An asymptotic 3/2-approximation algorithm for static-priority multiprocessor scheduling of implicit deadline tasks

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Real-time Scheduling

Given: synchronous tasks $\tau_1, \ldots, \tau_n$ where task τ_i

- is periodic with period $p(\tau_i)$
- has running time $c(\tau_i)$
- has implicit deadline

W.l.o.g.: Task τ_i releases job of length $c(\tau_i)$ at $0, p(\tau_i), 2p(\tau_i), \ldots$

Scheduling policy:

- multiprocessor
- fixed priority
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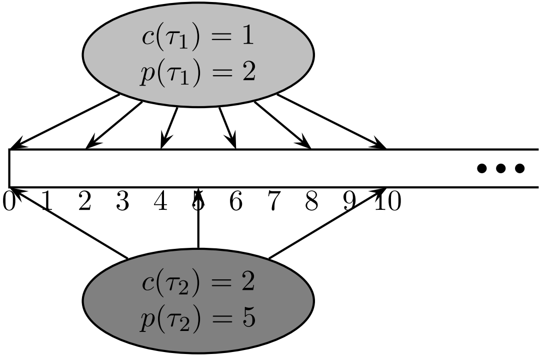
Definition

$u(\tau) = \frac{c(\tau)}{p(\tau)} =$ *utilization* of task τ

Example



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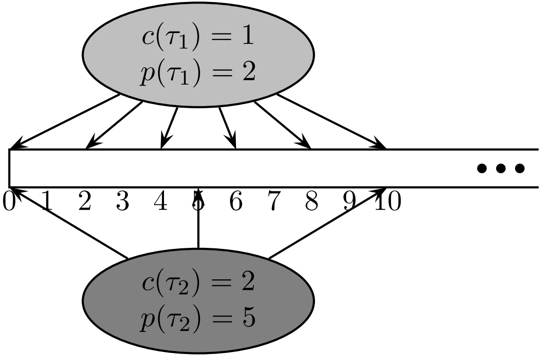
Theorem (Liu, Layland '73)

Optimal priorities: $\frac{1}{p(\tau_i)}$ *for* τ_i *(Rate-monotonic Schedule)*



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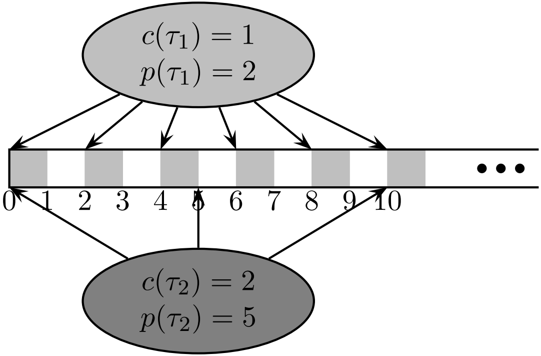

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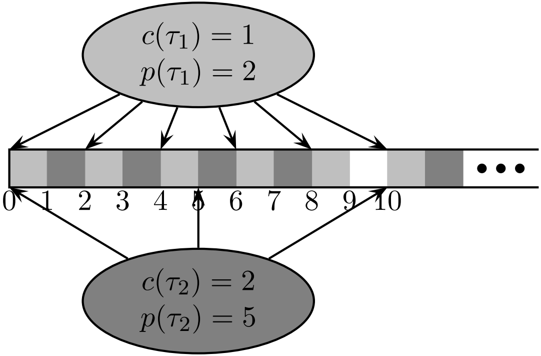

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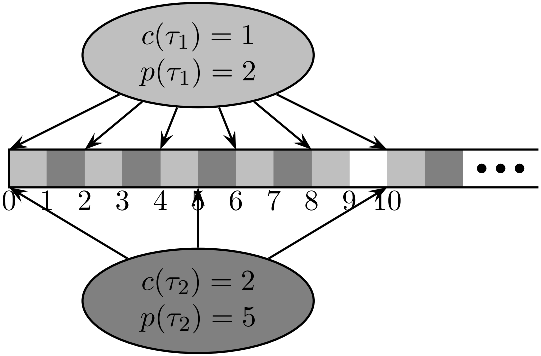

Theorem (Liu, Layland '73)

$u(\mathcal{S}) = \sum_{\tau \in \mathcal{S}} u(\tau) \leq \ln(2) \approx 0.69 \Rightarrow$ *RM-schedule feasible*

Response times

Definition

Response time $r(\tau_i)$ of τ_i := longest time that an instance of task τ_i waits for accomplishment

Theorem (Lehoczky et al. '89)

If $p(\tau_1) \leq \ldots \leq p(\tau_n)$ *then* $r(\tau_i)$ *is the smallest value s.t.*

$$c(\tau_i) + \sum_{j<i} \left\lceil \frac{r(\tau_i)}{p(\tau_j)} \right\rceil c(\tau_j) \leq r(\tau_i)$$

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Theorem (Eisenbrand & R. - RTSS'08)

Unless $\mathbf{NP} = \mathbf{P}$ response times $r(\tau_i)$ cannot be approximated within any constant.

