Conformal welding of dendrites

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Dedicated to the memory of Frederick W. Gehring

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Abstract

We characterize the laminations that arise from conformal mappings onto planar dendrites whose complement is a John domain. This generalizes the well-known characterization of quasicircles via conformal welding. Our construction of a John domain from a given lamination introduces a canonical realization of finite laminations, and generalizes to some Hölder domains. In particular, we show that the Continuum Random Tree admits conformal welding and yields such Hölder domains. We also show that this conformal CRT is the (distributional) limit of uniformly random Shabat-trees as the degree of the Shabat polynomial tends to infinity.

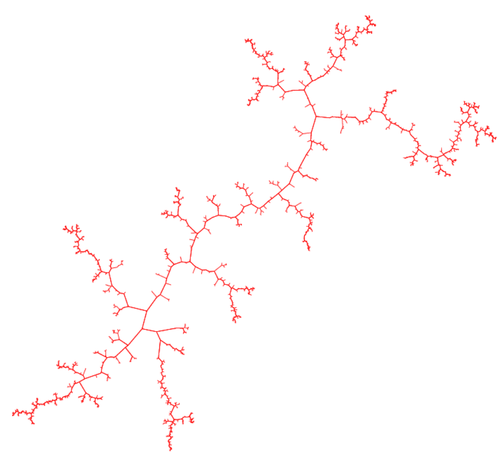

Figure 0.1: The conformal embedding of the Continuum Random Tree

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1 Introduction and Results

1.1 Statement of Results

Every conformal map f of the exterior of the unit disc $\Delta = \{|z| > 1\} \subset$ onto the complement of a dendrite $T \subset$, a locally connected continuum that contains no simple closed curves, extends continuously to the unit circle $\mathbb{T}$. It induces an equivalence relation $\mathcal{L} = \mathcal{L}_T$ on $\mathbb{T}$ via

$$\mathcal{L} = \{(z, w) \in \mathbb{T} \times \mathbb{T} : f(z) = f(w)\},$$

the *lamination* associated with T. We also write $z \sim w$ if $(z, w) \in \mathcal{L}$.

In this paper we consider the question: Which laminations arise from dendrites? In other words, given a lamination $\mathcal{L}$, does there exist a dendrite T such that $\mathcal{L} = \mathcal{L}_T$? In this generality, the question is out of reach of current methods: Already the classical conformal welding problem, which corresponds to the very special class of dendrites that are simple arcs, does not have a satisfying answer in this generality. On the other hand, under the relatively mild regularity assumption that the arc is quasiconformal, a satisfying theory exists and the lamination is given by a quasisymmetric map ([Leu96]). One of the main results of this paper is a characterization of the lamination in the setting of *Gehring trees*, which are dendrites whose complements are John domains. The methods that we develop also apply beyond this quasiconformal regularity and allow us to show that the *Brownian lamination* corresponds to a dendrite almost surely.

The Brownian lamination is a random equivalence relation on $\mathbb{T}$ obtained from the Brownian excursion as follows. Let $e : [0, 1] \to [0, \infty)$ be any continuous function with $e(0) = e(1) = 0$, i.e. an *excursion*. This induces a pseudometric $d_e(\cdot, \cdot)$ on $[0, 1]$ by

$$d_e(x, y) = e(x) + e(y) - 2 \min_{t \in [x,y]} e(t), \qquad x, y \in [0, 1]. \tag{1.1}$$

Define $x \sim_e y$ if and only if $d_e(x, y) = 0$. In particular $0 \sim_e 1$, so we can view $\sim_e$ as an equivalence relation on the unit circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. If e is taken to be the standard Brownian excursion, then the resulting random equivalence relation $\sim_e$ is called the Brownian lamination. The quotient of $\mathbb{T}$ under the Brownian lamination is known as the continuum random tree (CRT) [Ald91].

In order to state our results precisely, we need to introduce some terminology, aiming at a description of an annular neighborhood of a point $p \in T$ in terms of the lamination.

For $x \in \mathbb{T}$ and parameters $C > 0$, $N \in \mathbb{N}$ we call a number $0 < r < 1$ a *(very) good scale*, $r \in G(x, C, N)$, if the following holds (see Figures 1.1 and 4.1):

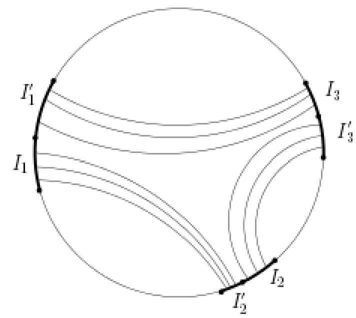

Figure 1.1: The definition of a *good scale*, depicted here near a triple point of a tree, requires many leaves of the lamination connecting a chain of intervals. See Figure 4.1 for the situation in the corresponding tree.

There are pairs of disjoint adjacent intervals I_j, I_j', $j = 1, ..., n$ (possibly $n = 1$) with $n \leq N$, uniformly perfect subsets $A_j \subset I_j, A_j' \subset I_j'$ where $A_{n+1}' := A_1'$, and decreasing (orientation reversing) quasisymmetric homeomorphisms $\phi_j : A_j \to A_{j+1}'$ such that $I_1 \cap I_1' = \{x\}$, $|I_1| \asymp_C r$, $|I_j| \asymp_C |I_j'| \asymp_C \operatorname{diam} A_j \asymp_C \operatorname{diam} A_j'$ for all j , and such that $(t, \phi_j(t)) \in \mathcal{L}$ for all $t \in A_j$.

Theorem 1.1. *If there are C, N and $\varepsilon > 0$ such that for every $x \in \mathbb{T}$, the set of good scales has lower density $\geq \varepsilon$,*

$$\frac{|\{k \leq n : 2^{-k} \in G(x, C, N)\}|}{n} \geq \varepsilon$$

for all n, then $\mathcal{L}$ is the lamination of a Hölder-tree. If every scale r is good, then $\mathcal{L}$ is the lamination of a Gehring tree.

It follows from P. Jones' removability theorem [Jon91] that the dendrite of Theorem 1.1 is unique up to a linear map: Indeed, given any two realizations f and g of the same lamination $\mathcal{L}$, the conformal map $g \circ f^{-1}$ of the complement of the first dendrite has a homeomorphic extension to the plane and hence is linear. By the Jones-Smirnov removability theorem [JS00], this argument generalizes to dendrites whose complement is a Hölder domain.

In the opposite direction, we will show

Theorem 1.2. *If $\mathcal{L}$ is the lamination of a Gehring tree, then there are C, N such that for every $x \in \mathbb{T}$, every scale r is good.*

Combining Theorems 1.1 and 1.2 we thus obtain a characterization of Gehring trees in terms their laminations. The existence of such a characterization was conjectured (unpublished) by Chris Bishop and Peter Jones.

There is another definition of "good scale" that applies in more generality and so that Theorems 1.1 and 1.2 are still true, see Definition 3.11 in Section 3.2.

We prove that this weaker condition is satisfied for the Brownian lamination, and obtain

Theorem 1.3. *Almost surely, the CRT admits conformal welding.*

Rather than proving this theorem directly, we prove a stronger result regarding convergence of Shabat trees. Shabat trees are the special case of Grothendieck's dessin d'enfant where the graph is a tree drawn on the sphere so that the Belyi function is a polynomial, known as a Shabat polynomial or generalized Chebychev polynomial. They arise in a number of different ways, and in particular can be shown ([Bia09, Bis14]) to be the dendrites whose laminations are given by non-crossing pairings of $2n$ arcs of equal size on the circle. Equivalently, these laminations are given by the Dyck paths S coding the trees via (1.1). Thus a Shabat tree with n edges chosen uniformly at random can be viewed as the welding solution to the lamination associated with simple random walk excursions of length $2n$ via (1.1). Roughly speaking, we prove that the (suitably normalized) uniform random Shabat tree of n edges converges (distributionally in the Hausdorff topology, and even stronger the topology induced by conformal parametrization) to the random dendrite of Theorem 1.3. More precisely, it is not hard to show that these laminations of simple random walk excursions converge to the Brownian lamination in distribution (similar to the convergence of rescaled simple random walk to Brownian motion), and we show that the solutions to the welding problems also converge.

Theorem 1.4. *There exists a constant $\alpha > 0$ such that the following holds. If $f_n : \Delta \to \overline{\mathbb{C}}$ is the welding map for the uniform random lamination (the random non-crossing pairing of $2n$ arcs), then f_n converges in distribution to a (random) conformal map f, with respect to uniform convergence on $\overline{\Delta}$. Furthermore, f is almost surely α-Hölder continuous. The law of the associated lamination $\mathcal{L}_f$ is that of the Brownian lamination.*

1.2 Motivation and Background

Already Koebe, in his work [Koe36] on conformal maps onto domains bounded by (touching) circles, considered the question of conformally realizing finite laminations. In geometric function theory, conformal welding of Jordan curves plays an important role, and the problem of conformally realizing a given lamination is a natural generalization. The starting point of our investigation was the desire to establish the analog of the well-understood quasiconformal theory of conformal welding in the setting of laminations. Theorems 1.1 and 1.2 constitute such an analog.

A *generalized Chebyshev polynomial* or *Shabat polynomial* is a polynomial $p(z)$ that has at most two critical values, see [BZ96] for the basic theory. If the critical values to are 0 and 1, then the preimage of the interval $[0, 1]$ is easily seen to be a tree T, consisting of $d = \deg(p)$ analytic arcs. Shabat polynomials (and, more generally, Belyi functions) have been intensely studied in their connection to algebraic geometry, number theory, and computational algebra. It is not hard to see that each 'combinatorial tree' (that is, a finite connected graph without loops) can be realized by a Shabat polynomial. The elegant description of these Shabat trees in terms of conformal laminations mentioned before Theorem 1.4 was given by Biane [Bia09] and independently [Bis14]. Small examples can be computed explicitly, for instance by solving the system of algebraic equations that the critical points satisfy. However, the computational complexity is exponential in the degree d, and explicit computations of large examples seem out of reach at present. Don Marshall's zipper algorithm [Mar] (see also [MR07]) provides numerical solutions to conformal welding problems and is remarkably accurate, particularly in the setting of quasisymmetric weldings. In the forthcoming [MR], Don Marshall and the second author describe a closely related zipper algorithm that allows to numerically approximate conformal realizations of laminations, and give applications to Shabat polynomials. The numerical accuracy of Don Marshall's implementation of this algorithm is most remarkable. It seems reasonable to believe, and is supported by the numerical computations for trees with thousands of edges discussed in [MR], that the algorithm converges at least for Gehring trees. Both figures 0.1 and 1.2 were generated with Marshall's program. Understanding the apparent and perhaps surprising numerical accuracy of this algorithm provides another motivation for the present work.

Conformal laminations arise naturally in complex dynamics: Beginning with Thurston [Thu09] and Douady-Hubbard [DH84], conformal laminations of Julia sets of quadratic polynomials have been studied and used to give combinatorial models of both Julia sets and the Mandelbrot set. The question which laminations allow for a conformal realization has been raised in numerous places and has been identified as difficult, see for instance the comments in [Dou93] and [Smi01b]. Some results about existence in a setting somewhat dual to ours, namely that of small support of $\mathcal{L}$ (logarithmic capacity zero of the set of endpoints) can be found in the Ph.D. theses [Leu96] and [Gup04]. In the setting of quadratic Julia sets, it has been proved that the set of bi-accessible points is always of harmonic measure zero [Zdu00, Smi01a] and even of dimension less then one [MS13], except for the Tchebyscheff polynomial z^2-2. In a different direction, Carleson, Jones and Yoccoz [CJY94] have shown that the domain of attraction to ∞ is a John-domain if and only if the polynomial is semi-hyperbolic, namely the critical point 0 is non-recurrent (and there is no parabolic point). The most prominent examples are the post-critically finite polynomials such as z^2+i, where the critical point is pre-periodic. The first author has given a combinatorial description of semi-hyperbolicity and proved that the conditions of Theorem 1.1 can be verified directly from the combinatorics, see [Lin18].

Theorem 1.5. *The lamination of a combinatorially semi-hyperbolic quadratic polynomial satisfies the condition of Theorem 1.1, hence the Julia set is a Gehring tree.*

In fact, in this case the uniformly perfect sets can be chosen as linear Cantor sets, and the quasisymmetric homeomorphisms are linear maps. As a corollary, this gives a new proof of the Carleson-Jones-Yoccoz theorem (in the quadratic setting).

In [HS94], Hubbard and Schleicher gave a proof of convergence of the spider-algorithm in the *periodic* case of critically finite quadratics. Based on Example 5.2, it is not hard to relate the spider-algorithm to our balloon animals, and then our proof of Theorem 1.1 yields a new proof of

Corollary 1.6. *The Hubbard-Schleicher spider-algorithm converges for nonperiodic postcritically finite quadratic polynomials.*

Another motivation comes from potential applications to questions related to random maps originating in probability theory and statistical physics. The description of SLE in terms of its conformal welding plays a central role in the Miller-Sheffield theory [MS16a, She16], and the (lack of) regularity of the welding still poses challenges for the deterministic machinery [AKSJ11].

In a different direction, groundbreaking work of Le Gall [LG07], Miermont [Mie13], Miller and Sheffield ([MS16b] and the references therein) established the Brownian map as the scaling limit of random planar maps as the number of faces tends to infinity, and suggests that a conformal structure on the Brownian map can be thought of as the conformal mating of two correlated copies of the conformal CRT described by our Theorem 1.3. We hope that the methods introduced in this paper will be useful in establishing the existence of the "slow mating" of trees in the sense of complex dynamics as described in [PM12].

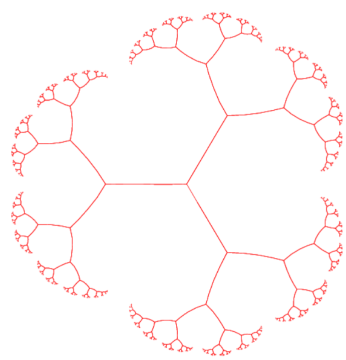

Figure 1.2: The conformal realization of the binary tree with 6141 edges.

1.3 Overview of proofs

The strategy of the proof of Theorem 1.2 can be summarized as follows. John domains can be viewed as one-sided quasidiscs (see Section 2.2) and have good localization properties, see [Jon91] for important examples. For the necessity of the condition, we give a geometric construction (Proposition 4.1) of a localization that provides a decomposition of an annulus into a chain of boundedly many topological squares as in Figure 4.1. The boundaries of two consecutive squares are connected through a subset of the tree. Typically, the harmonic measures of different "sides" of the tree are typically mutually singular, in other words the set of bi-accessible points is of harmonic measure zero. This is dealt with by proving the existence of a large set (in the sense of potential theory) of biaccessible points. Another difficulty is that the size alone of the set of biaccessible points provides no information: Indeed, in the classical welding problem (trees without branching) every point is biaccessible. What is needed in addition is control over the *quality* of the welding. We show that there is a large subset of the set of biaccessible points on which the welding is quasisymmetric. This relies on a result of Väisälä that conformal maps between John domains are quasisymmetric in the *internal* metric, combined with a construction of a subset of the tree where the euclidean and the internal metric are comparable.

Now we turn to the proofs of the remaining theorems. These existence theorems are proved by constructing certain well-chosen approximations for the given lamination for which the welding solution is known to exist, and showing that the welding solutions for these approximations converge.

Figure 1.2 shows the solution to the welding problem for the lamination corresponding a regular tree of depth $n = 11$, where each (half)-edge has the same harmonic measure with respect to infinity. It can be shown (see [Lin] for details) that as $n \to \infty$, the diameter of the middle edges stays bounded below. This implies that the associated conformal welding maps f_n cannot converge uniformly on $\overline{\Delta}$, as each edge is the image of an arc of size $\frac{1}{2}(3 \cdot 2^n - 3)^{-1}$ under f_n. However, along subsequences, the f_n do converge locally uniformly to a conformal limit (we believe that the limit exists and that is a conformal map onto a Jordan domain). An easier example of the same phenomenon is the conformal map $z \mapsto (z^n + z^{-n} + 2)^{1/n}$ onto the complement of a star with n edges; these map clearly do not converge uniformly on $\overline{\Delta}$, however they converge locally uniformly to the identity on Δ.

These examples illustrate the main difficulty in the proofs of Theorems 1.1, 1.3 and 1.4. To deal with this, we need to bound the diameter of images of arcs $I \subset \mathbb{T}$ under the welding f_n.

A standard idea in geometric function theory for doing this is to construct many thick conformal annuli surrounding the set $f_n(I)$. We will construct these annuli as conformal weldings of chains of rectangles in Δ. Control over the modulus of the resulting annulus relies on the following key observations (Propositions 3.2 and 3.4) that the conformal modulus of the welding of two squares along an edge can be bounded in terms of the regularity of the gluing on a sufficiently large subset. We use this in Theorem 3.9 to establish a condition ensuring the nondegeneracy of the annuli constructed from gluings of chains of rectangles.

For the proof of Theorem 1.1, another difficulty is in finding an appropriate way to approximate a lamination and its solution. A key idea is that any finite lamination has a canonical realization that we call a *balloon animal*, Proposition 5.1 and Figure 5.1: It is characterized by the property that harmonic measure at ∞ and the harmonic measures of the bounded components G_i are linearly related,

$$\frac{d\omega_\infty}{d\omega_i} \equiv \text{const} \quad \text{on} \quad \partial G_i.$$

This provides the tool for estimating the conformal modulus of the annuli obtained from gluing a chain of squares along (subsets of) their edges: Roughly speaking, loops surrounding the annulus described above correspond to loops that pass through the balloons in the discrete approximation, and the defining property of the ballon animal allows to pass information from the complement of a ballon (measured by ω_∞) to the interior of the balloon (measured by ω_i) without distortion. It is then fairly standard to translate such modulus estimates into analytic control.

It can be shown that the Brownian lamination satisfies the good scale condition of Theorem 1.1 (although it does not satisfy the *very* good scale good condition). Section 6 contains a sketch of a proof of Theorem 1.3 along these lines. However, we will not give all the details because the same strategy also gives a proof of the stronger Theorem 1.4, which we prove in detail in Section 7.

The strategy is to construct candidate annuli $\mathcal{A}_l$ (for $l = 1, 2, \dots$) in terms of ω_l where, roughly speaking, ω_l is the part of the excursion S lying between height λ^l and λ^{l+1}. The conditions of Theorem 3.9 that ensure the nondegeneracy of $\mathcal{A}_l$ translate into five simple conditions $\mathrm{Good}_{1,2,3,4,5}$ on ω_l, see Section 6.2, and its discrete counterpart, Section 7.3. We will define ω_l in such a way (see Sections 6.2 and 7.5) that $(\omega_l)_{l\geq 1}$ is a Markov chain. This allows us to use standard large deviation techniques (Theorem A.3) to show that, with very high probability, the density of the set of indices l for which $\omega_l \in \mathrm{Good}_{1,2,3,4,5}$ is greater than $\frac{1}{2}$.

The crux of the proof is a long computation (Lemma 7.8) showing that a certain function on the state space decreases sharply in expectation when the Markov chain transitions from a state that is not in $\mathrm{Good}_{1,2,3,4,5}$.

Section 2 contains basic definitions and facts from conformal geometry and potential theory. In Section 3 we state and prove the key conditions needed for the welding of two rectangles to have bounded modulus, see Propositions 3.2, 3.4. We then develop the notion of chains that allow us to easily state the good scale conditions, culminating in Theorem 3.9.

The remaining Sections 4, 5, 6, and 7 contain the proofs of Theorems 1.1, 1.2, 1.3, and 1.4 respectively, and are mutually independent, with the exception that Section 1.3 can be a viewed as a extended outline of the ideas in Section 1.4.

For the convenience of the reader, we give a self-contained presentation of the large deviations estimate that we need in Appendix A. Some technical estimates for discrete random walks needed for Section 1.4 are collected in Appendix B.

Acknowledgement The second author would like to thank Jean-Francois Le Gall for pointing out Biane's paper and for raising the question if the CRT is a conformal lamination. This paper evolved over several years and is related to several ongoing projects of the second author with Don Marshall, whom he would like to thank for countless inspiring discussions. In an early stage, a collaboration with James Gill and Don Marshall on questions related to this paper led to a proof of the sunny Lemma 4.5 in the case $N = 1$. We would like to thank them for their contribution.

Some of the ideas of this paper were developed at the Random Shapes program at MSRI in 2012. The second author is very grateful for the inspiring atmosphere and the generous support, and would like to thank Curt McMullen for stimulating discussions during that time. He also benefitted immensely from participation in the special program on Geometric Analysis at IPAM in 2013, where further progress on the deterministic part was made, and is grateful for their generous support.

