New Hardness Results for Diophantine Approximation

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ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Simultaneous Diophantine Approximation (SDA)

Given:

- $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}$
- bound $N \in \mathbb{N}$
- error bound $\varepsilon > 0$

Decide:

$$\exists Q \in \{1, \ldots, N\} : \max_{i=1,\ldots,n} \left| \alpha_i - \frac{\mathbb{Z}}{Q} \right| \leq \frac{\varepsilon}{Q}$$

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- Gap version **NP**-hard [Rössner & Seifert '96, Chen & Meng '07]

Inapproximability

Theorem (Rössner & Seifert '96, Chen & Meng '07)

Given $\alpha_1, \ldots, \alpha_n$, N, $\varepsilon > 0$ *it is* **NP**-*hard to distinguish*

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Given $\alpha_1, \ldots, \alpha_n$, N, $\varepsilon > 0$ *it is* **NP**-*hard to distinguish*

- $\exists Q \in \{N/2, \ldots, N\} : \max_{i=1,\ldots,n} |\lceil Q\alpha_i \rfloor - Q\alpha_i| \leq \varepsilon$
- $\nexists Q \in \{1, \ldots, 2^n \cdot N\} : \max_{i=1,\ldots,n} |\lceil Q\alpha_i \rfloor - Q\alpha_i| \leq n^{\frac{O(1)}{\log\log n}} \cdot \varepsilon$

even if ε *small*.

Reduction SDA$\to$ DDA

Theorem (Directed Diophantine Approximation (DDA))

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- Given SDA instance $\alpha_1, \ldots, \alpha_n, \varepsilon, N$, choose $\delta := \frac{2\varepsilon}{N}$

$$\begin{aligned}
\alpha'_i &:= \alpha_i - \delta & \forall i = 1, \ldots, n \\
\alpha'_{i+n} &:= -(\alpha_i + \delta) & \forall i = 1, \ldots, n \\
\varepsilon' &:= 3\varepsilon &
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- Note that $|\lceil -x \rceil - (-x)| = |\lfloor x \rfloor - x|$

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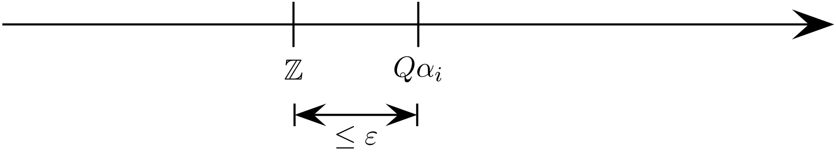
- Note that $|\lceil -x \rceil - (-x)| = |\lfloor x \rfloor - x|$
- Assume $\varepsilon < \frac{1}{6} \cdot (\frac{1}{2})^n$

Proof: SDA-YES ⇒ DDA-YES

- Let $Q \in \{N/2, \ldots, N\} : \max_{i=1,\ldots,n} |\lceil Q\alpha_i \rfloor - Q\alpha_i| \leq \varepsilon$.



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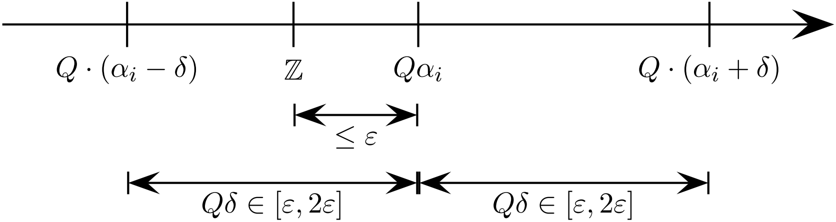

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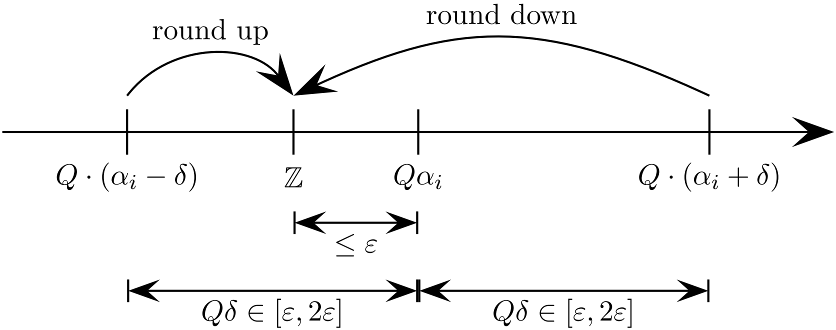