A sufficient feasibility criterium

Observation:

If periods multiples of each other:
Tasks $\mathcal{S}$ RM-schedulable $\Leftrightarrow u(\mathcal{S}) \leq 1$ ($\rightarrow$ Bin Packing)

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Lemma (Burchard et al '95)

For tasks $\mathcal{S} = \{\tau_1, \ldots, \tau_n\}$ *define*

$$S(\tau_i) = \log p(\tau_i) - \lfloor \log p(\tau_i) \rfloor$$

and

$$\beta(\mathcal{S}) = \max_{\tau \in \mathcal{S}} S(\tau) - \min_{\tau \in \mathcal{S}} S(\tau)$$

Then the tasks can be RM-scheduled if

$$u(\mathcal{S}) \leq 1 - \beta(\mathcal{S}) \cdot \ln(2).$$

A simple First Fit algorithm

First Fit:

1. Sort tasks s.t. $0 \leq S(\tau_1) \leq ... \leq S(\tau_n) < 1$
2. Distribute tasks via First Fit (using $u(\mathcal{S}) \leq 1 - \beta(\mathcal{S}) \cdot \ln(2)$ feasibility criterium)

Lemma

Given periodic tasks $\mathcal{S} = \{\tau_1, \ldots, \tau_n\}$ and $\varepsilon > 0$.

(1) *Let $u(\tau_i) \leq \alpha$ then #proc. $\leq \frac{1}{1-\alpha} u(\mathcal{S}) + 3$.*

(2) *Let $u(\tau_i) \leq \frac{1}{2} - \varepsilon$ then #proc. $\leq \frac{n}{2} + \frac{1}{\varepsilon}$*

(3) *First Fit gives an asymptotic 2-approx.*

Known results for multiprocessor case

- An asymptotic 7/4-approximation algo [Burchard '95]

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Theorem (Karrenbauer & R. '09)

There is an asymptotic 3/2-approximation algorithm for Multiprocessor Real-time Scheduling with running time $O(n^3)$.

Using the power of matchings

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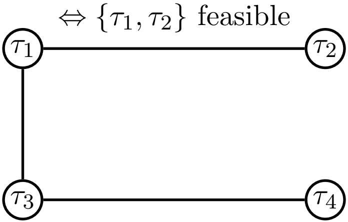

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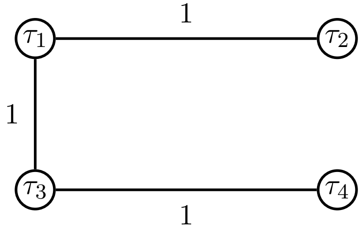

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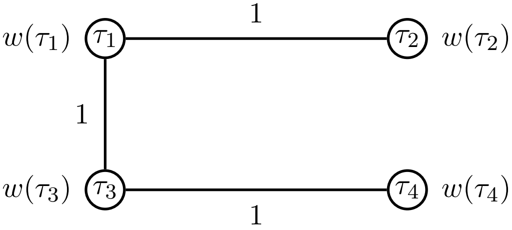

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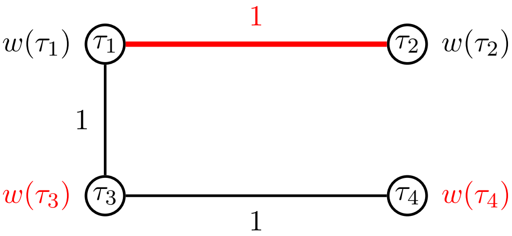

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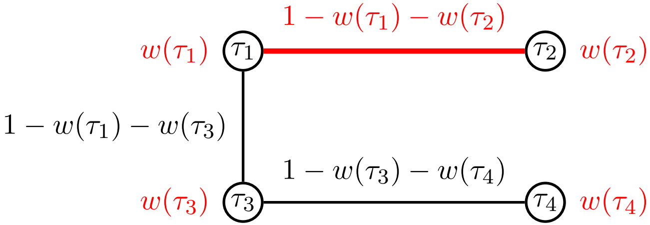

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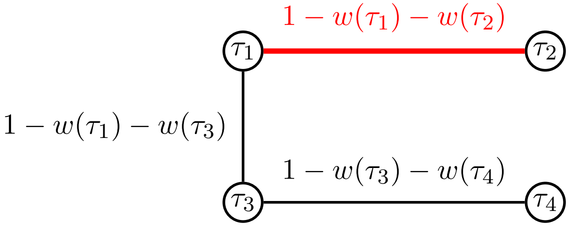