Finally, we would also like to thank Chris Bishop, Mario Bonk, Peter Jones, Nick Makarov, Daniel Meyer, Brent Werness and many others for discussions related to this work.

2 Preliminaries

In this section we collect definitions and facts about John domains, quasisymmetric maps, logarithmic capacity, conformal modulus and uniformly perfect sets, as can be found for instance in the monographs [Ahl06], [Ahl10], [And04], [GM05], [Hei01], [Pom79]. The expert may skip this section and return to these results when needed.

2.1 Notation

Throughout this paper we will use the following notation:

$\overline{\mathbb{C}} = \cup\{\infty\}$ is the extended complex plane, $\mathbb{D}$ is the (open) unit disc, $\Delta = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, $\mathbb{T} = \partial\mathbb{D}$ the unit circle. $D_r(z) = D(z, r)$ is the disc of radius r centered at z, $C_r(z) = \partial D_r(z)$, and $C_r = \partial D_r(0)$.

We denote line segments by $[a, b]$, arcs (intervals) on $\mathbb{T}$ by (a, b), and geodesics in hyperbolic domains by $< a, b >$, and the length of a line segment or arc by $|I|$.

We write $a \lesssim_\lambda b$ to designate the existence of a function $C(\lambda)$ such that $a/b \le C(\lambda)$, and $a \lesssim b$ if $a \le Cb$ for some constant $C > 0$. We sometimes say that a statement holds *quantitatively* if the associated parameters (domain constants etc.) only depend on the parameters associated with the data.

2.2 John domains, quasisymmetric maps and Gehring trees

A connected open subset D of the Riemann sphere is a *John-domain* if there is a point $z_0 \in D$ (the John-center) and a constant C (the John-constant) such that for every $z \in D$ there is a curve $\gamma \subset D$ from z_0 to z such that

$$\mathrm{dist}(\gamma(t), z) \le C \mathrm{dist}(\gamma(t), \partial D)$$

for all t. If $\infty \in D$, then $z_0 = \infty$. An equivalent definition ([Pom92]) is that

$$\operatorname{diam} D(\sigma) \le C' \operatorname{diam} \sigma \tag{2.1}$$

for every crosscut σ of D, where $D(\sigma)$ denotes the component of $D \setminus \sigma$ that does not contain z_0. Moreover, it is enough to consider crosscuts that are line segments.

John domains were introduced in [Joh61] and are ubiquitous in analysis. Simply connected planar John domains can be viewed as one-sided quasidiscs. Indeed, a Jordan curve is a quasicircle if and only if both complementary components are John domains. Our main reference is the exposition [NV93] of Näkki and Väisälä. Important work related to John domains can be found in [And04],[Jon91],[CJY94],[NV93],[Väi89] and a large number of references in these works. A *planar dendrite* is a compact, connected, locally connected subset T of the plane with trivial fundamental group.

Definition 2.1. A Gehring tree is a planar dendrite such that the complement is a John-domain.

Gehring trees are easily described as planar dendrites built from quasiconformal arcs, see [Ahl06], [Ast09] and [Hei01] for basic definitions and an introduction to quasiconformal maps. A $K-$*quasiarc* is the image of a straight line segment under a $K-$quasiconformal homeomorphism of the plane. For every arc γ and all $x, y \in \gamma$, denote $\gamma(x, y)$ the subarc with endpoints x, y. Quasiarcs are characterized by Ahlfors'-condition

$$\operatorname{diam} \gamma(x, y) \le K|x - y| \quad \text{for all} \quad x, y, \tag{2.2}$$

see [Geh12] for a wealth of properties and characterizations of quasiconformal arcs and discs.

Proposition 2.2. *A dendrite T is a Gehring-tree if and only if there is K such that every subarc $\alpha \subset T$ is a $K-$quasiarc.*

This follows from the well-known fact [NV93] that the complement of a John disk is of bounded turning. We give a simple direct proof:

Proof. If $\alpha \subset T$ is a subarc and $x, y \in \alpha$, then $[x, y] \setminus T$ is a collection of intervals $\sigma_j = [x_j, y_j]$ that are crosscuts of $D = \setminus T$. It is easy to see that

$$\alpha \subset [x, y] \cup \bigcup_j \overline{D(\sigma_j)}$$

so that α satisfies the Ahlfors condition by (2.1).

Conversely, if T is a tree consisting of $K-$quasiarcs, and if $[x, y]$ is a crosscut of $D = \setminus T$, then the outer boundary of $D([x, y])$ is $[x, y] \cup T(x, y)$, and (2.1) follows from the quasiarc property of $T(x, y)$. $\square$

The notion of *quasisymmetry* is a generalization of quasiconformality to the setting of metric spaces, see [Hei01]. An embedding $f : X \to Y$ of metric spaces (X, d_X) and (Y, d_Y) is quasisymmetric if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that $d_Y(f(x), f(z)) \leq \eta(t) d_Y(f(y), f(z))$ whenever $d_X(x, z) \leq t d_X(y, z)$.

If $D \subsetneq \hat{}$ is open and connected, the *internal metric* is defined as

$$\delta_D(x, y) = \inf_\gamma \operatorname{diam}(\gamma),$$

where the infimum is over all curves $\gamma \subset D$ with endpoints x and y. If ∂D is locally connected, or equivalently if a conformal map f from the disc onto D has a continuous extension to $\overline{D}$, then the completion of (D, δ_D) coincides with the completion of D via the prime end boundary, and f extends to a homeomorphism between $\overline{\mathbb{D}}$ and this completion. For instance, if D is a slit disc, then both sides of the slit give rise to different points in the closure of (D, δ_D). John domains are characterized by the quasisymmetry of this extension:

Theorem 2.3 ([NV93], Section 7)**.** *A conformal disc D with $\infty \in D$ is a $c-$John domain if and only if the conformal map $f : \Delta \to D$ that fixes ∞ is quasisymmetric in the internal metric. Here η depends only on c and vice versa.*

John domains are intimately related to the doubling property for harmonic measure. We will use the following characterization. The proof in ([Pom92], Theorem 5.2) for bounded domains can easily be modified to cover our situation.

Theorem 2.4. *Let $f : \Delta \to G$ be a conformal map fixing ∞. Then G is a John domain if and only if there is a constant $\beta > 0$ such that*

$$\operatorname{diam} f(A) \leq \frac{1}{2} \operatorname{diam} f(I)$$

whenever $A \subset I \subset \mathbb{T}$ are arcs of length $|A| \leq \beta|I|$.

Finally, we will need the following result of P. Jones, [Jon91],[NV93].

Lemma 2.5. *If D is a John domain, $f : \mathbb{D} \to D$ a conformal map sending 0 to the John center, and if $D' \subset \mathbb{D}$ is a John domain, then $f(D')$ is a John domain, quantitatively.*

2.3 Logarithmic capacity, conformal modulus and uniformly perfect sets

Let μ be a Borel measure of finite nonzero mass on . The *(logarithmic) energy* $\mathcal{E}(\mu)$ of the measure μ is the extended real number

$$\mathcal{E}(\mu) = \iint_{2} -\log|x-y| \, d\mu(x) d\mu(y). \tag{2.3}$$

The *(logarithmic) capacity* of a compact set $E \subset$ is the real number

$$\operatorname{cap}(E) = e^{-\inf_\mu \mathcal{E}(\mu)},$$

where the infimum is taken over all Borel probability measures supported on E. If the capacity is non-zero, the unique minimizing measure is the harmonic measure at infinity ω_∞, and the Green's function of $\setminus E$ is given by

$$g_E(z) = \int_E \log|z-w| d\omega_\infty(w) - \log(\operatorname{cap} E) = \log|z| - \log(\operatorname{cap} E) + O(\frac{1}{|z|}).$$

Important examples are $\operatorname{cap} B(x,r) = r$ and $\operatorname{cap}[a,b] = |b-a|/4$. The capacity of a Borel set is defined as the supremum of the capacities of compact subsets.

The *conformal modulus* $M(\Gamma)$ of a family of curves Γ is an important conformal invariant, see [Ahl10, GM05, Pom92]. It is defined as

$$M(\Gamma) = \inf_\rho \operatorname{Area}(\rho) = \inf_\rho \int \rho^2 dx dy$$

where the infimum is over all *admissible metrics* ρ, namely all Borel measurable functions ρ with the property that the $\rho-$length $\int_\gamma \rho|dz| \geq 1$ for all $\gamma \in \Gamma$.

The proof of Theorem 1.1 relies on estimates of the modulus of continuity of some conformal maps. We will employ a standard technique to obtain such estimates. It is based on the following relation between the conformal modulus of an annulus and its euclidean dimensions. Consider a topological annulus $A \subset$ with boundary components A_1, A_2 and set $r(A) = \min(\operatorname{diam} A_1, \operatorname{diam} A_2)$, $R(A) = \operatorname{dist}(A_1, A_2)$. The conformal modulus $M(A)$ is defined as the modulus of the family of all closed curves $\gamma \subset A$ that separate A_1 and A_2. For example, $M(A(x, r, R)) = \log(R/r)/2\pi$.

Lemma 2.6. *There is a constant C such that*

$$|M(A) - \frac{1}{2\pi}\log(1 + \frac{R(A)}{r(A)})| \leq C.$$

See for instance ([Roh97], Lemma 2.1) for a discussion and references. We will use it in combination with the subadditivity property of the modulus:

Lemma 2.7. *If A_j are disjoint annuli that separate the boundary components of an annulus A, then $M(A) \geq \sum_j M(A_j)$.*

We will also use Pfluger's theorem which quantifies a close connection between capacity and modulus, see [Pom92]:

Theorem 2.8. *If $E \subset \partial\mathbb{D}$ is a Borel set and if Γ_E is the set of all curves $\gamma \subset \mathbb{D}$ joining the circle C_r to the set E, then*

$$\operatorname{cap} E \asymp e^{-\pi/M(\Gamma_E)}$$

with constants only depending on $0 < r < 1$.

Specifically, we will use the following variant whose proof we leave as an exercise: If $D = [0, M] \times [0, 1]$ is a rectangle, if $E \subset \{M\} \times [0, 1]$ is Borel, and if Γ_E is the family of curves that join the left side $\{0\} \times [0, 1]$ to E in D, then

$$\operatorname{cap} E \leq C(M) e^{-\pi/M(\Gamma_E)}. \tag{2.4}$$

The compact set A is called *uniformly perfect* if there is a constant $c > 0$ such that no annulus $A(x, cr, r)$ with $r < \operatorname{diam} A$ separates A : If $A \cap A(x, cr, r) = \emptyset$, then $A \subset B(x, cr)$ or $A \cap B(x, r) = \emptyset$. An equivalent definition is the existence of a different but quantitatively related constant $c > 0$ such that

$$\operatorname{cap} A \cap B(x, r) \geq cr \quad \text{for all} \quad x \in A \quad \text{and} \quad r < \operatorname{diam} A.$$

See Exercise IX.3 in [GM05] for 13 other equivalent definitions.

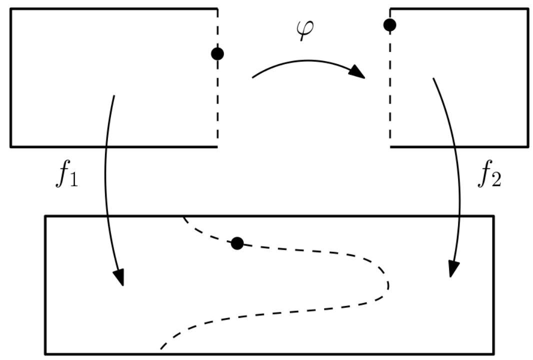

Figure 3.1: Conformal welding of two rectangles

3 Modulus of welded rectangles and annuli

In this section we will develop one of the main tools of the paper, namely a condition on the conformal gluing of rectangles under which the conformal modulus stays controlled.

3.1 Conformal rectangles

Consider two rectangles $D_1 = [0, M_1] \times [0, 1]$ and $D_2 = [0, M_2] \times [0, 1]$ together with their left boundaries $L_j = \{0\} \times [0, 1]$ and their right boundaries $R_j = \{M_j\} \times [0, 1], j = 1, 2$.

It is well-known that, if φ is a quasisymmetric homeomorphism between R_1 and L_2, then the *conformal welding* of D_1 and D_2 via φ yields a conformal rectangle of modulus bounded in terms of M_1, M_2 and the quasi-symmetry constant. More precisely, there are conformal rectangles D_1', D_2' with disjoint interiors and conformal maps $f_j : D_j \to D_j'$ sending corners to corners such that $f_1 = f_2 \circ \varphi$ on the right boundary R_1, and such that $D_1' \cup D_2'$ is a rectangle $[0, M] \times [0, 1]$ of modulus $M \geq M_1 + M_2$ bounded above in terms of M_1, M_2 and K, see Figure 3.1. Note that

$$M = \mathrm{M}(\Gamma)^{-1},$$

where Γ is the family of all pairs of curves (γ_1, γ_2) such that γ_i connects L_i and R_i in D_i, and the image of the endpoint of γ_1 under φ coincides with the initial point of γ_2 (in other words, $f_1(\gamma_1)$ together with $f_2(\gamma_2)$ form a single continuous curve in $D_1' \cup D_2'$).

For our results concerning Gehring trees, it will be sufficient to generalize this to the setting where a uniformly perfect subset of R_1 is welded to a uniformly perfect subset of L_2 via a quasisymmetric map. Thus we make the following definition:

Definition 3.1. A $C-$*rectangle* is a triple (D, A, B), where $D = [0, M] \times [0, 1]$ is a rectangle with $1/C \leq M \leq C$, where A (resp. B) is a $1/C-$uniformly perfect subset of the left (resp. right) boundary $\{0\}\times[0,1]$ (resp. $\{M\}\times[0,1]$), and where $\operatorname{diam} A \geq 1/C$ and $\operatorname{diam} B \geq 1/C$. Any conformal image onto a simply connected domain is called a *conformal $C-$rectangle*, where now A and B have to be interpreted as sets of prime ends.

Proposition 3.2. *If (D_1, A_1, B_1) and (D_2, A_2, B_2) are $C-$rectangles and if $\varphi : B_1 \to A_2$ is an increasing $K-$quasisymmetric homeomorphism, then*

$$\mathrm{M}(\Gamma)^{-1} \leq M(C, K),$$

where Γ is the family of pairs of curves (γ_1, γ_2) joining A_1 to B_2 through $\varphi(B_1) = A_2$ such that the image of the endpoint of γ_1 under φ is the initial point of γ_2.

Notice that this implies, by the monotonicity of the modulus of curve families, that *every* welding of D_1 and D_2 which extends φ has modulus bounded above and below, independent of the extension of φ from B_2 to A_1.

For applications to less regular settings, we need the following further generalization to an equivalence relation $\sim$ on $R_1 \cup L_2$, and the family Γ of pairs (γ_1, γ_2) joining L_1 to R_2 in the disjoint union of D_1 and D_2 such that the endpoint of γ_1 is equivalent to the initial point of γ_2.

Definition 3.3. A pair μ_1, μ_2 of probability measures supported on R_1 and L_2 respectively is called a *gluing pair* for (D_1, L_1, R_1) and (D_2, L_2, R_2) if there is a measure preserving bijection ϕ between measurable subsets B_1 of R_1 and A_2 of L_2 of full measure, $\mu_1(B_1) = \mu_2(A_2) = 1$, such that $\phi(x) \sim x$ for all $x \in B_1$.

Proposition 3.4. *The family Γ of pairs (γ_1, γ_2) joining L_1 to R_2 such that the endpoint of γ_1 is equivalent to the initial point of γ_2 satisfies*

$$\mathrm{M}(\Gamma)^{-1} \leq C_0(M_1, M_2) \inf_{\mu_1,\mu_2} \max\left(\mathcal{E}(\mu_1), \mathcal{E}(\mu_2)\right), \tag{3.1}$$

where the infimum is taken over all gluing pairs of measures as above, and $\mathcal{E}$ denotes logarithmic energy.

Proof. Let $\rho = (\rho_1, \rho_2)$ be a metric on the disjoint union $D_1 \sqcup D_2$ where $\rho_i : D_i \to \mathbb{R}_+$ and let μ_1, μ_2 be Borel probability measures supported on $B_1 \subset R_1$ and $A_2 \subset L_2$ respectively together with a measure preserving bijection ϕ as above. Let $E_1 \subset B_1$ be the set of points p such that every curve $\gamma_1 \subset D_1$ from L_1 to p has ρ_1-length at least 1, and define $E_2 \subset A_2$ similarly so that the ρ_2-length of all curves from E_2 to R_2 is at least 1. By Pfluger's theorem (2.4) we have

$$\operatorname{cap} E_i \leq C(M_i) e^{-\pi/\mathrm{M}(\Gamma_{E_i})} \leq C(M_i) e^{-\pi/\mathrm{Area}(\rho_i)} \leq e^{-\pi/\mathrm{Area}(\rho_i)+C_0}$$

for suitable C_0. Assume that

$$\text{Area}(\rho_i) \leq \frac{\pi}{16\ \mathcal{E}(\mu_i) + C_0} \tag{3.2}$$

for both $i = 1, 2$, so that $\operatorname{cap} E_i \leq \exp(-16\ \mathcal{E}(\mu_i))$. Then

$$\mu_i(E_i) \leq 1/4 : \tag{3.3}$$

Indeed, since $-\log|x-y| \geq 0$ for all $x, y \in [0,1]$, we have $\mathcal{E}(\mu_i) \geq \mathcal{E}(\mu_i|_{E_i})$, and since $\mu_i|_{E_i}/\mu_i(E_i)$ is a probability measure, we have

$$-\log \operatorname{cap} E_i \leq \mathcal{E}(\mu_i|_{E_i}/\mu_i(E_i)) = \frac{\mathcal{E}(\mu_i|_{E_i})}{\mu_i(E_i)^2} \leq \frac{\mathcal{E}(\mu_i)}{\mu_i(E_i)^2} \leq \frac{-\log \operatorname{cap} E_i}{16\mu_i(E_i)^2}.$$

Since ϕ is measure preserving we thus have $\mu_1(E_1 \cup \phi^{-1}(E_2)) \leq 1/2$ and in particular $B_1 \setminus (E_1 \cup \phi^{-1}(E_2))$ is non-empty. Let p be a point of this set. Then there are curves $\gamma_1 \subset D_1$ respectively $\gamma_2 \subset D_2$ joining L_1 to p respectively $\phi(p)$ to R_2 such that both ρ_i-lengths of γ_i are < 1. Thus the metric $\rho/2 = (\rho_1/2, \rho_2/2)$ is not admissible for Γ. It follows that every admissible metric ρ has

$$4\text{Area}(\rho) \geq \max_i \text{Area}(2\rho_i) \geq \frac{\pi}{16\ \mathcal{E}(\mu_i) + C_0}$$

and the proposition follows. $\square$

Proposition 3.2 can be proved similarly, using the simple fact that φ has a quasisymmetric extension Φ to R_1 and therefore is Hölder continuous, and sends sets of small capacity to sets of small capacity. It can also be viewed as a direct consequence of Proposition 3.4, since the energy of the equilibrium measure μ_1 of B_1 is bounded, and by the Hölder continuity of ϕ^{-1}, the energy of the push-forward μ_2 of μ_1 under φ has bounded energy as well.

The following lemma explains how Proposition 3.4 will be applied to the Brownian lamination. For the remainder of the section, we will write $D^+ = [0, M^+] \times [0,1], D^-$ and μ^+, μ^- instead of D_1, D_2, μ_1, μ_2 and so on.

Lemma 3.5. *Fix $x \in (0,1)$ and let $I^- = (0,x) \subset \mathbb{T}$ and let $I^+ = (x,1) \subset \mathbb{T}$. Let $e : [0,1] \to \mathbb{R}^+$ be a (C,α)-Hölder continuous excursion and let $\sim_e$ be the lamination on $\mathbb{T}$ induced by e. There are measures μ^-, μ^+ supported on measurable subsets E^-, E^+ of I^-, I^+ and a measurable measure preserving bijection f between E^- and E^+ such that $t \sim_e f(t)$ for all $t \in E^-$ and*

$$\max\left(\mathcal{E}(\mu^+), \mathcal{E}(\mu^-)\right) \leq \frac{1}{\alpha}\left(3/2 + \log e(x)^{-1} + \log C\right). \tag{3.4}$$

Proof. Let m be the Lebesgue measure on $[0, e(x)]$ normalized so that $|m| = 1$. Note that $\mathcal{E}(m) = \mathcal{E}(m_{[0,1]}) - \log e(x) = 3/2 - \log e(x)$ where $m_{[0,1]}$ is the Lebesgue measure on $[0, 1]$.

