bridge Lemma)

For T terminal spanning tree, x LP solution:

$$\sum_{C \in \mathbf{C}} x_C \cdot c(Br_T(C)) \geq c(T)$$



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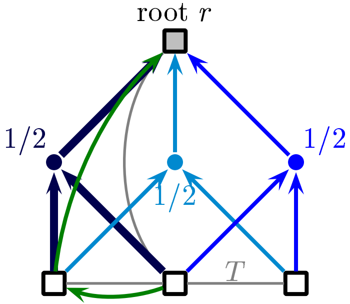

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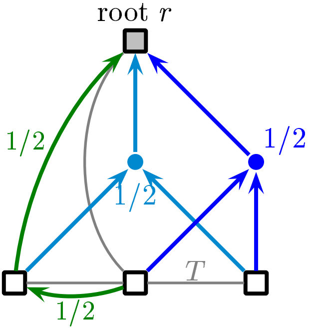

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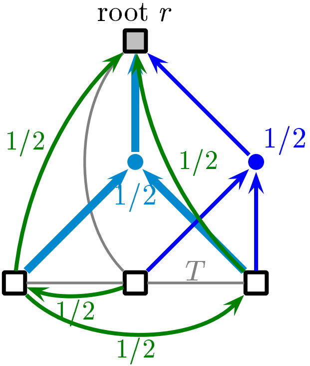

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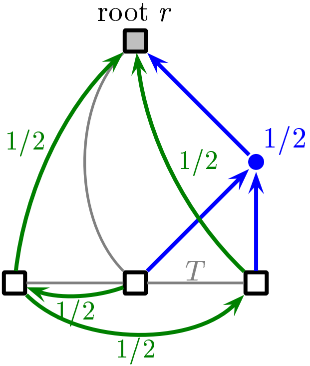

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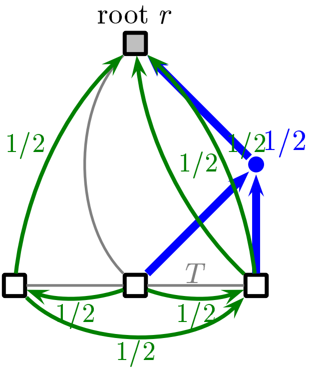

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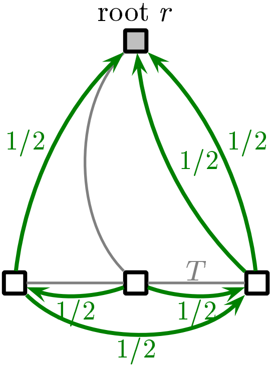

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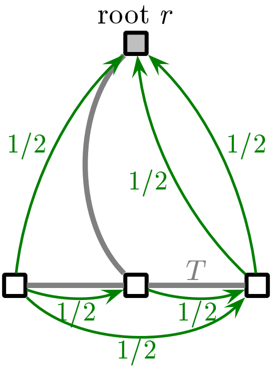

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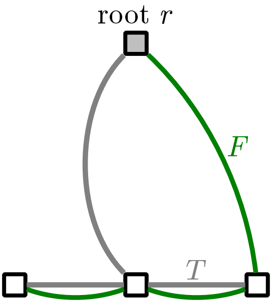

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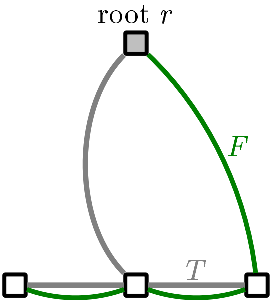
Edmonds Thm

Cycle rule



A 1st bound on OPT

Lemma

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A 2nd bound on OPT

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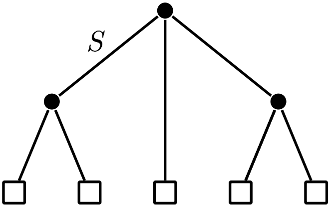
- Let S be opt. Steiner tree