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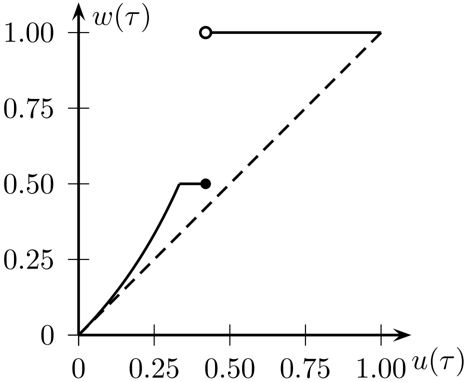
The vertex costs



$$w(\tau) = \lim_{k\to\infty} \frac{\#\text{proc. needed to schedule } k \text{ copies of } \tau}{k}$$

$$= \begin{cases} u(\tau) \cdot \frac{1}{1-u(\tau)} & u(\tau) \leq \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} < u(\tau) \leq \frac{1}{2} - \frac{1}{12k} \\ 1 & u(\tau) > \frac{1}{2} - \frac{1}{12k} \end{cases}$$

The algorithm

1. Construct $G = (\mathcal{S}, E)$ with edges $(\tau_1, \tau_2) \in E \Leftrightarrow \{\tau_1, \tau_2\}$ RM-schedulable. Choose edge costs $c_e = 1$ and vertex costs

$$w(\tau) = \begin{cases} u(\tau) \cdot \frac{1}{1-u(\tau)} & u(\tau) \leq \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} < u(\tau) \leq \frac{1}{2} - \frac{1}{12k} \\ 1 & u(\tau) > \frac{1}{2} - \frac{1}{12k} \end{cases}$$

2. Solve

$$\min\{\sum_{e \in M} c(e) + \sum_{\tau \in \mathcal{S} \backslash V(M)} w(\tau) \mid M \text{ is matching}\}$$

3. For all $\{\tau_1, \tau_2\} \in M$ create a processor with $\{\tau_1, \tau_2\}$
4. Define
   - $\mathcal{S}_i = \{\tau \in \mathcal{S} \backslash V(M) \mid \frac{1}{3} \cdot \frac{i-1}{k} \leq u(\tau) < \frac{1}{3} \cdot \frac{i}{k}\}$ $\forall i = 1, \ldots, k$
   - $\mathcal{S}_{k+1} = \{\tau \in \mathcal{S} \backslash V(M) \mid \frac{1}{3} \leq u(\tau) \leq \frac{1}{2} - \frac{1}{12k}\}$
   - $\mathcal{S}_{k+2} = \{\tau \in \mathcal{S} \backslash V(M) \mid u(\tau) > \frac{1}{2} - \frac{1}{12k}\}$
5. Distribute $\mathcal{S}_1, \ldots, \mathcal{S}_{k+2}$ via First Fit.

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The algorithm yields a solution of cost $(\frac{3}{2} + o(1)) \cdot OPT$.

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- Let $P_1, \ldots, P_m$ optimum solution. Each processor P_i contributes at most $(\frac{3}{2} + o(1))$ to the matching

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- Let $P_1, \ldots, P_m$ optimum solution. Each processor P_i contributes at most $(\frac{3}{2} + o(1))$ to the matching
- Let $P_i = \{\tau_1, \ldots, \tau_q\}$ with $u(\tau_1) \geq \ldots \geq u(\tau_q)$.

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$$1 + \sum_{j=3}^{q} w(\tau_j) = 1 + \underbrace{\sum_{j=3}^{q} u(\tau_j)}_{\leq 1/3} \cdot \overbrace{\frac{1}{1 - \underbrace{u(\tau_j)}_{\leq 1/3}}}^{\leq 3/2} \leq \frac{3}{2}.$$

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- Case: $u(\tau_2) \leq \frac{1}{3} < u(\tau_1) \leq \frac{1}{2} - \frac{1}{12k}$. Leave tasks uncovered. Contr.: $\leq \frac{3}{2}$

Average case behavior

Define the **waste** of a solution as the cumulated ratios of idle times ($= APX - u(\mathcal{S})$).

Theorem (Karrenbauer & R. - ESA'09)

Generate n tasks with

- *periods arbitrary*
- *draw $u(\tau_i) \in [0, 1]$ uniformly at random.*

Then

$$E[\textit{waste of First Fit}] \leq O(n^{3/4}(\log n)^{3/8}).$$

Corollary (Karrenbauer & R. - ESA'09)

Given n tasks $\mathcal{S}$ w.r.t. above probability distribution, the best solution in which at most 2 task share a processor, costs in expectation at most

$$u(\mathcal{S}) + O(n^{3/4}(\log n)^{3/8})$$

Average case behavior (2)

Corollary (Karrenbauer & R. '09)

Given n tasks w.r.t. above probability distribution then

$$E[\textit{waste of matching algo}] \leq O(n^{3/4}(\log n)^{3/8})$$

Open problems

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