Define the one-sided inverse functions e^-, e^+ on $[0, e(x)]$ by $e^-(y) = \sup\{t \leq x : e(t) = y\}$ and $e^+(y) = \inf\{t \geq x : e(t) = y\}$, see Figure 3.2. Let $E^- \subset I^-$ and $E^+ \subset I^+$ be the image of $[0, e(x)]$ under e^- and e^+ respectively. Define the measures μ^+, μ^- on $[0, 1]$ by $\mu^+(A) = m(e(A \cap E^+))$ and $\mu^-(A) = m(e(A \cap E^-))$. Since e is (C, α)-Hölder continuous, (2.3) shows that μ^+ and μ^- both have energy bounded above by $\frac{1}{\alpha}\mathcal{E}(m) + \frac{1}{\alpha}\log C = \frac{1}{\alpha}\left(3/2 + \log e(x)^{-1} + \log C\right)$.

Define the bijection $f : E^- \to E^+$ by $f(t) = e^+(e(t))$. Notice that $\min_{t \leq s \leq f(t)} e(s) = e(t) = e(f(t))$, so by definition of $\sim_e$ we have $t \sim_e f(t)$. We also have $\mu^-(A) = m(e(A \cap E^-)) = m(e(e^-(e(A \cap E^-)))) = m(ef^{-1}(A \cap E^-)) = \mu^+(f^{-1}(A))$ for all Borel $A \subset [0, 1]$. Thus μ^- and μ^+ are the desired measures. $\square$

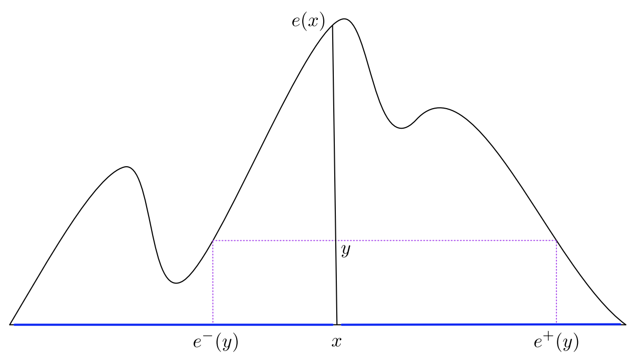

Figure 3.2: If e is a continuous excursion then for any fixed x we can construct a gluing pair of measures (μ^+, μ^-) by pushing forward normalized Lebesgue measure on $[0, e(x)]$ via the one-sided inverses of e.

In the deterministic setting of Section 5 below we will need the following variant of the modulus estimate Proposition 3.4. It gives control over the modulus in the setting where finitely many equivalence classes have been conformally contracted to points.

Consider finite partitions

$$R^+ = \bigcup_j I_j \quad \text{and} \quad L^- = \bigcup_k J_k$$

of the boundary segments of D^+ and D^- into disjoint intervals such that each endpoint of each I_j is equivalent under $\sim$ to an endpoint of some J_k, and conversely every endpoint of J_k is $\sim$ to some endpoint of some I_j. For each $I_j = [\alpha_j, \beta_j]$, either both endpoints are $\sim$ to the same point on L^- (and consequently $\alpha_j \sim \beta_j$), or there is a unique interval $J_k = [\gamma_k, \delta_k]$ of the second partition such that $\alpha_j \sim \gamma_k$ and $\beta_j \sim \delta_k$. We may re-label the I_j and J_k so that

$j = k$ for some m and $1 \leq j \leq m$, and for $j > m$ the I_j are of the first type ($\alpha_j \sim \beta_j$). Consider the abstract Riemann surface $\mathcal{R}$ obtained from the disjoint union $D^+ \sqcup D^-$ as follows: For each of these pairs I_j, J_j with $j \leq m$, glue a unit disc B_j, to $D^+ \sqcup D^-$ along $I_j \sqcup J_j$, such that the arclength of ∂B_j is proportional to arclength on I_j and on J_j, where the proportionality constants are determined by the requirement that the normalized arclength (harmonic measure ω_{B_j} at the center of the disc) of I_j and J_j on ∂B_j is $1/2$ each. And for each arc I_j, J_j with $j > m$ just glue in a disc arbitrarily (for instance again with harmonic measure proportional to arclength), see Figure 3.3, and see also Figure 5.3 below for motivation. We have

Proposition 3.6 (Balloon animal lemma)**.** *Let Γ be the family of curves that join L^+ and R^- in $\mathcal{R}$. Then*

$$\mathrm{M}(\Gamma)^{-1} \leq C_0(M^+, M^-) \inf_{\mu^+,\mu^-} \max\left(\mathcal{E}(\mu^+), \mathcal{E}(\mu^-)\right) \tag{3.5}$$

where the infimum is taken over all gluing pairs of measures.

We prepare the proof with the following

Lemma 3.7. *Fix $0 < a < 1$ and suppose B is a Borel subset of $[0, a]$. Then for any Borel probability measure μ on $[0, 1]$, we have*

$$\mu(B) \leq C_1 \mathcal{E}(\mu)^{1/2} \mathrm{M}(\Gamma_B)^{1/2},$$

where Γ_B is the set of paths in the square $[0, a]^2$ joining the "top edge" $[0, a] + ai$ to B, and C_1 is a universal constant.

Proof. Since $\mathcal{E}(r\mu)^{1/2} = r\mathcal{E}(\mu)^{1/2}$ we may assume that μ is a probability measure supported on B. First assume that $a = 1$. Then the definition of capacity and Pfluger's theorem (2.4) yield

$$e^{-\mathcal{E}(\mu)} \leq \operatorname{cap}(B) \leq e^{-\pi/\mathrm{M}(\Gamma_B) + C_0}$$

where C_0 is a universal constant. Thus $\mathcal{E}(\mu) + C_0 \geq \frac{\pi}{\mathrm{M}(\Gamma_B)}$ and we obtain $\mathcal{E}(\mu)\mathrm{M}(\Gamma_B) \geq \pi/(1 + C_0/\log 4)$ since $\mathcal{E}(\mu) \geq -\log \operatorname{cap}[0, 1] = \log 4$.

Now suppose $a \neq 1$. The map $x \mapsto x/a$ maps B onto a subset B/a of $[0, 1]$ and maps μ onto a measure μ^a with energy $\mathcal{E}(\mu) + \log a \leq \mathcal{E}(\mu)$. The inequality for $a = 1$ yields

$$\mu^a(B/a) \leq C_1(\mathcal{E}(\mu))^{1/2} \mathrm{M}(\Gamma_{B/a})^{1/2}$$

which implies the desired result since the modulus is dilation invariant. $\square$

Now we are ready to prove Proposition 3.6

Proof. Let μ^-, μ^+ be finite energy Borel probability measures on R^+, L^- which form a gluing pair as above. The Riemann surface $\mathcal{R}$ is a quotient of the two squares D^+, D^-, a number of discs B_i, $i = 1, \dots, m$, and further discs (corresponding to intervals with equivalent endpoints) that do not play a role in the proof. For each disk B_i, we cut out a concentric disk B_i° of radius $e^{-\pi}$ so that the remaining annulus is a quotient of two squares as in Figure 3.3. Let B_i^+ denote the "left square" and let B_i^- be the "right square" in this decomposition.

Let ρ be a metric on $\mathcal{R}$, given by a non-negative function on D^+, D^-, the B_i, and the remaining discs. As in the proof of Proposition 3.4, let $E^+ \subset R^+$ be the set of points p such that every curve $\gamma_1 \subset D^+$ from L^+ to p has $\rho-$length at least 1, and similarly $E^- \subset L^-$ so that the $\rho-$length of curves from E^- to R^- is ≥ 1. By (3.2) and (3.3) we have $\mu^+(E^+ \cup \phi^{-1}(E^-)) \leq 1/2$ if $\mathrm{Area}(\rho) \leq \min(\frac{\pi}{16\ \mathcal{E}(\mu^+)+C_0}, \frac{\pi}{16\ \mathcal{E}(\mu^-)+C_0})$, where C_0 depends on the moduli $M^\pm$ of $D^\pm$.

Note that for each i, the interval I_i is the left edge of the square B_i^+. Let $E_i^+ \subset I_i$ be the set of points $p \in I_i$ for which $\inf_\gamma \ell_\rho(\gamma) \geq 1$, where the infimum is taken over all paths γ from x to the right edge $\partial B_i^+ \cap B_i^\circ$ of B_i^+. Applying Lemma 3.7 to the set E_i^+, the measure $\mu^+|_{I_i}$ and the family of curves $\Gamma_{E_i^+}$ joining points of E_i^+ to right edge of B_i^+ inside B_i^+ yields

$$\mu^+(E_i^+) \leq C_1 \mathcal{E}(\mu^+|_{I_i})^{1/2} \mathrm{M}(\Gamma_{E_i^+})^{1/2}.$$

On the other hand, the definition of E_i^+ implies that $\rho|_{B_i}$ is admissible for $\Gamma_{E_i^+}$ so that $\mathrm{M}_{B_i^+}(\Gamma_{E_i^+}) \leq \mathrm{Area}(\rho|_{B_i})$. Summing over i and using Cauchy-Schwarz yields

$$\sum_i \mu^+(E_i^+) \leq C_1 \sum_i \mathcal{E}(\mu^+|_{I_i}) \sum_i \mathrm{Area}(\rho|_{B_i}) \leq C_1 \mathrm{Area}(\rho) \mathcal{E}(\mu^+).$$

It follows that $\mu^+\left(\cup_i E_i{}^+\right) \leq \frac{1}{8}$ if $\mathrm{Area}(\rho) \leq \frac{1}{8C_1}\mathcal{E}(\mu^+)^{-1}$, and similarly for μ^-.

The rest of the proof proceeds in a similar way to Proposition 3.4: If

$$\mathrm{Area}(\rho) \leq \min(\frac{\pi}{16\ \mathcal{E}(\mu^+) + C_0}, \frac{\pi}{16\ \mathcal{E}(\mu^-) + C_0}, \frac{1}{8C_1}\mathcal{E}(\mu^-)^{-1}, \frac{1}{8C_1}\mathcal{E}(\mu^+)^{-1}),$$

then there exists $i = 1, \dots, m$ and paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ in $\mathcal{R}$ such that

- $\ell_\rho(\gamma_j) \leq 1$ for each j.
- γ_1 is a continuous path from the left edge L^+ to the right edge R^+ of D^+.
- γ_2 is a continuous path from R^+ to the "right" edge of B_i^+.
- γ_3 is a continuous path from "left" edge of B_i^- to L^-.
- γ_4 is a continuous path from L^- to R^-.

- The concatenations $\gamma_1\gamma_2$ and $\gamma_3\gamma_4$ are continuous.

Since the modulus of the annulus $B_i \backslash B_i^\circ$ is $1/2$, (being the gluing of two squares), if $\text{Area}(\rho) \le 2$, then there exists a loop γ_5 of ρ-length less than 1 around this annulus. This loop must intersect both γ_2 and γ_3.

We conclude that if $\text{Area}(\rho) \le C(M^+, M^-) \min(\mathcal{E}(\mu^-)^{-1}, \mathcal{E}(\mu^+)^{-1}) \wedge 2$, there is a path γ from L_1 to R_2 in $\mathcal{R}$ with ρ-length less than 5. $\square$

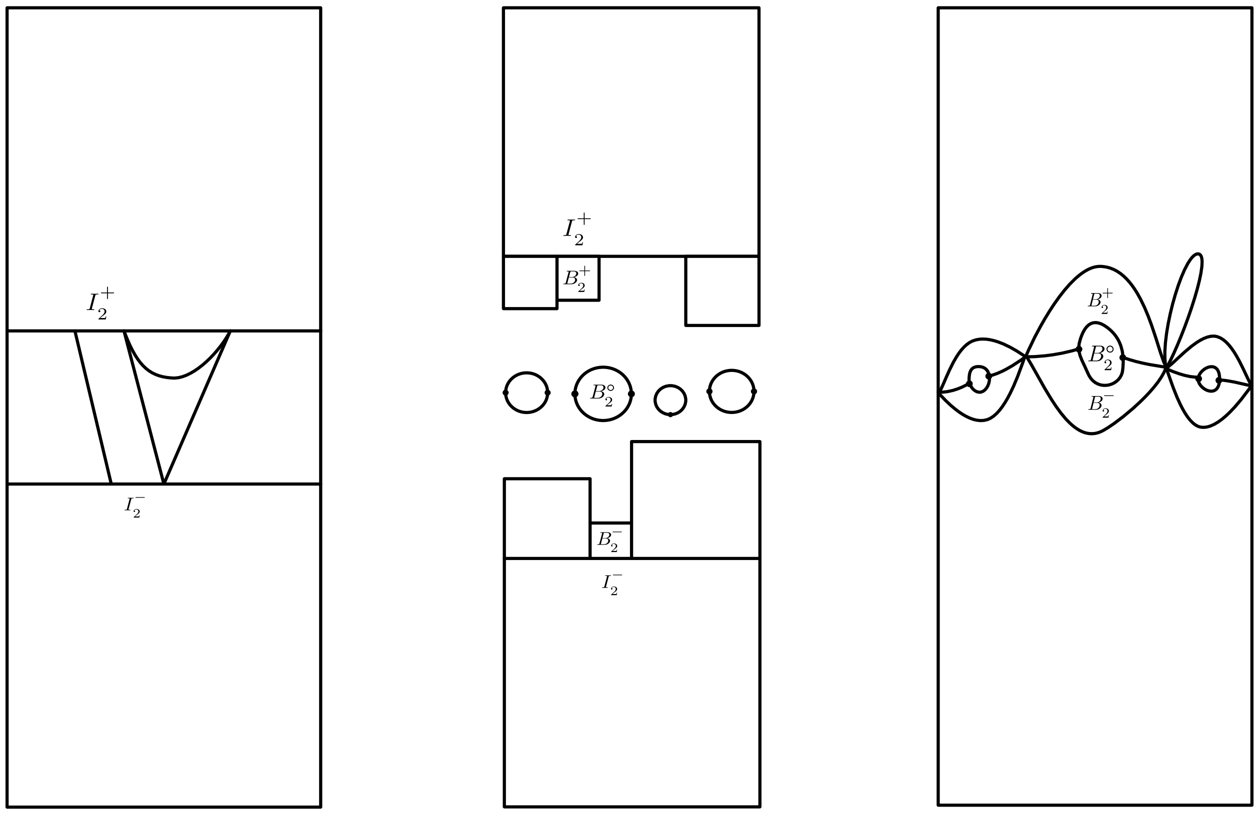

Figure 3.3: Every pair of partitions with equivalent end points gives rise to a Riemann surface $\mathcal{R}$ (right). This "balloon animal" can be viewed as a gluing of squares and disks (middle).

3.2 Chains of rectangles and conformal annuli

In this section we will apply the results of the previous section to obtain modulus estimates for annuli that are formed by conformal welding of several rectangles. We will first set up some notation and then describe sets of conditions that allow control of the modulus. Fix an equivalence relation $\sim$ on $\mathbb{T}$.

Definition 3.8. A *chain link* is a pair of disjoint closed intervals $J^- = [a^-, b^-], J^+ = [a^+, b^+] \subset \mathbb{T}$ such that some point of J^- is equivalent to some point of J^+. Often (but not necessarily) the endpoints will be equivalent, $a^- \sim b^+$ and $a^+ \sim b^-$, which explains some of our terminology. For $m \geq 1$, an $m-$*chain* $\mathcal{C}$ is a collection of m mutually disjoint chain links J_i^-, J_i^+ such that the intervals are in cyclic order on the circle, $a_1^- < b_1^- < a_1^+ < b_1^+ < a_2^- < ... < a_m^+$. Associate with a chain the conformal rectangles $D_i \subset$ bounded by J_i^+, J_{i+1}^- and the two hyperbolic geodesics from a_i^+ to b_{i+1}^- and from b_i^+ to a_{i+1}^-, where indices are to be taken mod m so that $J_{m+1} = J_1$. The *chain-annulus* is

$$\mathcal{A}(\mathcal{C}) = \bigcup_{i=1}^{m} D_i.$$

The *curve family* $\Gamma(\mathcal{C})$ consists of the $m-$tuples $(\gamma_1, \gamma_2, ..., \gamma_m)$ of curves $\gamma_i \subset D_i$ such that for each $1 \leq i \leq m$, the endpoint of γ_i is equivalent to the initial point of γ_{i+1}.

Note that if $\sim$ is the conformal lamination of a dendrite and if the endpoints of the links of a chain are equivalent, then the conformal image of $\mathcal{A}(\mathcal{C})$ is a topological annulus. As we are interested in the modulus of such an annulus, we want to estimate the modulus of $\Gamma(\mathcal{C})$. We will now describe a set of conditions, involving a parameter L, that give control over $\mathrm{M}(\Gamma(\mathcal{C}))$.

Condition 1: All (D_i, J_i^+, J_{i+1}^-) are conformal $L-$rectangles. This is easily seen to follow from an estimate of the form

$$|a_{i+1}^- - b_i^+| \lesssim_L |J_i^+| \asymp_L |J_{i+1}^-|. \tag{3.6}$$

Condition 2: The lamination is $L-$*thick* between J_i^- and J_i^+ : We require that there are Borel measures μ_i^- and μ_i^+ supported on J_i^- and J_i^+, together with a measure preserving bijection ϕ_i between measurable subsets B_i^- of J_i^- and A_i^+ of J_i^+ of full measure, such that

$$\max\left(\mathcal{E}(\hat{\mu_i^-}), \mathcal{E}(\hat{\mu_i^+})\right) \leq L, \tag{3.7}$$

where $\hat{\mu}$ denotes the probability measure on $[0, 1]$ obtained from μ by linear scaling.

Condition 3: We require a bound on the number of links of the chain,

$$m \leq L. \tag{3.8}$$

Applying Proposition 3.4 $m-$times yields

Theorem 3.9. *If Conditions 1-3 are met, then*

$$\mathrm{M}(\Gamma(\mathcal{C}))^{-1} \lesssim_L 1. \tag{3.9}$$

In particular, if f is a conformal map realizing the lamination $\sim$, then $f(\mathcal{A}(\mathcal{C}))$ contains an annulus of modulus bounded away from 0.

Applying Proposition 3.6 instead of Proposition 3.4, we obtain the following variant.

Theorem 3.10. *Given an $m-$chain together with partitions of the $J_i^{\pm}$ into disjoint intervals such that the endpoints of intervals in J^- are equivalent to endpoints of intervals in J^+ and vice versa. Form a Riemann surface by gluing D_i to D_{i+1} along discs as in Proposition 3.6, for all $1 \le i \le m$. If Conditions 1-3 are met, then the family Γ of closed curves that visit $L_1, R_1, L_2, R_2, ..., R_m$ in this order has*

$$\mathrm{M}(\Gamma)^{-1} \lesssim_L 1. \tag{3.10}$$

We conclude this section by stating the definition of *good scales* that depends on a parameter L, in the notation that is most suitable for the probabilistic context.

Definition 3.11. A number $0 < r < 1$ is a *good scale* for a point $x \in \mathbb{T}$, if there is an $m-$chain (J_i^-, J_i^+) with $b_1^+ < x < a_2^-$ and $|J_1^+| \asymp_L r$ such that the above conditions 1-3 hold.

Notice that in the notation of Theorem 1.1, the $m-$chain is formed by the chain links (I_j, I'_{j+1}).

Also, a 'good scale' in the sense of Theorem 1.1 is a good scale in the sense of Definition 3.11, but not necessarily vice-versa. Calling the 'good scales' of the introduction 'very good scales', Theorem 1.1 and Theorem 1.2 are both true with either definition. In particular, combining Theorem 1.1 for good scales and Theorem 1.2 for very good scales gives the self improvement that if every scale is good then actually every scale is very good.

4 Gehring trees have quasisymmetric weldings

4.1 Localization of Gehring trees

The main result of this section is the following decomposition of annuli centered at a Gehring tree into conformal rectangles that are cyclically glued as in Proposition 3.2. Throughout this section we denote $K = K(T)$ the constant of the Gehring tree.