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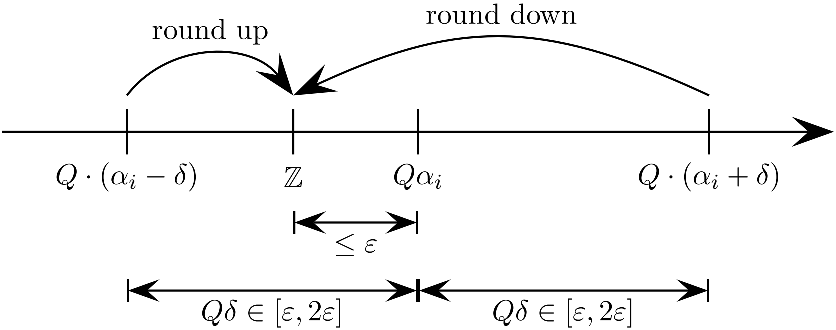

- Conclusion: Q is DDA solution

$$\max_{j=1,\ldots,2n} : |\lceil Q\alpha'_j \rceil - Q\alpha'_j| = \max_{i=1,\ldots,n} \left\{ \begin{array}{c} |\lceil Q(\alpha_i - \delta) \rceil - Q(\alpha_i - \delta)|, \\ |Q(\alpha_i + \delta) - \lfloor Q(\alpha_i + \delta) \rfloor| \end{array} \right\} \leq 3\varepsilon$$

Proof: SDA-NO $\Rightarrow$ DDA-NO

- Show: $\neg$DDA-NO $\Rightarrow$ $\neg$SDA-NO

Proof: SDA-NO ⇒ DDA-NO

- Show: ¬DDA-NO ⇒ ¬SDA-NO
- Let $Q \leq n^{O(\frac{1}{\log \log n})} N$ with

$$\max_{j=1,\ldots,2n} (\lceil Q\alpha'_j \rceil - Q\alpha'_j) \leq 2^n \cdot 3\varepsilon < 1/2$$

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- Suppose $\nexists$ integer in $[Q(\alpha_i - \delta), Q(\alpha_i + \delta)]$

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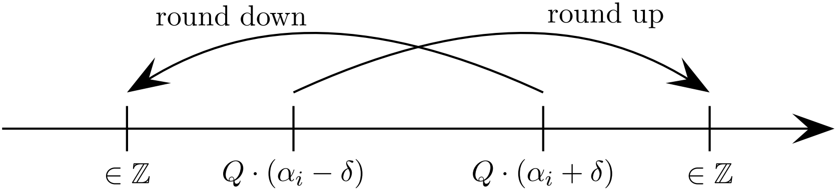

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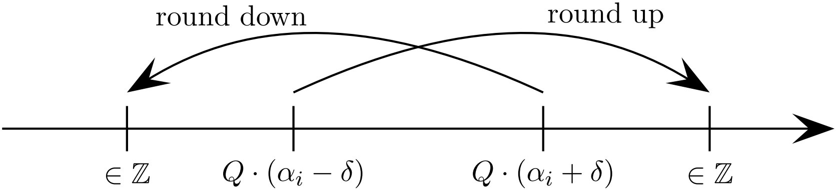

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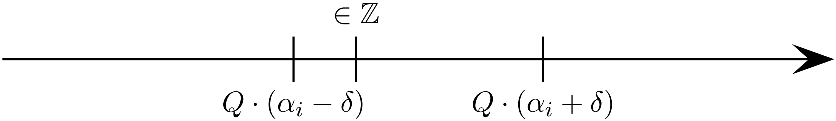

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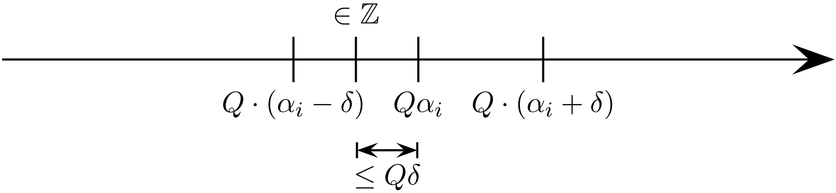