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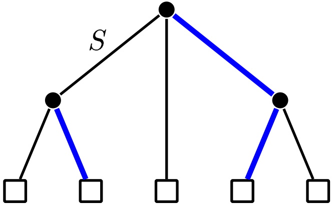

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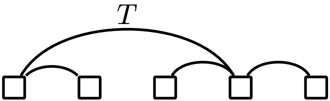

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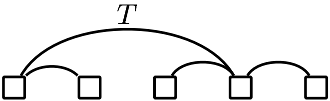


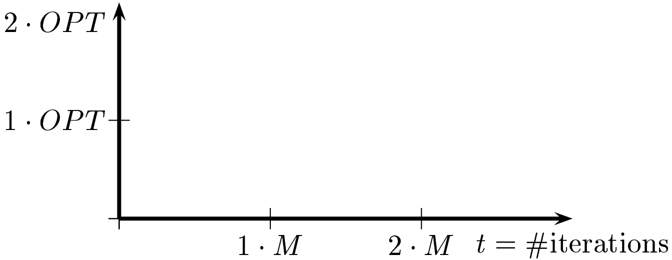
$$E[\text{save on } S] \geq E[\text{save on } T] \overset{\text{Bridge Lem}}{\geq} \frac{1}{M} \cdot \underbrace{c(T)}_{\geq \frac{1}{2} c(S)} \geq \frac{1}{2M} \cdot c(S)$$

□

The approximation guarantee

Theorem

$E[APX] \leq (1.5+\varepsilon) \cdot OPT.$



- Cost of sampled components:

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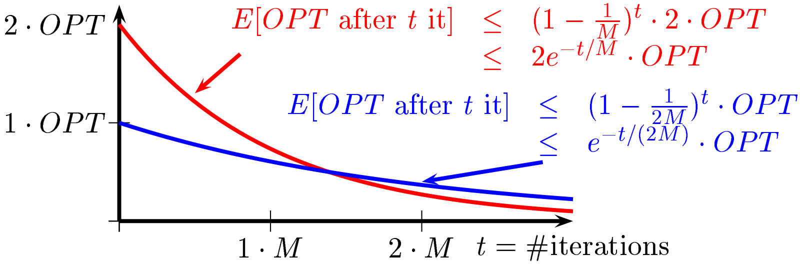

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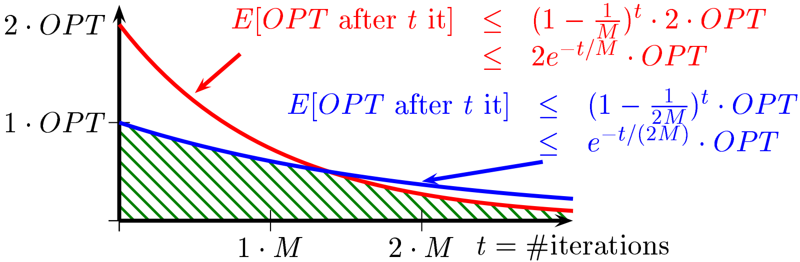

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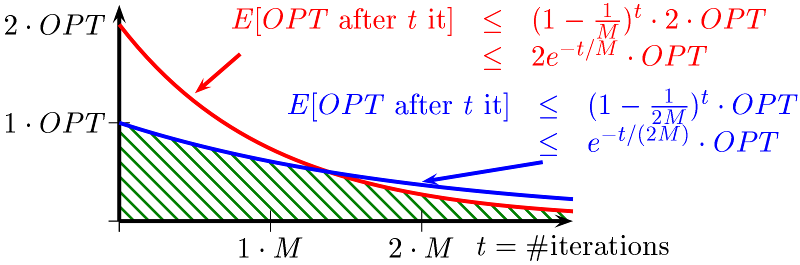

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Open problems

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- Byrka, Grandoni, Rothvoß, Sanità - STOC'10:
  An improved LP-based approximation for Steiner Tree

http://infoscience.epfl.ch/record/148220/files/SteinerTree-STOC2010.pdf

Thanks for your attention