Proposition 4.1. *For every Gehring tree T there are constants K_1, N, M and C (depending only on K) such that for every $p \in T$ and every $0 < r < \operatorname{diam} T/C$ there are disjoint* *conformal $C-$rectangles $(D_i, A_i, B_i), i = 1, 2, ..., n$ where*

$$n \leq N, \tag{4.1}$$

$$D_i \subset (\overline{\mathbb{C}} \setminus T) \cap \{z : r < |z - p| < Cr\}, \tag{4.2}$$

$$A_i, B_i \subset T, \text{ and } A_{i+1} = B_i \tag{4.3}$$

$$\text{the “horizontal sides” of the } D_i \text{ are geodesics of } \overline{\mathbb{C}} \setminus T, \tag{4.4}$$

$$\text{The set } \cup_{i=1}^{n} \overline{D}_i \text{ separates } 0 \text{ and } \infty. \tag{4.5}$$

Moreover, for every i, if (D_i', A_i', B_i') denotes a rectangle conformally equivalent to (D_i, A_i, B_i), then

$$\text{there is a } K_1 - \text{quasisymmetric homeomorphism between } B_i' \text{ and } A_{i+1}'. \tag{4.6}$$

Notice that the index $i+1$ in (4.3) and (4.6) has to be interpreted modulo n, so that D_n glues back to D_1. In particular, if $n = 1$, the two vertical sides of D_1 are glued together.

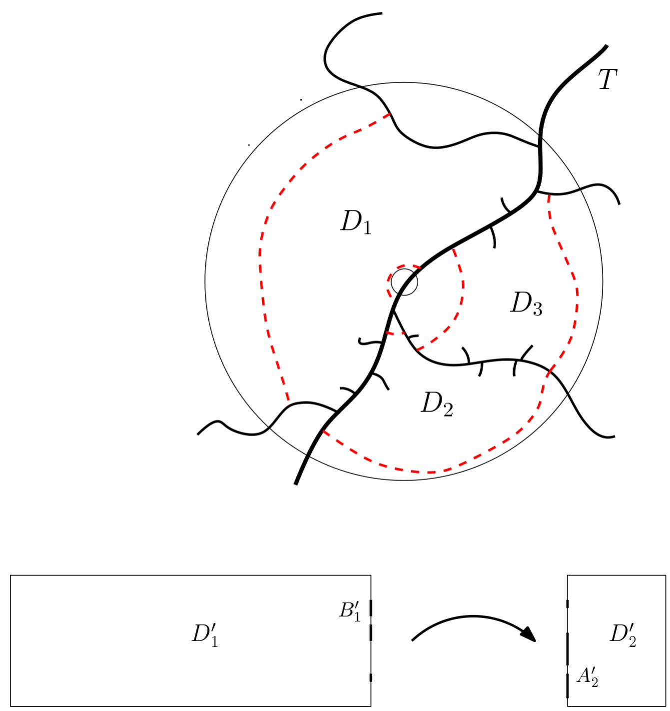

Figure 4.1: The conformal rectangles D_i of Proposition 4.1. The dashed boundary components are hyperbolic geodesics of ˆ $\setminus T$ and correspond to the horizontal boundary of the rectangles.

The proof of Proposition 4.1 requires some preparation. Assume $p = 0 \in T$, $\operatorname{diam} T > R$, and denote $A(r,R) = \{r < |z| < R\}$. Fix a connected component D of $A(r,R) \setminus T$. We say that D *crosses* $A(r,R)$ if $\overline{D} \cap C_r$ and $\overline{D} \cap C_R$ contain non-trivial intervals. The boundary of D contains two arcs $\partial_\ell D$ and $\partial_h D$ of T joining C_r and C_R, where ℓ stands for "lower" and h for "higher" in logarithmic coordinates. They can be defined formally as follows: Since $\operatorname{diam} T > R$, the closed set $T \cup C_r \cup C_R$ is connected and D is simply connected, so that a continuous branch of $\log z$ is well-defined on D. There is an arc $\gamma' = \log \gamma$ in $\log D$ joining the lines $x = \log r$ and $x = \log R$. The set $\{\log r < x < \log R\} \setminus \log D$ has two unbounded components. Both unbounded components have boundary consisting of a crosscut σ together with two half-infinite vertical lines. Denote σ_ℓ resp. σ_h the lower resp. higher of these two crosscuts (they are disjoint because they belong to different complementary components of γ'). Finally denote $\partial_\ell D$ and $\partial_h D$ the images of σ_ℓ and σ_h under the exponential function. Note that they can be disjoint, identical, or neither.

The following lemma is the key to the inductive construction of the conformal rectangles. Consider three disjoint curves γ_j that cross the annulus $A(r_1, r_2)$. We say that γ_1 lies *between* γ_2 and γ_3 if the endpoints of $\gamma_2, \gamma_1, \gamma_3$ are positively oriented on the circle. Thus γ_1 either crosses between γ_2 and γ_3, or between γ_3 and γ_2, but not both. We use the same terminology for disjoint connected subsets of $A(r_1, r_2)$ that cross the annulus.

Lemma 4.2. *For every $M > 0$ there exists a constant $C = C(M, K(T))$ such that the following holds: If D_1 and D_2 are not necessarily distinct components of $A(r,R) \setminus T$ that cross $A(r,R)$ and if $\partial_h D_1 \cap \partial_\ell D_2$ does not contain an arc of diameter $> Mr$, then there exists a component D_3 crossing $A(Cr, R)$ between $\partial_h D_1$ and $\partial_\ell D_2$.*

Proof. If $\partial_h D_1 \cap \partial_\ell D_2$ does not contain an arc of diameter $> Mr$, then $\partial_h D_1$ and $\partial_\ell D_2$ are disjoint in $A(Cr, R)$ for sufficiently large C : Suppose not, then there would be a point $x \in \partial_h D_1 \cap \partial_\ell D_2 \cap A(Cr, R)$. Consider the points $x_h \in \partial_h D_1 \cap C_r$, $x_\ell \in \partial_\ell D_2 \cap C_r$ and the "center" x' of the "triangle" on T with vertices x_ℓ, x_h and x,

$$x' = T(x_\ell, x_h) \cap T(x_\ell, x) \cap T(x_h, x).$$

Since $T(x, x') \subset \partial_h D_1 \cap \partial_\ell D_2$, it follows that $\operatorname{diam} T(x, x') \leq Mr$, hence $|x'| > (C - M)r$. Thus

$$\operatorname{diam} T(x_\ell, x_h) > (C - M - 1)r \geq \frac{C - M - 1}{2} |x_\ell - x_h|,$$

contradicting the quasiarc property (2.2) if $(C - M - 1)/2 > K$.

A similar argument shows that there is no subarc of T joining $\partial_h D_1$ and $\partial_\ell D_2$ inside $A(Cr, R)$. Hence there is a curve $\gamma \subset A(Cr, R) \setminus (T \cup D_1 \cup D_2)$ between $\partial_h D_1$ and $\partial_\ell D_2$ from C_{Cr} to C_R. The component of $A(Cr, R) \setminus T$ containing γ is the required D_3. ☐

We will need the following bound on the number of components that can cross an annulus.

Lemma 4.3. *If $R/r \geq 4K$, then the number of components of $A(r,R) \setminus T$ that cross $A(r,R)$ is bounded above by $N = 8K\pi$.*

Proof. Denote $D_1, ..., D_n$ the components that cross $A(r,R)$. The arcs $\partial_\ell D_i$ meet C_r resp. C_R in points $a_{\ell,i}$ resp. $b_{\ell,i}$ and similarly $\partial_h D_i$ meet C_r resp. C_R in points $a_{h,i}$ resp. $b_{h,i}$. The n arcs of C_R between $b_{\ell,i}$ and $b_{h,i}$ are pairwise disjoint. Consequently, if $d_j = |b_{\ell,j} - b_{h,j}|$ is minimal among $d_i, 1 \leq i \leq n$, then $d_j \leq 2\pi R/n$. By the Ahlfors condition (2.2),

$$\operatorname{diam} T(b_{\ell,j}, b_{h,j}) \leq K \frac{2\pi R}{n}.$$

In particular, the arc $\gamma \subset T$ that joins $\partial_\ell D_j$ and $\partial_h D_j$ is of distance $\leq K \frac{2\pi R}{n}$ from C_R, hence of distance

$$d \geq R - r - K \frac{2\pi R}{n}$$

from C_r. Since $a_{\ell,j}$ and $a_{h,j}$ connect through the same arc γ, we have

$$\operatorname{diam} T(a_{\ell,j}, a_{h,j}) \geq d \geq \frac{R}{2}$$

if $R > 4r$ and $n > 8\pi K$. On the other hand,

$$\operatorname{diam} T(a_{\ell,j}, a_{h,j}) \leq K|a_{\ell,j} - a_{h,j}| \leq K2r < \frac{R}{2}$$

if $R > 4Kr$, and we conclude that $n \leq 8\pi K$ whenever $R/r > 4K$. □

Next, still assuming that $0 \in T$ we will construct a chain of domains D_i that already has most of the desired features of Proposition 4.1, namely (4.1),(4.2), (4.4) and (4.5).

Lemma 4.4. *For all M and K there are constants C_1 and C_2 such that whenever $R = C_2 r <$ diam T, then there are domains $D_1, D_2, ..., D_n$ in $A(r,R)$ with $n \leq N = 8K\pi$ that are all crossing $A(C_1 r, R)$ and satisfy the following: Each D_i is a connected component of $A(C^{n_i} r, R) \setminus T$ for some $0 \leq n_i \leq N$, each intersection $\partial_h D_i \cap \partial_\ell D_{i+1}$ (where $D_{n+1} = D_1$) contains an arc α_i of diameter $\geq M \max(C^{n_i} r, C^{n_{i+1}} r)$, and $\cup_i \overline{D_i}$ separates $A(r,R)$.*

Proof. Set $C_1 = C^{8\pi K}$ where C is the constant of Lemma 4.2, and $C_2 = 4KC_1$. Fix a component D_1 crossing $A(r,R)$. If $\partial_h D_1 \cap \partial_\ell D_1$ contains an arc of diameter $\geq Mr$, set $n = 1$ and we are done. Else, by Lemma 4.2, there is a component D_2 crossing $A(Cr,R)$ between $\partial_h D_1$ and $\partial_\ell D_1$. Note that D_1 also crosses $A(Cr,R)$. If the pair D_1, D_2 does not satisfy the claim, repeated application of Lemma 4.2 (relabeling the domains if necessary to keep the cyclic order) yields a sequence of disjoint domains $D_1, ..., D_n$ that all cross $A(C^{n-1} r, R)$. By Lemma 4.3, this process has to stop when $n \leq N = 8K\pi$. Joining a point of $\partial_h D_{i-1} \cap \partial_\ell D_i$ with a point of $\partial_h D_i \cap \partial_\ell D_{i+1}$ by a Jordan arc in D_i, we obtain a Jordan curve in $\cup_i \overline{D_i}$ separating C_r and C_R. □

Next, we slightly modify the D_i to turn them into conformal rectangles of controlled modulus with geodesic boundaries. Roughly speaking, since we have no lower bound for the size of the "inner boundary" $\partial D_i \cap C_r$ of D_i, we just replace it by a larger arc: By construction, each D_i is a connected component of $A(C^{n_i}r, R) \setminus T$. Consider the component of $D_i \cap A((C^{n_i}+1)r, R)$ that joins the two boundary circles, and let I_1 be the arc on $C_{(C^{n_i}+1)r}$ that separates $C_{C^{n_i}r}$ and C_R. By the Ahlfors condition, the length of I_1 is comparable to $C^{n_i}r$. Let γ_1 be the hyperbolic geodesic of $\setminus T$ joining the two endpoints $a_{1,i}, b_{1,i}$ of I_1 (viewed as prime ends). Then the diameter of γ_1 is comparable to $C^{n_i}r$ (the upper bound follows from the Gehring-Hayman inequality, the lower bound from the Ahlfors-condition),

$$\operatorname{diam} \gamma_1 \leq C_0 C^{n_i} r.$$

Furthermore, the distance of γ_1 to 0 is comparable to $C^{n_i}r$ and hence greater than a constant times r. Similarly, let I_2 be the arc on C_R that separates $C_{C^{n_i}r}$ and ∞, and let γ_2 be the geodesic joining the endpoints $a_{2,i}, b_{2,i}$ of I_2. The diameter of γ_2 is comparable to R, again by the Gehring-Hayman inequality. Now let D_i' be the connected component of $\setminus T$ bounded by γ_1 and γ_2. These are the domains of Proposition 4.1. Notice that properties (4.1),(4.2), (4.4) and (4.5) are satisfied by Lemma 4.4. Notice also that the set

$$\alpha_i \setminus D(0, (C^{n_i}+1)r)$$

contains an arc α_i' of diameter comparable to r,

$$\operatorname{diam} \alpha_i' \geq (MC^{n_i}r - 2(C^{n_i}+1)r)/2,$$

and that this arc is contained in $\partial_h D_i' \cap \partial_\ell D_{i+1}'$.

To simplify notation, from now on we will drop the prime and will simply write D_i and α_i instead of D_i' and α_i'. Assume without loss of generality counterclockwise ordering of a_1, b_1, b_2, a_2, and notice that the conformal modulus of the topological rectangles $(D_i, a_1, b_1, b_2, a_2)$ is bounded above and below : This can easily be seen using Rengel's inequality

$$\frac{w^2}{\operatorname{area} D} \leq M \leq \frac{\operatorname{area} D}{h^2},$$

where w and h are the distances between the opposite pairs of sides. Indeed, the sides (a_1, b_1) and (b_2, a_2) trivially have distance of order $R - C^{n_i}r$, and the sides (b_1, b_2) and (a_2, a_1) have distance of order $C^{n_i}r$ again by the Ahlfors condition. The area is of order R^2, since the diameter of γ_2 is of order R.

We will now turn to the construction of the sets A_i, B_i. Since D_i and D_{i+1} connect through the arc $\alpha_i \subset T$, we will define B_i as a subset of α_i. It can be shown that $(D_i, \alpha_{i-1}, \alpha_i)$ is a conformal (M, C)-rectangle for suitable constants (only the uniform perfectness of the image of α under a conformal map onto a rectangle would require proof). But this is NOT the appropriate choice of B_i: Indeed, every triple point on α corresponds to two prime ends on one side and one prime end on the other side of α, so that the choice $B_i = A_{i+1} = \alpha$ does not even allow for a bijection between the corresponding sets in rectangular coordinates, see Figure 4.1 . To obtain a homeomorphism, we need to avoid the branch points of T. Roughly speaking, to obtain a quasisymmetric homeomorphism, we need to stay away from the branch points in the following quantitative way: If β is a branch of T branching off a point x in the interior of α (not one of the two endpoints), then we need to remove an interval of size proportional to $\operatorname{diam}\beta$ from α. More precisely, since α is a quasiconformal arc, there is a global quasiconformal map F that sends α to the interval $[0,1]$ on the real line. For every component β of $T \setminus \alpha$ with $\overline{\beta} \cap \alpha \neq \emptyset$, the intersection consists of a single point x_β. Denote

$$\ell_\beta = \operatorname{diam} F(\beta).$$

We will show that for sufficiently small $\delta > 0$, the set

$$B_i = A_{i+1} = F^{-1}\Big([0,1] \setminus \bigcup_\beta [F(x_\beta) - \delta\ell_\beta, F(x_\beta) + \delta\ell_\beta] \Big) \tag{4.7}$$

satisfies the requirements of Proposition 4.1. We begin by describing a set of assumptions guaranteeing that such a set it non-empty. A picturesque description of the situation is as follows: Suppose trees (vertical line segments of length ℓ_β) grow from the forest floor (the interval $[0,1]$) in such a way that they are not too close to each other: The distance between two trees is bounded below by a constant times the height of the smaller tree. Each tree casts a shadow of size proportional to its height. Then there are many sunny places on the forest floor (assuming this proportionality constant is small). More generally (corresponding to branch points of order more than three) we need to allow for a bounded number of trees to get close. A precise statement is the

Sunny Lemma 4.5. *For every integer $N \geq 1$ and real $M > 0$ there are $\delta = \delta(N,M)$ and $c = c(N,M)$ such that the following holds: Suppose $S = \{(x_n, \ell_n)\}$ is a collection of pairs $(x,\ell) \in [0,1] \times (0,M]$ with the property that for every interval $I = [a,b] \subset [0,1]$, the number of pairs $(x,\ell) \in S$ with $x \in [a,b]$ and $\ell \geq M|b-a|$ is $\leq N$. Then*

$$[0,1] \setminus \bigcup_{(x,\ell)\in S} [x - \delta\ell, x + \delta\ell]$$

contains a $c-$uniformly perfect set of diameter at least $1/2$.

Proof. Set

$$r = \frac{1}{4N+4}$$

and fix

$$\delta < \frac{1}{2(4N+4)M}.$$

Beginning with $\mathcal{L}_0 = \{[0,1]\}$, inductively construct a nested collection $\mathcal{L}_n$ of disjoint "potentially sunny intervals" of size r^n as follows: Fix $I \in \mathcal{L}_n$ and subdivide I into $4N+4$ equal sized intervals of size r^{n+1}. By assumption, there are at most N intervals ("shadows") $[x-\delta\ell, x+\delta\ell]$ for which $x \in I$ and $Mr^n < \ell \leq Mr^{n-1}$. By the definition of δ, each of these shadows $[x-\delta\ell, x+\delta\ell]$ will intersect at most two (adjacent) of the $4N+4$ intervals. In addition, there are at most two more of the $4N+4$ intervals that intersect a shadow $[x-\delta\ell, x+\delta\ell]$ with $x \notin I$ and $Mr^n < \ell \leq Mr^{n-1}$ (one on either endpoints of I). Hence there are at least $2N+2$ intervals remaining, and they constitute the collection of the intervals of $\mathcal{L}_{n+1}$ that are in I. In particular, it follows that

$$\operatorname{diam} \bigcup_{J \in \mathcal{L}_{n+1}, J \subset I} \geq 1/2 \operatorname{diam} I. \tag{4.8}$$

By construction

$$A = \bigcap_n \bigcup_{I \in \mathcal{L}_n} I \quad \subset \quad [0,1] \setminus \bigcup_{(x,\ell) \in S} [x-\delta\ell, x+\delta\ell],$$

and it is easy to see that A is uniformly perfect: If $x \in A$ and if the annulus $A(x; R_1, R_2)$ separates A where $R_1 < R_2 \leq 1$, then consider the minimal n for which $r^n < 4R_1$, and denote $I_n(x)$ the interval of $\mathcal{L}_n$ that contains x. By (4.8), the interval $I_{n-1}(x)$ contains a second interval $J \in \mathcal{L}_n$ of distance more than $1/4r^{n-1}$ and less than r^{n-1} from $I_n(x)$. Thus $A(x; R_1, R_2)$ separates $I \cap A$ and $J \cap A$, which yields an upper bound on R_2/R_1. □

Returning to the definition (4.7) of B_i, we claim that the collection $\{(F(x_\beta), \ell_\beta)\}$ satisfies the assumptions of Lemma 4.5, where $\ell_\beta = \operatorname{diam} F(\beta)$. To this end, let us first notice that by the quasisymmetry of F, there is an upper bound $\ell_\beta \leq M$ that only depends on $K(T)$: Indeed, all arcs β are contained in $D_i \cup D_{i+1}$ so that $\operatorname{diam} \beta \leq \operatorname{diam} D_i \cup D_{i+1} \lesssim \operatorname{diam} \alpha$. Next, fix an interval $[a,b] \subset [0,1]$ and consider an arc β with $F(x_\beta) \in [a,b]$ and $\ell_\beta > M|b-a|$. Then the quasisymmetry of F implies that

$$\operatorname{diam} \beta \geq M' \operatorname{diam} F^{-1}[a,b]$$

where M' is large when M is large, and since T is a Gehring tree there is an upper bound N on the number of such arcs. Thus the assumptions of the sunny Lemma 4.5 are satisfied, and for sufficiently small δ the set $[0,1] \setminus \bigcup_\beta [F(x_\beta) - \delta\ell_\beta, F(x_\beta) + \delta\ell_\beta]$ is uniformly perfect and of diameter $\geq 1/2$. Consequently the image under F^{-1} is uniformly perfect (quasiconformal maps distort moduli of annuli only boundedly) and of diameter comparable to $\operatorname{diam} \alpha_i$. We will also need the following

Lemma 4.6. *The internal distance d_i of D_i and the euclidean distance are comparable on $A_i \cup B_i$.*

Proof. Let $x, y \in B_i$, viewed as prime ends of D_i. We need to show that $d_i(x,y) \lesssim |x-y|$. Recall the bounded turning property of the John domain $\setminus T$: By Theorem 6.3 of [NV93], the smaller of the two arcs of $\partial^* T$ between x and y has diameter comparable to the internal distance between x and y in $\setminus T$, which is easily seen to be comparable to the internal distance of D_i. This smaller arc is of the form

$$\alpha(x,y) \cup \bigcup_{x_\beta \in \alpha(x,y)} \beta,$$

where the union is over all components β of $T \setminus \alpha$ with $\overline{\beta} \cap \alpha \neq \emptyset$ and $\beta \subset \overline{D}_i$. Since $x \in B_i$ we have

$$|F(x) - F(x_\beta)| \geq \delta \operatorname{diam} F(\beta)$$

so that

$$|x - x_\beta| \geq \delta' \operatorname{diam} \beta$$

by quasisymmetry of F, and similarly for y. It follows that any arc β with x_β between x and y has diameter bounded by a constant times $|x-y|$, and the Lemma follows in this case. If both x and y are in A_i, the proof is the same. In case $x \in B_i$ and $y \in A_i$, both the euclidean and the internal distance of x and y are bounded above and below, and there is nothing to prove. □

Conclusion of the Proof of Proposition 4.1. We have already established the existence of constants K_1, N, M, C and defined the domains D_i and sets $A_i, B_i \subset \partial D_i \cap T$ satisfying (4.1) - (4.5). It only remains to show that the topological rectangles (D_i, A_i, B_i) are conformal $(M, C)-$rectangles, and to verify (4.6). Recall that we already showed that the conformal modulus of the topological rectangle $(D_i, a_1, b_1, b_2, a_2)$ is bounded above and below, where the four marked points (a_1, b_1, b_2, a_2) are the endpoints of α_i and α_{i-1}, namely $\alpha_i = T(a_1, a_2)$ and $\alpha_{i-1} = T(b_1, b_2)$. Since the diameter of B_i is comparable to the diameter of α_i, and the diameter of A_i is comparable to $\operatorname{diam} \alpha_{i-1}$, the same argument using Rengels inequality shows that the conformal modulus of (D_i, A_i, B_i) is bounded above and below (away from zero).