- $\max_{j=1,\ldots,2n} |\lceil Q\alpha'_j \rceil - Q\alpha'_j| \geq 1/2$ Contradiction!
- $|Q\alpha_i - \mathbb{Z}| \leq Q\delta \leq n^{O(\frac{1}{\log\log n})} \cdot \varepsilon$ hence ¬SDA-NO

Mixing Set

$$\begin{aligned} \min c_s s + c^T y & & \\ s + a_i y_i &\geq b_i \quad \forall i = 1, \ldots, n \\ s &\in \mathbb{R}_{\geq 0} \\ y &\in \mathbb{Z}^n \end{aligned}$$

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- Conforti et al. '08: Poly-time?
- $\rightarrow$ **NP**-hard by reduction from:

$$\exists Q \in \{1, \ldots, N\} : \sum_{i=1}^{n} |Q\alpha_i - \lfloor Q\alpha_i \rfloor| \leq \varepsilon$$

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\begin{aligned}
\min \sum_{i=1}^{n} (Q\alpha_i - y_i) & & & \qquad\qquad (IP) \\
Q\alpha_i - y_i &\geq 0 & \forall i = 1, \ldots, n \\
Q &\geq 1 \\
Q &\leq N \\
Q &\in \mathbb{Z} \\
y_1, \ldots, y_n &\in \mathbb{Z}
\end{aligned}
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Theorem

Polytime optimization $\Leftrightarrow$ *Polytime optimization over any non-empty face*

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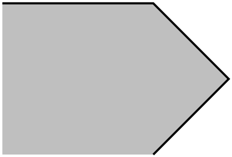

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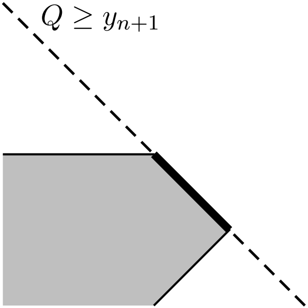

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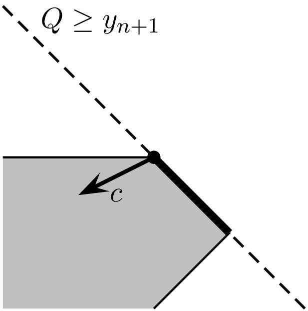

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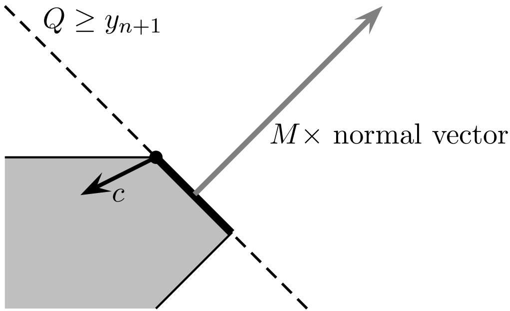

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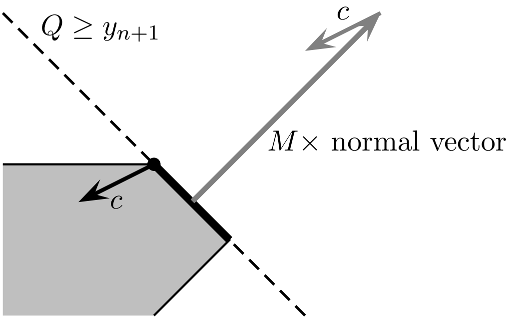

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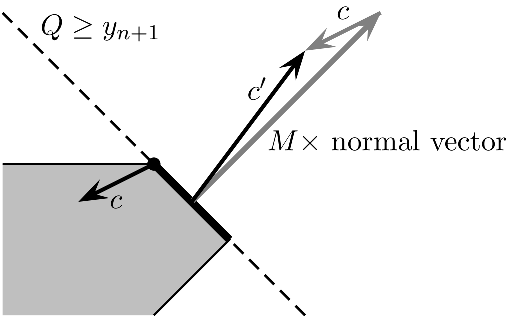

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- to good for NO-case $\Rightarrow$ YES case of DDA

Other applications

Theorem (Submitted to SODA 2010)

Testing EDF-schedulability of n periodic tasks (C_i, D_i, P_i) (C_i = running time, D_i = deadline, P_i = period), i.e.

$$\forall t \geq 0 : \sum_{i=1}^{n} \left(\left\lfloor \frac{t - D_i}{P_i} \right\rfloor + 1 \right) \cdot C_i \leq t$$

is **coNP**-*hard.*

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Thanks for your attention