Now let g_i be the conformal map of D_i onto the rectangle $R_i = [0, X_i] \times [0,1]$ that takes the extreme points of A_i resp B_i to the left resp. right vertices. Since the modulus X_i is bounded above and below, the rectangle is a John domain with bounded constant. By Theorem 7.4 of [NV93], g is a quasisymmetric map between D_i and R with respect to the internal metric d_i of D_i. It easily follows that $g_i(A_i)$ and $g_i(B_i)$ are uniformly perfect with constant only depending on those of A_i and B_i, and the quasisymmetry data. Finally, the required homeomorphism between $g_i(B_i)$ and $g_{i+1}(A_{i+1})$ is simply given by the restriction of $g_{i+1} \circ g_i$ to B_i, which is quasisymmetric by two applications of the aforementioned Theorem 7.4 of [NV93], combined with the bilipschitz-continuity of the restriction to B_i of the identity map on $\partial D_i \cap \partial D_{i+1}$ with respect to the internal metrics of D_i and of D_{i+1}, again using Lemma 4.6. □

4.2 Proof of Theorem 1.2

Most of the work has already been done by proving Proposition 4.1. Let $\mathcal{L}$ be the lamination of a Gehring tree T and $f : \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \overline{\mathbb{C}} \setminus T$ a conformal map fixing ∞. Given $x \in \mathbb{T}$ and a scale $0 < \rho < \rho_0$ (where ρ_0 will be specified later), let I be the arc of $\mathbb{T}$ of length ρ with initial point x. Applying Proposition 4.1 to $p = f(x)$ and

$$r = \frac{\operatorname{diam} f(I)}{C},$$

we obtain conformal (M, C)-rectangles (D_i, A_i, B_i) separating the annulus $A(p; r, Cr)$, arranged in counter-clockwise order. We may assume the labeling is such that the prime end boundary of D_1 contains points of $f(I)$. Recall the definition of the points $a_{1,i}, a_{2,i}, b_{1,i}, b_{2,i}$, introduced during the construction of D_i right after Lemma 4.4, and denote $\hat{a}_{1,i}, \hat{a}_{2,i}, \hat{b}_{1,i}, \hat{b}_{2,i}$ their preimages under f. The preimages

$$\hat{D}_i = \ f^{-1}(D_i)$$

are bounded by the two arcs $[\hat{a}_{1,i}, \hat{a}_{2,i}]$ and $[\hat{b}_{1,i}, \hat{b}_{2,i}]$ on $\mathbb{T}$ together with the hyperbolic geodesics $< \hat{a}_{1,i}, \hat{b}_{1,i} >$ and $< \hat{a}_{2,i}, \hat{b}_{2,i} >$ of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, see Figure 4.2.

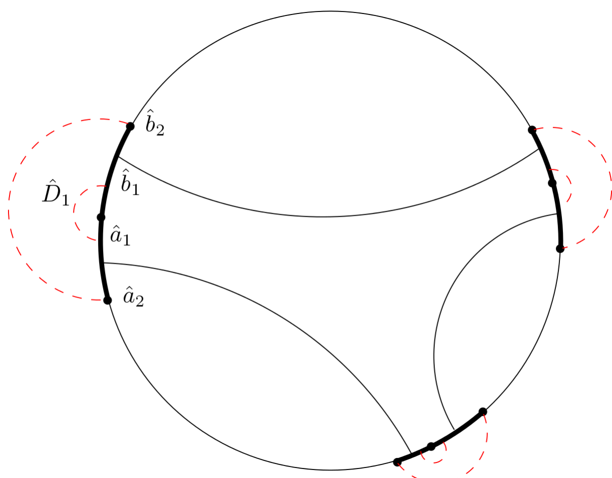

Figure 4.2: The preimages $\hat{D}_i = \ f^{-1}(D_i)$

Set $p_1 = p$, for each $i = 2, 3, ..., n$ fix any point $p_i \in [a_{1,i}, b_{1,i}] \subset \mathbb{T}$ (for instance $p_i = [a_{1,i}]$), and for $i = 1, 2, ..., n$ let

$$I_i = [b_{2,i}, p_i] \ , \quad I_i' = [p_i, a_{2,i}].$$

Since the conformal modulus of (D, a_1, b_1, b_2, a_2) is bounded above and below for each i, the lengths $|I_i|$ and I_i' are comparable. Let

$$\hat{A}_i = f^{-1} A_i \cap I_i \ , \quad \hat{A}_i' = f^{-1} B_i \cap I_i'$$

so that for each i, $(\hat{D}, \hat{A}, \hat{A}')$ is a conformal image of (D, A, B) and hence a conformal (M, C)-rectangle. In particular, $\hat{A}$ and $\hat{A}'$ are uniformly perfect and

$$\operatorname{diam} \hat{A}_i \asymp |I_i| \asymp |I_i'| \asymp \hat{\operatorname{diam}} \hat{A}_i'.$$

The conformal maps of $(\hat{D}, \hat{A}, \hat{A}')$ onto a rectangle that send the points a_1, a_2, b_1, b_2 to the corners is quasisymmetric (this is clear since both domains are John with bounded constants, and can be verified directly since it is just a composition of a Mobius transformation with a logarithm). It remains to notice that the quasisymmetric homeomorphisms of (4.6) conjugates to a quasisymmetric homeomorphism between $\hat{A}_i'$ and $\hat{A}_{i+1}$. Thus the sets $\hat{A}_i, \hat{A}_i'$ satisfy all requirements of Theorem 1.2 and we have proved that Gehring trees satisfy the conditions of Theorem 1.2.

5 Thick laminations admit welding

Our proof of Theorem 1.1 is based on a canonical solution to a finite welding problem which we will describe first. Then in Section 5.2 we will approximate a given lamination by an increasing sequence of finite ones and employ the modulus estimates of Section 3.2 to get control of the regularity of their conformal solutions. The proof is finished in Section 5.3.

5.1 Finite laminations and balloon animals

Given a finite lamination $\mathcal{L}$, there are many solutions to the realization problem of finding a conformal map f of Δ such that $\mathcal{L}_f = \mathcal{L}$. Koebe [Koe36] showed that laminations without triple points can be realized as conformal laminations of circle domains ($\overline{\mathbb{C}} \setminus f(\Delta)$ is a union of discs with disjoint interiors), and also Lemma 19 in Bishops important work [Bis07] which partly motivated our approach. There is nothing really special about discs, and it is feasible that instead of discs one could prescribe any say strictly convex shape (up to homothety). However, our method relies on the existence of a realization for which the harmonic measures of complementary components are linearly related. The purpose of this section is to prove the following proposition, see Figure 5.1.

Proposition 5.1. *a) For every finite lamination $\mathcal{L}$, there is a simply connected domain $G \subset \overline{\mathbb{C}}$ and a conformal map $f : \Delta \to G$ fixing ∞ such that $\overline{\mathbb{C}} \setminus \overline{G}$ consists of Jordan domains G_i, and such that $f(x) = f(y)$ if and only if $x = y$ or $(x, y) \in \mathcal{L}$.*

b) Moreover, G can be chosen such that there are points $z_i \in G_i$ for which the harmonic

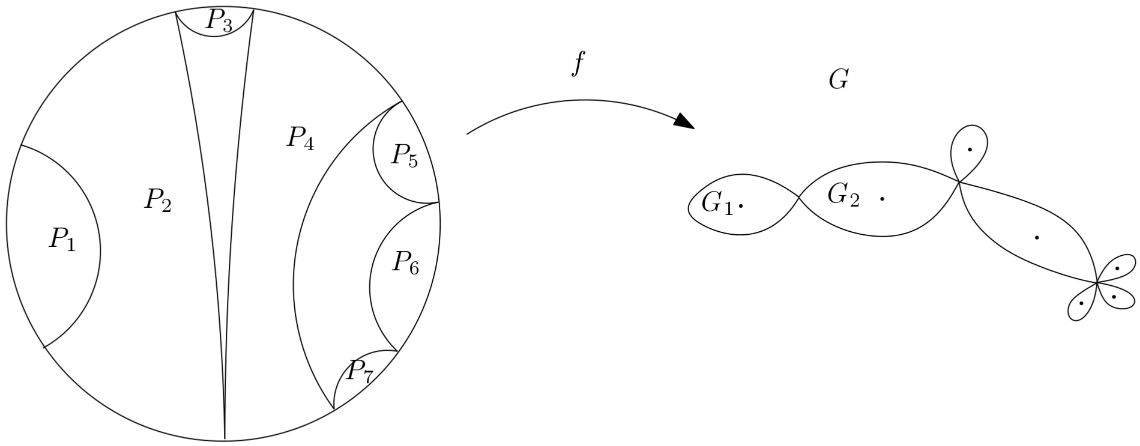

Figure 5.1: A finite lamination with seven pieces and two gaps, and its balloon animal G.

measures ω_i of ∂G_i at z_i and ω_∞ of G at ∞ have the property

$$\frac{d\omega_\infty}{d\omega_i} \equiv p_i \quad \text{on} \quad \partial G_i$$

for all i, where $p_i = \omega_\infty(\partial G_i)$.

c) More generally, given positive numbers $p_{i,j}$ for each boundary arc $\alpha_{i,j}$ of P_i such that $\sum_j p_{i,j} = p_i$, then G and z_i can be chosen such that

$$\frac{d\omega_\infty}{d\omega_i} \equiv p_{i,j} \quad \text{on} \quad \beta_{i,j} = f(\alpha_{i,j}) \quad \text{for all} \quad i, j.$$

In b) and c), the domain G and the z_i are unique up to normalization by a linear map $z \mapsto az + b$.

Remark. Boundedness of $d\omega_\infty/d\omega_i$ fails dramatically for circle domains, as the harmonic measure of a disc is just normalized length measure, while the harmonic measure from the outside at the intersection point of two discs decays exponentially. In fact, we will see that the boundaries of the G_i are piecewise analytic arcs that make up equal angles at the contact points.

The connected components of $\mathbb{D} \setminus \mathcal{L}$ either have finite hyperbolic area, or meet the circle in one or more non-degenerate intervals. The former are called *gaps* and correspond to points of multiplicity at least three. The latter are in one-to-one correspondence with the G_i, We call them the *pieces* of the lamination and label them P_i.

The *balloon animal* of $\mathcal{L}$ and its *balloons* are the domains G and G_i of Proposition 5.1, where we normalize f hydrodynamically ($f(z) = z + O(1/z)$ near ∞).

Proof of Proposition 5.1. The proof of b) and c) is essentially an exercise in conformal welding: Take any smoothly bounded solution to a), form an abstract Riemann surface by gluing discs to ∂G according to the harmonic measure ω_∞, and invoke the uniformization theorem. Because there are singularities at the cut points, we will give a more detailed proof based on the measurable Riemann mapping theorem: First, construct a domain H bounded by C^2-arcs realizing $\mathcal{L}$ and such that near each cut point, ∂H is a star in some local coordinate, $\phi(\partial H) = \cup_{k=1}^{2n}[0, e^{\pi i/k}]$: This can easily be done by starting with disjoint actual stars, inductively joining their endpoints by sufficiently smooth curves (for instance hyperbolic geodesics of the complementary domain), and finally correcting the harmonic measures at ∞ of the boundary arcs by an appropriate smooth quasiconformal map as described below. Next, if $h : \Delta \to H$ is a conformal map fixing ∞ and if $h_i : \mathbb{D} \to H_i$ are conformal maps to the bounded complementary components of $\overline{H}$, define homeomorphisms $\phi_i : \mathbb{T} \to \mathbb{T}$ by setting

$$|\phi_i'(x)| = |(h^{-1} \circ h_i)'(x)|/p_{i,j} \quad \text{for} \quad x \in \alpha_{i,j}$$

and normalizing by $\phi_i(1) = 1$, say. Then the ϕ_i are smooth and admit quasiconformal extensions $\Phi_i : \mathbb{D} \to \mathbb{D}$. The Beltrami equation

$$\frac{\overline{\partial} F}{\partial F}(z) = \begin{cases} 0 \quad \text{if} \quad z \in H, \\ \frac{\overline{\partial}}{\partial}\Phi_i \circ h_i^{-1} \quad \text{if} \quad z \in H_i \end{cases}$$

has a quasiconformal solution F, and it is easy to check that $G = F(H)$ satisfies the claim. $\square$

Example 5.2. If $\mathcal{L}$ consists of one chord only, say $(-1, 1)$, then G is the unbounded component of the lemniscate $\{\sqrt{z-1} : |z| = 1\}$ and f is the square root of a quadratic polynomial.

Call a finite lamination $\mathcal{L}$ *well-branched* if for each piece P, $\overline{P} \cap \mathbb{T}$ has one or two connected components. In other words, $\mathcal{L}$ is well-branched if there is no balloon G_i for which ∂G_i contains more than two cut points of $\overline{G}$. The lamination of Figure 5.1 is well-branched. Since we prefer to work with well-branched laminations, we observe that every finite lamination has a well-branched refinement:

Lemma 5.3. *If $\mathcal{L}'$ is a finite sub-lamination of a maximal lamination $\mathcal{L}$, then there is a finite well-branched lamination $\mathcal{L}''$ with $\mathcal{L}' \subset \mathcal{L}'' \subset \mathcal{L}$.*

This can easily be proved by induction over the number of pieces with more than two cut points: In every such piece, there is a gap subdividing the piece into smaller pieces with fewer cutpoints.

We conclude this section with a simple criterion that guarantees existence of a solution to the realization problem. Let $\mathcal{L}_n$ be an increasing sequence of finite laminations converging to a lamination $\mathcal{L} \supset \cup_n \mathcal{L}_n$ in the sense that for every chord $(a, b) \in \mathcal{L}$ there is a sequence of chords $(a_n, b_n) \in \mathcal{L}_n$ with $a_n \to a$ and $b_n \to b$. Denote $\mathcal{P}_n$ the set of pieces of $\mathcal{L}_n$, and let f_n

be a hydrodynamically normalized conformal map of Δ realizing $\mathcal{L}_n$ (that is, $f_n(a) = f_n(b)$ for each $(a, b) \in \mathcal{L}_n$). Denote

$$m_n = \max_{P \in \mathcal{P}_n} \sup_{k \geq n} \operatorname{diam} f_k(\overline{P} \cap \mathbb{T}) \tag{5.1}$$

the largest diameter amongst all images of pieces of generation n. Notice that, by our assumption $\mathcal{L}_n \subset \mathcal{L}_{n+1}$, each of the sets $f_k(\overline{P} \cap \mathbb{T})$ is a finite union of balloons of generation k.

Proposition 5.4. *If $m_n \to 0$ as $n \to \infty$, and if f is any subsequential limit of $(f_n)_{n \geq 1}$ under compact convergence in Δ, then f extends continuously to $\overline{\Delta}$, convergence is uniform in Δ, and f realizes $\mathcal{L}$.*

Proof. Each piece of $\mathcal{P}_n$ intersects $\partial\mathbb{D}$ in finitely many arcs. Denote s_n the size of the smallest such arc amongst all pieces of $\mathcal{P}_n$. Notice that $s_n \to 0$ by Beurling's projection theorem, since f_n are normalized and $m_n \to 0$. Then every interval $I \subset \partial\mathbb{D}$ of size $\leq s_n$ is contained in at most two such arcs, hence

$$\operatorname{diam} f_k(I) \leq 2m_n \quad \text{for all} \quad k \geq n.$$

It follows that

$$|f_k(z) - f_k(w)| \leq m'_n \quad \text{for all} \quad k \geq n, z, w \in \Delta, |z - w| < s_n$$

for a sequence $m'_n \to 0$. By pointwise convergence, this also holds for f, so that f is uniformly continuous and extends to $\overline{\Delta}$. It also easily follows that the compact convergence is in fact uniform. Finally, if $(a, b) \in \mathcal{L}$ and $(a_n, b_n) \in \mathcal{L}$ converges to (a, b), then $f(a) = \lim f_n(a_n) = \lim f_n(b_n) = f(b)$ so that f realizes $\mathcal{L}$. □

5.2 The Modulus estimate for finite approximations to $\mathcal{L}$

We inductively construct a sequence $\mathcal{L}_k$ of finite approximations of $\mathcal{L}$ as follows. Set $\mathcal{L}_0 = \emptyset$ and fix $k \geq 1$. For each dyadic point $x = x_{\ell,k} = \ell/2^k \in \mathbb{T}$ with $1 \leq \ell \leq 2^k$ for which $r = 2^{-k}$ is a good scale, consider the sets and intervals $A_j = A_j(x, k) \subset I_j = I_j(x, k), A'_j \subset I'_j, 1 \leq j \leq n = n(x, k) \leq N$ and homeomorphisms ϕ_j of Theorem 1.1. Denote $a_j = a_j(x, k) \in A_j, a'_j \in A'_j$ the point of maximal distance from the point of intersection $x_j \in I_j \cap I'_j$. By the monotonicity of ϕ_j we have $(a_j, a'_{j+1}) \in \mathcal{L}$ for each j. Next, since A_j is uniformly perfect and of size comparable to I_j, it is easy to see that there is a point $b_j \in A_j \cap [a_j, x_j]$ with $|a_j - b_j| \asymp |b_j - x_j|$ such that $[a_j, b_j] \cap A_j$ is uniformly perfect, with constants only depending on the constant of $\mathcal{L}$. For each j, set $b'_{j+1} = \phi(b_j) \in A'_j \cap [x, a'_j]$ so that $(b_j, b'_{j+1}) \in \mathcal{L}$, see Figure 5.2. By the quasisymmetry of ϕ_j we have

$$|a_j - b_j| \asymp |b_j - x_j| \asymp |b'_j - x_j| \asymp |a'_j - b'_j|,$$

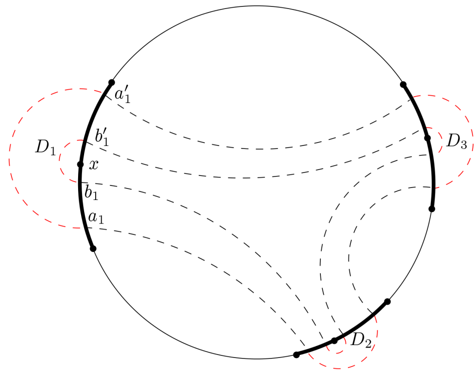

Figure 5.2: The definition of $\mathcal{L}_k$ and the annular neighborhood $\mathcal{A}_k(x)$.

and $[a_j', b_j'] \cap A_j'$ is uniformly perfect as well. Now form the set $\hat{\mathcal{L}}_k$ of all the chords (a_j, a_{j+1}') and (b_j, b_{j+1}') , namely

$$\hat{\mathcal{L}}_k = \bigcup_{\ell=1}^{2^k} \bigcup_{j=1}^{n(x_\ell,k)} (a_j, a_{j+1}') \cup (b_j, b_{j+1}').$$

Applying Lemma 5.3 to the finite sub-lamination $\mathcal{L}' = \mathcal{L}_{k-1} \cup \hat{\mathcal{L}}_k$ of the maximal lamination $\mathcal{L}$, we obtain a well-branched lamination $\mathcal{L}' \subset \mathcal{L}'' \subset \mathcal{L}$ and set

$$\mathcal{L}_k = \mathcal{L}''.$$

Denoting $D_j = D_{j,k}(x)$ the hyperbolic convex hull (with respect to Δ) of $[a_j, b_j] \cup [a_j', b_j']$, we set

$$\mathcal{A}_k(x) = \bigcup_{j=1}^{n(x,k)} D_j, \tag{5.2}$$

see Figure 5.2.

We think of $\mathcal{A}_k(x)$ as an annular neighborhood of x at scale $\asymp 2^{-k}$, and leave the details of the proof of the following Lemma to the reader.

Lemma 5.5. *With*

$$A_{j,k} = A_j \cap [a_j, b_j] \quad \textit{and} \quad A_{j,k}' = A_j' \cap [b_1', a_1'],$$

the $(D_{j,k}, A_{j,k}, A_{j,k}')$ *are conformal* $C-$*rectangles. Moreover,*

$$\mathcal{A}_k(x) \cap \mathcal{A}_{k'}(x') = \emptyset$$

whenever $|k - k'| \geq C'$ *and* $|x - x'| \leq C' 2^{-k}$*. Here* C *and* C' *only depend on the constant of* $\mathcal{L}$*.*

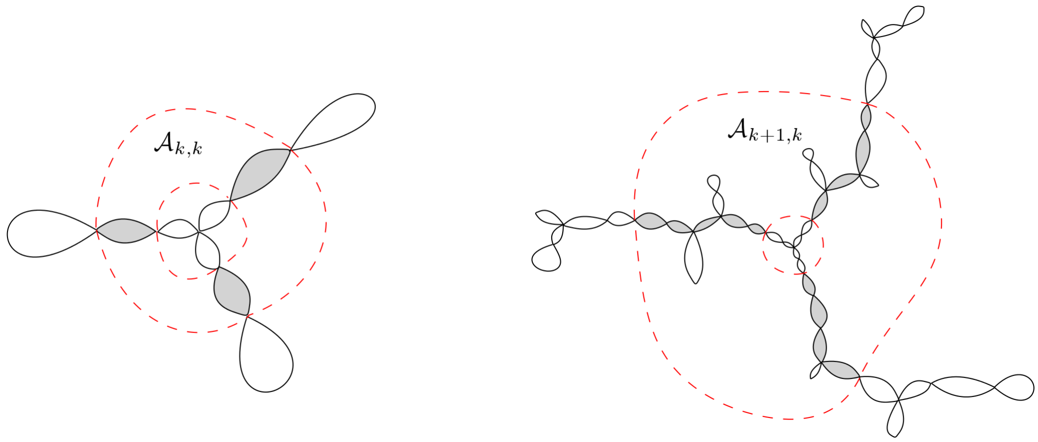

Figure 5.3: Two consecutive balloon animals and annuli $\mathcal{A}_{m,k}(x)$. Shaded are those G_i that allow for a crossing from $f_m(D_j)$ to $f_m(D_j+1)$.

We now turn to the key modulus estimate. Form the balloon animal $\mathcal{G}_m$ corresponding to $\mathcal{L}_m$. More precisely, apply Proposition 5.1 c) to $\mathcal{L}_m$ with

$$p_{i,j} = 2\omega_\infty(\beta_{i,j})$$

so that

$$\omega(z_i, \beta_{i,1}, G_i) = \omega(z_i, \beta_{i,2}, G_i) = \frac{1}{2}$$

for those i for which the balloon G_i has two boundary arcs (since $\mathcal{L}_m$ is well branched, each G_i either has one or two boundary arcs). Denote $f_m : \Delta \to \setminus \mathcal{G}_m$ the corresponding conformal map. Let $k \leq m$, let $x = \ell/2^k \in \mathbb{T}$ be a dyadic point, and consider the image $f_m(\mathcal{A}_k(x))$. The hyperbolic geodesics $f_m(< a_j, a'_j >), 1 \leq j \leq n$ are Jordan arcs whose union forms a Jordan curve surrounding $f_m(x)$, and similarly for the union $\cup_j f_m(< b_j, b'_j >)$. Together these Jordan curves bound a topological annulus

$$\mathcal{A}_{m,k}(x) \supset f_m(\mathcal{A}_k(x)).$$

This annulus can also be obtained from $f_m(\mathcal{A}_k(x))$ by adding those G_i that correspond to the chords of $\mathcal{L}_m$ with endpoints in $\bigcup [a_j, b_j] \cup \bigcup [a'_j, b'_j]$. See Figure 5.3. Now Theorem 3.10 implies

Proposition 5.6. *If r is a good scale for x, then the conformal modulus $M(\mathcal{A}_{m,k}(x))$ is bounded away from zero, with bound depending only on the constant of $\mathcal{L}$.*

5.3 Proof of Theorem 1.1

We now have all ingredients to finish the proof of our main result Theorem 1.1.

Proof of Theorem 1.1. Given a maximal non-degenerate lamination $\mathcal{L}$, form the approximations $\mathcal{L}_k$ described in the previous Section 5.2, together with their conformal realizations $f_k : \Delta \to \backslash\mathcal{G}_k$ and annuli $\mathcal{A}_{m,k}$ obtained from $f_m(\mathcal{A}_k)$. Denote again $\mathcal{P}_n$ the set of pieces of $\mathcal{L}_n$. Then

$$m_n = \max_{P\in\mathcal{P}_n} \sup_{k\geq n} \operatorname{diam} f_k(\overline{P}\cap\mathbb{T})$$

tends to zero exponentially fast: Indeed, by Lemma 5.5, every piece $P \in \mathcal{P}_n$ is surrounded by n/C' disjoint "annular neighborhoods" of the form $\mathcal{A}_\ell(x)$ (where $x \in \overline{P}\cap\mathbb{T}$ and $\ell = jC', 1 \leq j \leq n/C'$), so that $f_k(\overline{P}\cap\mathbb{T})$ is surrounded by n/C' nested annuli $\mathcal{A}_{k,\ell}(x)$. By Proposition 5.6, all of these annuli have modulus $\geq M_0$, and the claim follows from Lemma 2.6. Let $f = \lim_{j\to\infty} f_{k_j}$ be an arbitrary subsequential limit. By Proposition 5.4, f has a continuous extension to $\overline{\Delta}$ and realizes $\mathcal{L}$, namely $\mathcal{L} = \mathcal{L}_f$.

Notice that the exponential decay of the diameters of the balloons implies the Hölder continuity of f. The stronger John property of $\mathcal{G} = f(\Delta)$ follows from the same modulus estimate, applied to the characterization Theorem 2.4: Indeed, if $A \subset I \subset \mathbb{T}$ are arcs of length $|A| \leq \beta|I|$, then there is a point $x = \ell/2^m \in A$ and a scale $2^m \sim |A|$ such that the annular neighborhood $\mathcal{A}_m(x)$ surrounds A. By Lemma 5.5, there are disjoint nested annular neighborhoods $\mathcal{A}_{m-jC'}(x_j)$. If $\beta \lesssim 2^{-nC'}$, the interval I crosses all $\mathcal{A}_{m-jC'}(x), 1 \leq j \leq n$. Consequently, for every k, $f_k(I)$ crosses the annuli $\mathcal{A}_{k,m-jC'}(x_j)$. Since $f_k(A)$ is surrounded by these annuli, by Lemmas 2.6 and 2.7 we have that

$$\log(1 + \frac{\operatorname{diam} f(I)}{\operatorname{diam} f(A)}) \geq nM_0 - c$$

and we obtain $\operatorname{diam} f(A) \leq 1/2 \operatorname{diam} f(I)$ if β is sufficiently small. □

6 Proof strategy for Theorem 1.3

Fix a standard Brownian excursion $\mathbb{e} : [0,1] \to \mathbb{R}_+$. We would like to employ Theorem 1.1 and show that almost surely for every $x \in \mathbb{T}$ and every $n \geq 1$ there are $n/2$ nested annuli centered at x of scale $\geq 2^{-n}$ that satisfy the Conditions 1-3 of Section 3.2. By rotation invariance of the CRT, it suffices to consider $x = 0$ and show that the probability of not having $n/2$ good scales decays faster than 2^{-n}.

6.1 Decomposition of Brownian excursion

The annuli will be obtained from a decomposition of $\mathbb{e}$ into excursions away from H_j that reach height H_{j+1}, where essentially $H_j = \lambda^{j-n}$ for some fixed $\lambda > 1$ and $1 \leq j \leq n$. For

Figure 6.1: The decomposition of a Brownian excursion with respect to heights h_1 and h_2. Here, there are $k = 2$ excursions from height h_1 to height h_2, over the intervals U_1 and U_2. The lengths of the intervals in this decomposition are given by the a_i. The indexing is always chosen so that a_1 and a_2 are the lengths of the intervals on the end, then $a_3, \ldots, a_{k+1}$ are the lengths of the intervals in between the excursion intervals, and finally $a_{k+2}, \ldots, a_{2k+1}$ are the lengths of the excursion intervals themselves.

ease of notation, we fix j and write $h_1 = H_j, h_2 = H_{j+1}$ and so on in our description of the decomposition below, see Figure 6.1.

Let $\mathcal{X} = \{t : \mathbb{e}(t) = h_1\}$. Consider those connected components U of $[0,1]\backslash\mathcal{X}$ on which $\mathbb{e}$ is an excursion that reaches level h_2, $\mathbb{e}|_U \geq h_1$ and $\sup \mathbb{e}|_U \geq h_2$. Suppose there are k such components $U_1, ..., U_k$ (by continuity there are finitely many of these intervals). Then there are $k+1$ components $U_{k+1}, ..., U_{2k+1}$ of the complement $[0,1] \setminus \bigcup_j U_j$. Notice that

- Conditioned on the leftmost interval, the law of $\mathbb{e}$ on that interval is that of a Brownian meander conditioned on ending at h_1 and staying below h_2. Similarly, the conditional law of $\mathbb{e}$ on the rightmost interval is that of a time-reversal of such a meander.
- On the U_j with $1 \leq j \leq k$, the (conditional) law of $\mathbb{e} - h_1$ is that of an excursion that reaches height $h_2 - h_1$.
- On the remaining U_j, the (conditional) law of $\mathbb{e}$ is that of a Brownian bridge from h_1 to h_1, conditioned to stay between 0 and h_2.

If $k \geq 1$, the lengths of the intervals U_j can be viewed as a $(2k+1)-$ dimensional vector $\mathbf{a} = (a_1, \dots, a_{2k+1})$. Re-label the indices so that

- a_1 and a_2 denote the lengths of the left- and rightmost interval.
- $a_3, \dots, a_{k+1}$ denotes the lengths of the $k-1$ bridges, in left to right order.
- $a_{k+2}, \dots, a_{2k+1}$ denotes the lengths of the k excursions, in left to right order.

Thus for example $a_{k+2} = |U_1|$. It is not hard to write down an explicit expression for the density of the random variable $\mathbf{a}$, see Proposition 7.2 below for the analog in the discrete setting.

Here are some intuitive statements about $\mathbf{a}$. By Brownian scaling we may assume $h_1 = 1$ and $h_2 = \lambda h_1$ for some fixed $\lambda > 0$. Let $W > 0$ be the length of the excursion after this rescaling.

- If W is very small, then $k = 0$ with high probability.
- k has exponential tails, uniformly as $W \to \infty$.
- It is very unlikely for a Brownian bridge to stay in an interval of size h_1 over a time period much longer than h_1^2. It follows that it is very unlikely for the lengths $a_3, \dots, a_{k+1}$ to be much longer than h_1^2.
- For $W/h_1^2 \gg 1$, it is likely that most of the mass of the interval $[0, W]$ goes to a single a_i: For example, it is much more likely that there is an a_i with $a_i \approx W_i$ than it is to have a_i and a_j with $a_i \approx a_j \approx W/2$.

6.2 Constructing chains for the Brownian excursion

We now explain how the decomposition defined in the previous section can be used to construct chains of large modulus (see Section 3.2 for definitions).

Fix N large and for $l = 0, 1, 2, \dots$ consider the geometric sequence of scales $H_0 = 0$ and

$$H_{l+1} = H_l + \lambda^{-N} \lambda^l$$

so that

$$H_{l+2} - H_{l+1} = \lambda(H_{l+1} - H_l).$$

Let $\mathbb{e} : [0, 1] \to [0, \infty)$ be an excursion.

Fix a "scale" $h_0 = H_l, h_1 = H_{l+1}, h_2 = H_{l+2}$ and denote $H_{l+.5} = h_{1.5}$ the point in between h_1 and h_2 satisfying

$$\frac{h_2 - h_{1.5}}{h_{1.5} - h_1} = \Lambda$$

where Λ is a large parameter that is determined later. Fix an excursion interval $U_j \subset \mathbb{T}$, $1 \le j \le k$, so that by our definition $\inf \mathbb{e}|_{U_j} = h_1$ and $\sup \mathbb{e}|_{U_j} \ge h_2$. Let $\tau = \tau_j = \inf\{t \in U_j : \mathbb{e}|_{U_j}(t) = h_2\}$ and $\tilde{\tau} = \sup\{t \in U_j : \mathbb{e}|_{U_j}(t) = h_2\}$ be the first and last times respectively that $\mathbb{e}|_{U_j}$ visits h_2. Let $\tau^- = \sup\{t \in U_j : t < \tau, \mathbb{e}(t) = h_{1.5}\}$ be the last time that $\mathbb{e}|_{U_j}$ visits $h_{1.5}$ before visiting h_2. Let $\tau^+ = \inf\{t \in U_j : t > \tilde{\tau}, \mathbb{e}(t) = h_{1.5}\}$ be the first time that $\mathbb{e}|_{U_j}$ visits $h_{1.5}$ after visiting h_2 for the last time, see Figure 7.1 below.

The endpoints of U_j, θ^- and θ^+, are equivalent, so if τ^- and τ^+ are equivalent, the pair of intervals $\mathcal{C}^{(l,j)} := ([\theta^-, \tau^-], [\tau^+, \theta^+])$ form a chain link (Definition 3.8). Define the chain $\mathcal{C}^{(l)}$ as the sequence of chain links $\mathcal{C}^{(l,j)}$ for $j = 1, \dots, k$, see Figure 6.2.

The following conditions $\mathrm{Good}_1, \mathrm{Good}_2, ..., \mathrm{Good}_5$ guarantee the desired lower bound on $\mathrm{M}(\Gamma(\mathcal{C}^{(l)}))$. They all involve the parameter $L > 0$, where larger L corresponds to less restrictive conditions. See Section 7.3 for the detailed definitions on these conditions in the discrete setting.

We say that $S|_{U_j} \in \mathrm{good}_1(h_1, h_2)$ if the restriction of the jth excursion to $[\tau^-, \tau^+]$ does not dip below height $h_{1.5}$, so that τ^- and τ^+ are identified via $\sim$ (Figure 6.2), and we say that $S \in \mathrm{Good}_1(h_1, h_2)$ if $S|_{U_j} \in \mathrm{good}_1(h_1, h_2)$holds for all $1 \le j \le k_l$.

If Good_1 holds then by the discussion above, we get a well defined chain link $\mathcal{C}^{(l)}$. The remaining conditions $\mathrm{Good}_{2,3,4,5}$ ensure that this chain link satisfies the conditions of Theorem 3.9. We say that $S|_{U_j} \in \mathrm{good}_2(h_1, h_2)$ if the diameters of the two intervals in the chain link $C^{(l,j)}$ are comparable to h_1^{-2}, and we say that $S \in \mathrm{Good}_2$ if $S|_{U_j} \in \mathrm{Good}_1(h_1, h_2)$ for all $j = 1, \dots, k$.

We say that $S|_{U_j} \in \text{good}_3(h_1, h_2)$ if the excursion is Hölder on the chain link intervals of $C^{(j)}$. This yields the regularity of $\sim$ on the corresponding pair of chain link intervals via Lemma 3.5. We say that $S \in \text{Good}_3(h_1, h_2)$ if $S|_{U_j} \in \text{good}_3(h_1, h_2)$ for all $j = 1, \ldots, k$. The Good_4-condition is satisfied if the total length of the non-excursion intervals is comparable to h_1^{-2}. The Good_5-condition means that the degree $k = k_l$ of the chain from this construction is bounded.

By Theorem 3.9 (more precisely, by discrete approximation of the lamination as in Section 5.2 together with Theorem 3.10), every scale that satisfies the Good conditions gives rise to a good scale in Theorem 1.1. Thus the proof of Theorem 1.3 reduces to showing that the Good conditions hold at many scales l. If the scales were independent, it would suffice to estimate the probability that a given scale satisfies the Good conditions. Since the scales are not independent, we have to work a little harder. We analyze them via a discrete time Markov exploration process ω_l, where ω_l consists of the following information:

- The excursion intervals U_j of $\mathbb{e}$ from H_l to H_{l+1} (described in Section 6.1).
- The excursion intervals V_i from H_{l+1} to H_{l+2}
- The restriction of $\mathbb{e}$ to the U_j, modulo the restriction onto the V_i. In other words, we keep track of what happens on the U_j, but we 'forget' what happens on the V_i intervals.

See https://sites.math.washington.edu/~peterlin/excursion-exploration for a interactive demonstration of this exploration.

If $\mathbb{e}$ is distributed as a Brownian excursion, then it easy to see that $(l, \omega_l)_{l\geq 0}$ is a Markov chain. Some aspects of this Markov chain can be computed explicitly. For example, let $\mathbf{a}(\omega_l)$ denote the sequence of lengths of the excursion intervals V_i above. This is also a Markov chain and its transition probabilities can be computed explicitly. See the next section for the details in the discrete setting.

Each of the Good_i conditions can be identified with a certain subset of the state space of this Markov chain, and the following large deviation estimate (proved in Appendix A) can be applied.

Theorem 6.1. *Let ω_l be a Markov chain on state space Ω with transition densities $\pi(x, dy)$. Let $A \subset \Omega$ and suppose $u : \Omega \to [1, \infty)$ is a function with*

$$\lambda_u(x) = \log\left(\frac{u(x)}{\int u(y)\pi(x, dy)}\right) \geq 0,$$

Then for each $\epsilon > 0$,

$$\mathbb{P}\left(\frac{1}{n}\,\left|\{k : \omega_k \in A\}\right| \geq \epsilon\right) \leq \mathbb{E}u(\omega_1)\exp\left(-n\epsilon \inf_{\omega\in A} \lambda_u(\omega)\right). \tag{6.1}$$

Figure 6.2: Left: we have a Brownian excursion which has $k = 3$ excursions from $h_1 = \lambda^l h$ to $h_2 = \lambda^{l+1} h$. The rest of the excursion is irrelevant and not shown here. The conditions $\mathrm{Good}_1(l, j)$ are satisfied for $j = 1, 2, 3$, and the resulting chain links are highlighted on the x-axis.

It remains to construct a test function u such that (6.1) is satisfied and such that $\inf_{\omega \notin \text{Good}} \lambda_u(\omega)$ is large. For the sake of exposition, we first describe how to create a test function that gives us a bound for the Good_5 condition.

In our exploration process, each excursion interval splits into multiple excursion intervals, independently of the other excursion intervals. Generically, there will be one large excursion interval which is sustained from level to level, and occasionally this excursion interval will have a few child intervals of short length. These shorter excursion intervals will tend to not have too many children (see Proposition 7.6), and so it is plausible that at most levels, the Good_5 condition is satisfied. To understand the choice of test function, it is helpful to use subcritical Galton-Watson branching with immigration as a simplified toy model of the process. Let $Z_0, Z_1, Z_2, \dots$ where $Z_0 = 0$ and

$$Z_{n+1} = 1 + \sum_{i=1}^{Z_n} \Xi_i,$$

and the Ξ_i are i.i.d. random variables of mean strictly less than 1, supported on the non-negative integers. The immigrant plays the role of the large excursion interval. In this case, the test function $u(Z_n) := \zeta^{Z_n}$ for some appropriately chosen constant $\zeta > 1$ can be used in Theorem 6.1 to get large deviations upper bounds on the density of generations for which Z_n is large. Indeed, the fact that each node has its children independently allows the exponent in the right hand side of (6.1) to be bounded. The same technique works for subcritical multitype Galton-Watson processes, where now the test function has to take into account the different types: $u(Z_n) = \prod_{i=1}^{Z_n} \zeta_{\text{Type}(i)}$ where ζ is now a real valued function of types. For the exploration process of excursion intervals, the 'type' of the excursion interval is the (scaled) length $\beta > 0$ of the excursion interval, and it turns out (c.f. Lemma 7.9) that $\zeta(\beta) = 2 + \beta^{1/4}$ works.

That is, if $\lambda > 1$ is sufficiently large, then it can be shown that the following test function $u = V$ satisfies the hypotheses of Theorem 6.1 when $A = \{\omega : \omega \text{ has more than } L \text{ excursion intervals}\}$ is the complement of the Good_5 states.

$$V(\omega, l) = \prod_{i=k+2}^{2k+1} (2 + (a_i g_l^{-2})^{1/4}),$$

where $g_l = H_{l+2} - H_{l+1}$ and the a_i are the lengths of the excursion intervals from H_{l+1} to H_{l+2}.

Now we write down the more complicated test function that we will actually use, to get large deviations for all the $\text{Good}_{1,2,3,4,5}$ conditions. Calculation shows that for suitable parameters $q, \lambda > 1$ large, and $s > 1$, $W_0 > 0$ small, the following function has the desired properties:

$$V(\omega,l) = s^{g_l^{-2}-\sum_{i=1}^{k_l}\alpha_i}\prod_{i=1}^{k_l} q^{\mathbb{1}(\alpha_i\le W_0)}(2+\alpha_i^{1/4})^{1/2}\prod_{j=1}^{k_{l-1}} q^{\mathbb{1}(S|_{U_j}\notin \mathrm{good}_{1,2,3}(H_l,H_{l+1})}. \tag{6.2}$$

where for $i = 1,\dots,k$, the $\alpha_i = \mathbf{a}(\omega_l)_{i+k+1}g_l^{-2}$ are the scaled lengths of the excursions intervals from H_{l+1} to H_{l+2}.

This definition depends on several different constants, some of which have already been introduced. We summarize them here for the reader's convenience. All these constants except for s and W_0 will be taken to be 'large'.

1. $q > 1$ is a penalty factor for violating the $\mathrm{good}_{1,2,3}$ condition and also a penalty for any excursion intervals that are too short. It will turn out that we need to take $q \asymp \lambda^{20}$.
2. $L > 1$ is a parameter that determines how restrictive the $\mathrm{Good}_{1,2,3,4,5}$ conditions are.
3. $W_0 > 0$ is a parameter that determines what constitutes a 'short' excursion interval. We need to penalize short excursion intervals so that we can ensure the $\mathrm{good}_{1,2,3}$ conditions are satisfied (see the hypotheses of Proposition 7.7).
4. $s > 1$ is a penalty factor for violating the Good_4 condition.
5. $\lambda \ge 2$ is the step size for the Markov exploration process.
6. $\Lambda \ge 2$ determines the relative distances between $H_l, H_{l+0.5}$, and H_{l+1}. Changing this parameter affects the definition of the $\mathrm{good}_{1,2,3}$ conditions.

In (6.2), the factor $s^{g_l^{-2}-\sum_{i=1}^{k_l}\alpha_i}$ penalizes states for which much of the interval $[0,1]$ is taken up by non-excursion-intervals. Whenever the Good_4 condition is violated, this factor is large. However, this tends to decrease under iteration of the Markov chain due to the extra factor of λ^{-2} from rescaling.

Using the explicit equations for the transition probabilities of the Markov chain, it can be shown that the test function (6.2) has the desired properties. We will not present the proof here. Instead, we will prove the analogous result (Lemma 7.8) for the discrete approximations to the Brownian excursion.

7 Proof of Theorem 1.4

In this section we present the details of the proof the proof of Theorem 1.4, following the strategy of the proof of Theorem 1.3 outlined in the previous section. As Theorem 1.4

implies Theorem 1.3, this also concludes a detailed proof of Theorem 1.3. Before adopting the decomposition described in Section 6.1 to the setting of random walks, we collect some notation and terminology.

7.1 Notation and terminology

A *(Bernoulli) walk* of length n is a map $S : \{0, \ldots, n\} \to \mathbb{Z}$ such that $S_{i+1} - S_i \in \{-1, 1\}$ for $i \geq 1$. For the rest of this paper we will assume that Bernoulli walks are defined on the whole interval $[0, n]$ by requiring that the walk is linear of slope 1 in between the integer points.

For $a \in \mathbb{Z}$ denote $\mathbf{W}_n(a)$ denote the collection of walks of length n with $S_0 = a$. Let $\mathbf{W}_n(a \to b) \subset \mathbf{W}_n(a)$ denote the set of walks S with $S_n = b$, so that $|\mathbf{W}_n(a)| = 2^n$ and

$$|\mathbf{W}_n(a \to b)| = \binom{n}{n/2 - (b-a)/2}. \tag{7.1}$$

For this formula to be true when n is odd, we abide by the convention that binomial coefficients with noninteger arguments are equal to zero.

Let $\mathbf{E}_n(a) \subset \mathbf{W}_n(a \to a)$ denote the collection of *excursions away from a of length n*, namely walks with $S_0 = S_n = a$, and $S_i \geq a$ for all i. Note that $\mathbf{E}_n(0)$ is the collection of Dyck paths of length n, and recall that $|\mathbf{E}_n(0)|$ is given by the Catalan number $\frac{1}{n/2+1}\binom{n}{n/2}$ (this can be deduced by taking $a = n+1$ and $g = 1$ in Corollary B.2).

Fix an excursion an excursion S from 0. As in the previous section, we will consider the excursions of S away from h_1 that exceed level h_2, where $0 = h_0 < h_1 < h_2$. As before, this naturally leads us to consider the partition of $[0, n]$ into disjoint (except for their endpoints) closed intervals, where the restriction of S onto each part corresponds to one of the following three types:

- For $n \geq 2$, let $\mathbf{W}_n^{\uparrow}(a \to b) \subset \mathbf{W}_n(a \to b)$ denote the walks which 'approach a and b from below', that is

  $$\mathbf{W}_n^{\uparrow}(a \to b) = \{S \in \mathbf{W}_n(a \to b) : S_1 = a - 1 \text{ and } S_{n-1} = b - 1\}.$$

  This definition is needed to guarantee uniqueness of the decomposition, Proposition 7.2.

  Note the natural bijection $\mathbf{W}_n^{\uparrow}(a \to b) \cong \mathbf{W}_{n-2}(a-1 \to b-1)$ which together with (7.1) yields $|\mathbf{W}_n^{\uparrow}(a \to b)| = \binom{n-2}{n/2-1-(b-a)/2}$. For $c < d$, let $\mathbf{W}_n^{\uparrow}(a \to b; c \leq \min < \max \leq d)$ be the set of walks S in $\mathbf{W}_n^{\uparrow}(a \to b)$ for which $c \leq S \leq d$.

- Let $\mathbf{Z}_n(a \uparrow b) \subset \mathbf{W}_n(a \to b)$ denote the walks that stay above the left endpoint a and 'approach the right from below', that is

$$\mathbf{Z}_n(a \uparrow b) = \{S \in \mathbf{W}_n(a \to b) : S \geq a \text{ and } S_{n-1} = b - 1\}.$$

  Similarly, let $\mathbf{Z}_n(a \downarrow b) \subset \mathbf{W}_n(a \to b)$ denote the walks that stay above the right endpoint and 'approach the left from below', that is

$$\mathbf{Z}_n(a \downarrow b) = \{S \in \mathbf{W}_n(a \to b) : S \geq b \text{ and } S_1 = a - 1\}.$$

  Notice that there is a natural bijection $\mathbf{Z}_n(a \uparrow b) \cong \mathbf{Z}_n(b \downarrow a)$ by time reversal. Corollary B.2 shows that $|\mathbf{Z}_w(a \uparrow b)| = \frac{b-a}{w}\binom{w}{\frac{w}{2}+\frac{b-a}{2}}$.

- For $b > a$, let $\mathbf{E}_n(a; \max \geq b)$ be the collection of excursions in $\mathbf{E}_n(a)$ with maximum greater than or equal to b.

We will often need to consider the uniform probability measure on these spaces of walks. We will use a subscript to denote the probability measure in question, and the variable S to denote the random variable; for instance $\mathbb{P}_{\mathbf{W}_n(a\to b)}(S \in \cdot)$ denotes the uniform probability distribution on $\mathbf{W}_n(a \to b)$. Thus for $A \subset \mathbf{W}_n(a \to b)$ we have $\mathbb{P}_{\mathbf{W}_n(a\to b)}(S \in A) = \frac{|A|}{|\mathbf{W}_n(a\to b)|}$.

In what follows, we will frequently deal with walks and excursions that are defined on intervals $I = [u, v]$ instead of on $[0, n]$. Therefore it will be convenient to use the notation $\mathbf{W}_I, \mathbf{E}_I$, and so on, with the obvious meaning. If there is no subscript, the union over all intervals (with integer endpoints) is taken. For example, $\mathbf{W}(a) = \bigcup_I \mathbf{W}_I(a)$.

7.2 Excursion decomposition of Dyck paths

Fix integers $h_2 > h_1 > h_0 = 0$ and suppose $S \in \mathbf{E}_n(0, \max \geq h_2)$ is an excursion that reaches level h_2. Let $k = k(S) \geq 1$ be the number of excursions away from h_1 that reach level h_2. Then we can decompose S into a concatenation of walks

$$S = Z_1 E_1 B_1 E_2 B_2 \cdots B_{k-1} E_k Z_2$$

where

- $Z_1 \in \mathbf{Z}(0 \uparrow h_1, \max < h_2)$ and $Z_2 \in \mathbf{Z}(h_1 \downarrow 0, \max < h_2)$
- For $i = 1, \ldots, k-1$, $B_i \in \mathbf{W}^{\uparrow}(h_1 \to h_1, 0 \leq \min < \max < h_2)$.
- For $i = 1, \ldots, k$, $E_i \in \mathbf{E}(h_1, \max \geq h_2)$.

Definition 7.1. We will refer to the walks in the decomposition as $[0 \uparrow h_1 \uparrow h_2]$-ends, $[0 \uparrow h_1 \uparrow h_2]$-bridges and $[0 \uparrow h_1 \uparrow h_2]$-excursions (which we often abbreviate as $[h_1 \uparrow h_2]$-excursions), respectively, of S. We will also call the intervals in this decomposition the $[0 \uparrow h_1 \uparrow h_2]$-end intervals of S and so on. We denote $\mathbf{a}(S) = \mathbf{a}_{[0\uparrow h_1 \uparrow h_2]}(S) = (a_1, \ldots, a_{2k+1}) \in \mathbb{Z}_{\geq 0}^{2k+1}$ the vector of lengths of the intervals in this decomposition, and will always choose the indexing of the a_i as in Section 6.1:

- a_1 and a_2 are the lengths of the $[0 \uparrow h_1 \uparrow h_2]$-end intervals.
- $a_3, \ldots, a_{k+1}$ are the lengths of the $[0 \uparrow h_1 \uparrow h_2]$-bridge intervals, in left to right order.
- $a_{k+2}, \ldots, a_{2k+1}$ are the lengths of the $[0 \uparrow h_1 \uparrow h_2]$-excursion intervals, in left to right order.

Similarly, we define the $[H_1 \uparrow H_2 \uparrow H_3]$-decomposition of a walk S by translation as the $[0 \uparrow H_2 - H_1 \uparrow H_3 - H_1]$-decomposition of $S - H_1$, and we often abbreviate $[H_1 \uparrow H_2 \uparrow H_3]$-excursion intervals to $[H_2 \uparrow H_3]$-excursion intervals.

The indexing of the a_i above is consistent with the notation in the following simple consequence of the uniqueness of the above decomposition:

Proposition 7.2. *Fix integers $h_2 > h_1 > a$ and $n \geq 2$. There is a bijection*

$$\mathbf{E}_n(0) \cong \mathbf{E}_n(0, \max < h_2) \sqcup \tag{7.2}$$

$$\sqcup \bigsqcup_{k=1}^{\infty} \quad \bigsqcup_{a_1+\cdots+a_{2k+1}=n} \quad \mathbf{Z}_{a_1}(0 \uparrow h_1, \max < h_2) \times \mathbf{Z}_{a_2}(h_1 \downarrow 0, \max < h_2) \times$$

$$\times \prod_{i=3}^{k+1} \mathbf{W}^{\uparrow}_{a_i}(h_1 \to h_1, 0 \leq \min < \max < h_2) \times \prod_{i=k+2}^{2k+1} \mathbf{E}_{a_i}(h_1, \max \geq h_2),$$

where the second disjoint union is taken over positive integers $a_i \geq 0$.

7.3 Chains and conditions for large modulus

In this section we show how the decomposition introduced in the previous Section 7.2 naturally leads to annuli. We then identify several conditions that the decomposition at a given level must satisfy for the corresponding annulus to have good modulus. In the subsequent sections we will show that these conditions are satisfied at many scales.

Fix $0 < h_1 < h_2$ integer. Let $S \in \mathbf{E}_n(0, \max \geq h_2)$ be an excursion that reaches height h_2. Let $U_1, \ldots, U_k$ be the $[0 \uparrow h_1 \uparrow h_2]$-excursion intervals of S so that all $S|_{U_j} \in \mathbf{E}(h_1, \max \geq h_2)$.

We now describe these various conditions as subsets of $\mathbf{E}(h_1)$, denoted by $\mathrm{Good}_i(h_1, h_2)$ where $1 \le i \le 5$. They involve a parameter $L > 1$ where larger choices of L make the conditions less restrictive. In what follows, let $\Lambda > 1$ be an integer and let

$$h_{1.5} = h_1 + \lfloor (h_2 - h_1)/\Lambda \rfloor.$$

The first three conditions are regularity conditions that each of the $[h_1 \uparrow h_2]$-excursions have to satisfy individually, $S \in \mathrm{Good}_{1,2,3}(h_1, h_2)$ if and only if for all j, $S|_{U_j} - h_1 \in \mathrm{good}_{1,2,3}(h_{1.5} - h_1, h_2 - h_1)$.

The $\mathrm{good}_{1,2,3}(g', g)$ condition on excursions T from 0 that exceed g are defined below in terms of their $[0 \uparrow g' \uparrow g]$-decomposition.

The first condition $T \in \mathrm{good}_1(g', g)$ is that T has only one $[g' \uparrow g]$-excursion. That is, T only makes a single excursion away from g' that reaches g. If this condition holds then we define a chain link (recall Definition 3.8) as the pair of left and right $[g' \uparrow g]$-end intervals $J_j^- = [l^-, r^-]$ and $J_j^+ = [l^+, r^+]$ of T. Notice that $S|_{U_j} - h_1 \in \mathrm{good}_1(h_{1.5} - h_1, h_2 - h_1)$ implies $r^- \sim l^+$, while $l^- \sim r^+$ always holds. See Figure 7.1.

If $S \in \mathrm{Good}_1(h_1, h_2)$, then the corresponding collection of chain links $\{(J_j^-, J_j^+)\}_{i=1}^k$ forms a chain of degree k around 0. We call this the (h_1, h_2)-chain associated to S.

Next, we say that $T \in \mathrm{good}_2(g', g)$ if $a_i g'^{-2} \in [L^{-1}, L]$ for $i = 1, 2$, where a_1, a_2 are the lengths of the $[0 \uparrow g' \uparrow g]$-end intervals of T. This condition controls the diameters of the chain link associated to T.

We say that $T \in \mathrm{good}_3(g', g)$ if $\mathbb{e}_{Z_1}$ and $\mathbb{e}_{Z_2}$ are $(L, 1/3)$-Hölder continuous on $[0, 1]$. Here

$$\mathbb{e}_{Z_i}(t) := a_i^{-1/2} Z_i(a_i t) \tag{7.3}$$

are the Brownian rescalings of the $[0 \uparrow g' \uparrow g]$-end intervals of T. This condition gives control over the regularity of the welding on the chain link associated to T.

The remaining conditions depends on all the excursion intervals $(S|_{U_j})_{j=1}^k$ at a given scale, simultaneously.

We say that $\mathrm{Good}_4(h_1, h_2)$ holds for S if $n - w_1 - \cdots - w_k \le L h_1^2$, where $w_1, \ldots, w_k$ are the lengths of the $[h_1 \uparrow h_2]$-excursion intervals, and $\mathrm{Good}_5(h_1, h_2)$ holds if the number k of $[h_1 \uparrow h_2]$-excursion intervals of S is less than L. Finally we say that $\mathrm{Good}_{\tilde{5}}(h_1, h_2)$ holds if $\mathrm{Good}_5(h_1, h_2)$ holds and there is at least one $[h_1 \uparrow h_2]$-excursion interval.

This gives a bound on the degree on the (h_1, h_2)-chain and the sum (and hence maximum) of the gaps between the chain links.

Figure 7.1: Definition of the $\text{good}_1(h_{1.5}-h_1, h_2-h_1)$ condition. We have drawn the excursion S over its $j-$th excursion interval U_j. τ and $\tilde{\tau}$ are the first and last hitting times in U_j of height h_2. τ^- is the last hitting time of g' in U_j before hitting h_2, and τ^+ is the first hitting time of g' after $\tilde{\tau}$. We say that $\text{good}_1(h_{1.5}-h_1, h_2-h_1)$ holds for $S|_{U_j} - h_1$ if the portion of the excursion between τ and $\tilde{\tau}$ does not dip below height $h_{1.5}$, so that τ^- and τ^+ are identified via $\sim$. If this holds, then the pair $([\theta^-, \tau^-], [\tau^+, \theta^+])$ is a chain link as defined in Section 3.2.

Definition 7.3. Fix $L, \Lambda > 1$ and $0 < h_1 < h_2$ integer, with $h_2 - h_1 \geq \Lambda$. Let S be an excursion in $\mathbf{E}_n(0)$ and let U_j be the $[0 \uparrow h_1 \uparrow h_2]$-excursion intervals of S. We say that S belongs to $\mathrm{Good}(h_1, h_2)$ if $S|_{U_j} \in \mathrm{Good}_i(h_1, h_2)$ for each $1 \leq j \leq k$ and $1 \leq i \leq 3$, and if $\mathrm{Good}_4(h_1, h_2)$ and $\mathrm{Good}_{\tilde{5}}(h_1, h_2)$ holds.

Proposition 7.4. *Suppose $h_2 \geq 2h_1$. If an excursion S belongs to* $\mathrm{Good}(h_1, h_2)$*, then the curve family $\Gamma(\mathcal{C})$ of the (h_1, h_2)-chain $\mathcal{C}$ associated to S satisfies*

$$\mathrm{M}\left(\Gamma(\mathcal{C})\right) \geq \delta_0$$

where $\delta_0 > 0$ depends only on L and Λ.

Proof. We would like to apply Theorem 3.9 and need to verify Conditions 1-3.

First, Good_2 implies $|J_j^+| \asymp_{L^2} |J_{j+1}^-|$ and Good_4 implies $|l_{j+1}^- - r_j^+| \lesssim_{L,\Lambda} |J_j^+|$ so that (3.6) and therefore Condition 1 holds.

Second, Condition 2 follows from Good_3 and (the proof of) Lemma 3.5.

And third, the existence of the chain itself and the boundedness of the degree, Condition 3, is the same as $\mathrm{Good}_{\tilde{5}}$. $\square$

7.4 The main estimate and setup: Positive density of good scales

We now formulate the main estimate for the probability that a fixed percentage of scales are good. Fix $\lambda, \Lambda \geq 2$ integer and and consider the sequence of scales $H_0 = 0$ and $H_{l+1} = H_l + \lambda^l$. Let $H_{l+0.5} = H_l + \lfloor \frac{H_{l+1} - H_l}{\Lambda} \rfloor$, for sufficiently large l this will be strictly between H_l and H_{l+1}.

Consider an excursion S of length n, fix $0 < r < 1$ small and consider the ball of radius r centered at the root in the tree metric $d = d_{\mathrm{graph}}/n^{1/2}$. We wish to show that it can be separated from a circle of fixed radius by $\gtrsim \log \frac{1}{r}$ annuli of modulus $\gtrsim 1$ with probability $1 - O(r^{T_0})$, where any $T_0 > 2$ suffices for our purpose. More precisely, define ρ such that $H_{\rho-1} < r\sqrt{n} \leq H_\rho$ and define $N \geq \rho$ such that $H_{N-1} < r^{1/2}\sqrt{n} \leq H_N$. Notice that

$$\frac{1}{2} \log_\lambda \frac{1}{r} - 2 \leq N - \rho < \frac{1}{2} \log_\lambda \frac{1}{r} + 2. \tag{7.4}$$

Then many of the associated (H_l, H_{l+1})-chains satisfy the Good conditions and therefore the assumption of Proposition 7.4 :

Proposition 7.5. *There are integers $\lambda, \Lambda, L > 1$ and $r_0 > 0$ such that for all n for which $r \leq r_0$, we have*

$$\mathbb{P}\left(\frac{|\{l = \rho, \dots, N : S \in \mathrm{Good}_{1,2,3,4,\tilde{5}}(H_l, H_{l+1})\}| + 1}{N - \rho} < \frac{1}{2}\right) \leq C_\lambda r^{2.25},$$

where S is a uniformly random excursion in $\mathbf{E}_n(0)$ and the constant C_λ only depends on λ.

Proposition 7.5 follows from a large deviations estimate applied to a Markov chain ω_l and a suitable test function V that we will define in the next section. In the remainder of this section, we prove that each individual $[H_l \uparrow H_{l+1}]-$excursion satisfies the $\mathrm{good}_{1,2,3}$ conditions with probability arbitrarily close to 1 if the parameters are chosen appropriately. We begin with a geometric upper bound on the number of large excursions inside a given excursion. It immediately implies that, at a fixed scale, the Good_5 condition holds with high probability, and later also provides us with control over the Markov chain exploration.

Lemma 7.6. *Let $g > 1$ and $\lambda > 1$ be integers. Let S be a uniformly random excursion of type $\mathbf{E}_w(0, \max \geq g)$. Let k be the number of $[g \uparrow \lambda g]$-excursions of S. There exists $p_{\lambda, wg^{-2}}, \tilde{p}_{\lambda, wg^{-2}} < 1$ such that*

$$\mathbb{P}(k \geq m) \leq \tilde{p}_\lambda^{m-1} p_{\lambda, wg^{-2}}$$

for $m \geq 1$. Moreover, we can choose $p_{\lambda, wg^{-2}}$ and $\tilde{p}_\lambda$ in such a way that

1. $p_{\lambda, wg^{-2}} \lesssim \exp\left(-c_0 \frac{(\lambda-1)^2 g^2}{w}\right)$
2. $\tilde{p}_{\lambda, wg^{-2}} \lesssim \exp\left(-c_0 \frac{(\lambda-1)^2 g^2}{w}\right)$
3. $\tilde{p}_{\lambda, wg^{-2}} \to 0$ *uniformly in wg^{-2} as $\lambda \to \infty$.*

Here c_0 is a universal constant and the first two statements are primarily useful when wg^{-2} is bounded.

Proof. Let τ^- and τ^+ be the first and last times respectively that S is at level g. Conditioned on τ^-, τ^+, the walk $S|_{[\tau^-, \tau^+]}$ is, up to translation of the domain, a uniform walk of type $\mathbf{W}_T(g \to g, \min \geq 0)$, where $T = \tau^+ - \tau -$. We have $k \geq 1$ if and only if this latter walk reaches level λg. By Lemma B.7, this probability is bounded by a quantity $p_{\lambda, wg^{-2}}$ which has the desired properties.

This proves the statement of the lemma for $m = 1$. For the general case, it suffices to prove the bound $\mathbb{P}(k \geq m+1 | k \geq m) \leq \tilde{p}_\lambda$ for $m \geq 1$ and use induction. Suppose S is conditioned on $k \geq m$. Let $U \subset [0, T]$ be the mth excursion interval, and let τ^- be the first time that S hits λg in U. Let τ^+ be the last time in $[0, T]$ that S hits λg. Conditioned on τ^-, τ^+, the walk $S|_{\tau^-, \tau^+}$ is (up to translation of the domain) a uniform walk of type $\mathbf{W}_T(\lambda g \to \lambda g, \min \geq 0)$

where $T = \tau^+ - \tau^-$. We have $k \geq m+1$ if and only if this latter walk hits level g. Thus item 2) follows from Lemma B.7, and item 3) follows from Lemma B.6. □

Now we are ready to estimate the probability of the $\mathrm{good}_{1,2,3}-$ conditions of a single excursion at a fixed level. Let $g, L, \Lambda > 0$ be integers and let $g' = \lfloor g/\Lambda \rfloor$.

Proposition 7.7. *For any $W_0 > 0, \epsilon > 0$, there exists $\Lambda_0 > 0$ such that the probability that a uniformly random excursion from 0 of length $w > W_0 g^2$ satisfies the good conditions $\mathrm{good}_{1,2,3}(g', g)$ is bounded below by $1-\epsilon$ when $\Lambda \geq \Lambda_0$ and $L \geq L_0(\Lambda)$.*

Proof. Fix $\epsilon > 0$ and note that Lemma 7.6 implies $\mathbb{P}(S \notin \mathrm{Good}_1(g', g)) \leq \epsilon$ for sufficiently large Λ. Next, recall the condition $\mathrm{Good}_2(g', g)$, which says that $a_i g'^{-2} \in [L^{-1}, L]$, where a_1, a_2 are the lengths of the $[g' \uparrow g]$-end intervals respectively. First we bound the probability p that $a_1 g'^{-2} \notin [L^{-1}, L]$. Notice that if $S \in \mathbf{E}_w(0, \max \geq g)$, the part of S after its first $[g' \uparrow g]$-end interval may be decomposed uniquely into the concatenation of a walk of type $\mathbf{E}(g', \max \geq g)$ and a walk of type $\mathbf{Z}(g' \downarrow 0)$. So, by Corollary B.2 and Proposition B.5,

$$\begin{aligned}
p &= \frac{1}{\mathbf{E}_w(0, \max \geq g)} \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1}, L]}} |\mathbf{Z}_{a_1}(0 \uparrow g', \max < g)| \cdot |\mathbf{E}_b(g', \max \geq g)| \cdot |\mathbf{Z}_c(g' \downarrow 0)| \\
&\leq \frac{1}{C_{\mathrm{stir}} w^{-3/2} c_{W_0}} \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1}, L]}} |\mathbf{Z}_{a_1}(0 \uparrow g', \max < g)| 2^{-a_1} \\
&\qquad \cdot |\mathbf{E}_b(0, \max \geq (\Lambda - 1) g')| 2^{-b} \cdot |\mathbf{Z}_c(g' \downarrow 0)| 2^{-c}.
\end{aligned}$$

Using the estimates from Proposition B.5, Lemma B.9, and Corollary B.2, we get

$$p \lesssim_{W_0} w^{3/2} \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1}, L]}} g' a_1^{-3/2} e^{-\frac{g'^2}{3a_1}} e^{-c_0 \frac{a_1}{g^2}} \cdot b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}}$$

for some universal constant c_0. Now if $a_1 + b + c = w$ then either $a_1 \geq w/3$, or $b \geq w/3$, or $c \geq w/3$, so the sum above can be bounded by splitting the region of summation over those three regions: we have

$$p \lesssim w^{3/2} (I_a + I_b + I_c) \tag{7.5}$$

where, for fixed $\epsilon > 0$ and sufficiently large L,

$$
\begin{aligned}
I_a &= \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1},L] \\ a_1 \geq w/3}} g' a_1^{-3/2} e^{-\frac{g'^2}{3a_1}} e^{-c_0 \frac{a_1}{g^2}} \cdot b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}} \\
&\leq \sup_{\substack{a_1 \geq w/3 \\ a_1 g'^{-2} \notin [L^{-1},L]}} a_1^{-3/2} e^{-\frac{g'^2}{3a_1}} e^{-c_0 \frac{a_1}{g^2}} \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1},L] \\ a_1 \geq w/3}} g' \cdot b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}} \\
&\leq (w/3)^{-3/2} \epsilon \sum_{b,c \leq w} g' \cdot b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}} \\
&\leq (w/3)^{-3/2} \epsilon \cdot \int_1^\infty x^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2}{x}} dx \cdot \int_1^\infty x^{-3/2} e^{-\frac{1}{3x}} dx \\
&\leq (w/3)^{-3/2} \cdot \epsilon \cdot C_0.
\end{aligned}
$$

Similarly, for fixed $\epsilon > 0$ and fixed $\Lambda > 1$, and sufficiently large L ,

$$
\begin{aligned}
I_b &= \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1},L] \\ b \geq w/3}} g' a_1^{-3/2} e^{-\frac{g'^2}{3a_1}} e^{-c_0 \frac{a_1}{g^2}} \cdot b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}} \\
&\leq \sup_{b \geq w/3} b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \sum_{\substack{a_1,c \leq w \\ a_1 g'^{-2} \notin [L^{-1},L]}} g' a_1^{-3/2} e^{-\frac{g'^2}{3a_1}} e^{-c_0 \frac{a_1}{g^2}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}} \\
&\leq (w/3)^{-3/2} \cdot \int_{[1,\infty) \setminus [L^{-1},L]} x^{-3/2} e^{-\frac{1}{2x}} e^{-c_0 x \cdot \frac{g'^2}{g^2}} dx \cdot \int_1^\infty x^{-3/2} e^{-\frac{1}{2x}} dx \\
&\leq (w/3)^{-3/2} \cdot \int_{[1,\infty) \setminus [L^{-1},L]} x^{-3/2} e^{-\frac{1}{2x}} e^{-c_0 x \cdot \frac{1}{2\Lambda}} dx \cdot \int_1^\infty x^{-3/2} e^{-\frac{1}{2x}} dx \\
&\leq (w/3)^{-3/2} \cdot \epsilon \cdot C_0.
\end{aligned}
$$

A similar argument gives $I_c \leq (w/3)^{-3/2} \cdot \epsilon \cdot C_0$. Using these estimates in (7.5) gives, for fixed $\epsilon > 0$ and fixed $\Lambda > 1$, and sufficiently large L, $p \leq \epsilon$. By the union bound, the probability that $\mathrm{Good}_2(g', g)$ does not hold is bounded by $2p \leq 2\epsilon$.

Finally, we have from Lemma B.4 that $\mathbb{P}(S \in \mathrm{Good}_3 | S \in \mathrm{Good}_2) \geq 1 - \epsilon$ for sufficiently large L. Hence $\mathbb{P}(S \in \mathrm{Good}_2 \cap \mathrm{Good}_3) \geq (1 - C\epsilon) \cdot (1 - \epsilon)$. By the union bound, we get (for fixed ϵ, for sufficiently large Λ and L),

$$
\mathbb{P}(S \in \mathrm{Bad}) \leq \mathbb{P}(S \notin \mathrm{Good}_1) + \mathbb{P}(S \notin \mathrm{Good}_2 \cap \mathrm{Good}_3) \lesssim \epsilon
$$

and the proposition follows. □

7.5 The Markov chain exploration

The key observation is that a uniformly random $S \in \mathbf{E}_n(0)$ may be explored via a Markov chain on a state space Ω consisting of finite tuples of *quotient excursions*. These are equivalence classes of walks defined via the following equivalence relation on excursions $\mathbf{E}_w(H_l, \max \geq H_{l+1})$: Declare two such excursions S, S' equivalent if they have the same $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals and they are equal on the complement of these excursion intervals. In particular, if S, S' do not reach height H_{l+2} then they are equivalent if and only if they are equal. Denote the equivalence class of S by $[S]$.

Recall the excursion decomposition of Section 7.2 and in particular Definition 7.1. Equivalence classes have well defined $[H_l \uparrow H_{l+1} \uparrow H_{l+2}]$-ends and -bridges, and well defined $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals. In particular, the conditions $\mathrm{Good}_{1,2,3}(H_l, H_{l+1})$ are well defined on quotient excursions.

If $S \in \mathbf{E}_n(0)$ and if $U_1, \dots, U_k$ are the $[H_l \uparrow H_{l+1}]$-excursion-intervals of S, then set

$$\omega_l := ([S|_{U_1}], \dots, [S|_{U_k}])$$

so that $(\omega_l)_{l\geq 1}$ is a Markov chain. To get ω_{l+1} from ω_l,

- Let $V_1, \dots, V_m$ be the $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals of ω_l (this is the collection of $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals over the k quotient excursions in ω_l).
- Independently sample uniformly random excursions in $\mathbf{E}_{|V_j|}(H_{l+1}, \max \geq H_{l+2})$.
- Take equivalence classes.

In particular, this shows that the distribution of ω_{l+1} given ω_l is entirely determined by the lengths of the $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals of ω_l. The transition probabilities of this Markov chain can therefore be deduced from (7.2).

We will use the notation $\mathbf{b} = \mathbf{b}(\omega_l)$ for the vector of the lengths of *all* the $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals, and denote $k_l = k(\omega_l)$ the total number of these intervals. Note that ω_l consists of k_{l-1} quotient excursions. We will also write

$$g_l = g(\omega_l) = H_{l+2} - H_{l+1}$$

and

$$\mathrm{Gap}(\omega_l) = n - \sum \beta_i$$

where the sum is over the components β_i of $\mathbf{b}(\omega_l)$. Note that $\mathrm{Gap}(\omega_{l-1})$ can also be determined from ω_l, because ω_l contains the information about the $[H_l \uparrow H_{l+1}]$-decomposition. Now we define the test function for the large deviations estimate. Define $V : \Omega \to \mathbb{R}^+$ by

(7.6)
$$V(\omega_l) = s^{\mathrm{Gap}(\omega_l) g_l^{-2}} \prod_{i=1}^{k_l} q^{\mathbb{1}(\beta_i g_l^{-2} \le W_0)} (2 + (\beta_i g_l^{-2})^{1/4})^{1/2} \prod_{j=1}^{k_{l-1}} q^{\mathbb{1}(S|_{U_j} - H_l \notin \mathrm{good}_{1,2,3}(H_{l+0.5} - H_l, H_{l+1} - H_l))}.$$

For the rest of the paper, we will abbreviate this last term to $q^{\mathbb{1}(S|_{U_j} \notin \mathrm{good}_{1,2,3})}$. See the end of Section 6 for some heuristic remarks about the function V.

It will be useful to write the test function in the form

$$V(\omega_l) = s^{\mathrm{Gap}(\omega_{l-1}) g_l^{-2}} \prod_{i=1}^{k_{l-1}} v_{g_{l-1} \uparrow (\lambda+1) g_{l-1}} (S|_{U_i} - H_l)^{1/2} q^{\mathbb{1}(S|_{U_i} \notin \mathrm{good}_{1,2,3})}. \tag{7.7}$$

where $v_{h_1 \uparrow h_2} : \mathbf{E}(0, \max \ge h_1) \to \mathbb{R}^+$ is defined by

$$v_{h_1 \uparrow h_2}(S) = s^{2(a_1 + \cdots + a_{k+1}) g^{-2}} \prod_{j=k+2}^{2k+1} (2 + (a_j g^{-2})^{1/4}) q^{2 \mathbb{1}(a_j g^{-2} \le W_0)}. \tag{7.8}$$

Here $a_1, \ldots, a_{2k+1}$ is the vector of lengths in the $[0 \uparrow h_1 \uparrow h_2]$-decomposition of S, and $g = h_2 - h_1$. Note that $v_{h_1 \uparrow h_2}$ is well defined on the quotient space of $\mathbf{E}(0, \max \ge h_1)$ described at the beginning of this section.

We need to show that V satisfies the assumptions of Theorem A.3. The proof of the following crucial Lemma will occupy the next section.

Lemma 7.8. *Set $q = \lambda^{20}$. For sufficiently large $\lambda > 1$ and sufficiently small $s > 1$, the following holds. For sufficiently large $L, \Lambda > 1$, sufficiently small $W_0 > 0$, we have*

$$\frac{\mathbb{E}[V(\omega_{l+1}) | \omega_l = \omega]}{V(\omega)} \le 1 \text{ for all } \omega \in \Omega. \tag{7.9}$$

If $S \notin \mathrm{Good}_{1,2,3}(H_l, H_{l+1})$ or $S \notin \mathrm{Good}_{4,5}(H_{l+1}, H_{l+2})$,

$$\frac{\mathbb{E}[V(\omega_{l+1}) | \omega_l = \omega]}{V(\omega)} < \lambda^{-20}. \tag{7.10}$$

Finally, for $0 < r < 1$,

$$\mathbb{E} V(\omega_\rho) \lesssim_\lambda r^{-1/4}. \tag{7.11}$$

where ρ satisfies $H_{\rho-1} < r\sqrt{n} \le H_\rho$.

Proof. Recall that ω_{l+1} is generated by the k_l independent excursions $T_1, \ldots, T_{k_l}$, where T_i is uniformly randomly chosen from $\mathbf{E}_{\beta_i}(H_{l+1}, \max \geq H_{l+2})$. Using (7.6) for the denominator and (7.7) for the numerator, we can write

$$\begin{aligned}\frac{\mathbb{E}[V(\omega_{l+1})|\omega_l = \omega]}{V(\omega)} &= \frac{s^{\mathrm{Gap}(\omega_l) g_{l+1}^{-2}} \prod_{i=1}^{k_l} \mathbb{E}\left[v_{g_l \uparrow (\lambda+1) g_l}(T_i)^{1/2} q^{\mathbb{1}(T_i \notin \mathrm{good}_{1,2,3})}\right]}{s^{\mathrm{Gap}(\omega_l) g_l^{-2}} \prod_{i=1}^{k_l} q^{\mathbb{1}(\beta_i g^{-2} \leq W_0)} (2 + (\beta_i g_l^{-2})^{1/4})^{1/2} \prod_{j=1}^{k_{l-1}} q^{\mathbb{1}(S|_{U_j} \notin \mathrm{good}_{1,2,3})}} \\ &= s^{-\mathrm{Gap}(\omega_l)(1-\lambda^{-2}) g_l^{-2}} \left(\prod_{i=1}^{k_l} \frac{\mathbb{E}\left[v_{g_l \uparrow (\lambda+1) g_l}(T_i)^{1/2} q^{\mathbb{1}(T_i \notin \mathrm{good}_{1,2,3})}\right]}{q^{\mathbb{1}(\beta_i g_l^{-2} \leq W_0)} (2 + (\beta_i g_l^{-2})^{1/4})^{1/2}} \right) \prod_{j=1}^{k_{l-1}} q^{-\mathbb{1}(S|_{U_j} \notin \mathrm{good}_{1,2,3})},\end{aligned}$$

where the expectations are with respect to independent, uniformly random $T_i \in \mathbf{E}_{\beta_i}(0, \max \geq g_l)$.

By the Cauchy-Schwarz inequality, each term in the middle product is bounded above by

$$\left(\frac{\mathbb{E} v_{g_l \uparrow (\lambda+1) g_l}(T_i)}{2 + (\beta_i g_l^{-2})^{1/4}} \right)^{1/2} \cdot \left(\frac{\mathbb{E} q^{2\mathbb{1}(T_i \notin \mathrm{good}_{1,2,3})}}{q^{2\mathbb{1}(\beta_i g_l^{-2} \leq W_0)}} \right)^{1/2}. \tag{7.12}$$

Set $\lambda = \lambda_0 + 1$ where λ_0 is the constant of Lemma 7.9, and set $q = \lambda^{20}$, and W_0 to be the constant of Lemma 7.9. Choose $s < s_\lambda$ small enough that

$$s^{2 g_{\rho-1}^{-2}} \cdot \exp\left(-c_0 \frac{1}{H_\rho^2} \right) \leq 1. \tag{7.13}$$

This last condition on s will only be used further below in the proof of (7.11).

For these parameters, we have from Lemma 7.9 that the first term in (7.12) is bounded by $\left(\frac{7}{8}\right)^{1/2}$.

Now we turn to the second term of (7.12). By Proposition 7.7 with $\epsilon = 0.01 q^{-2}$ there is a $\Lambda > 0$ such that for $L \geq L_0(\Lambda)$ we have

$$\frac{\mathbb{E} q^{2\mathbb{1}(T_i \notin \mathrm{good}_{1,2,3})}}{q^{2\mathbb{1}(\beta_i g_l^{-2} \leq W_0)}} \leq 1 + q^2 \mathbb{P}(T_i \notin \mathrm{good}_{1,2,3} | \beta_i g_l^{-2} > W_0) \leq 1.01.$$

It follows that the product (7.12) is bounded by 0.95. Thus we get

$$\frac{\mathbb{E}[V(\omega_{l+1})|\omega_l = \omega]}{V(\omega)} \leq s^{-\mathrm{Gap}(\omega)(1-\lambda^{-2}) g_l^{-2}} 0.95^{k_l} \prod_{j=1}^{k_{l-1}} q^{-\mathbb{1}(S|_{U_j} \notin \mathrm{good}_{1,2,3})}. \tag{7.14}$$

This immediately implies (7.9). Now suppose $S|_{U_j} \notin \mathrm{good}_{1,2,3}$ for some j, or $S \notin \mathrm{Good}_{4,5}(H_{l+1}, H_{l+2})$. In the latter case this implies that $k_l \geq L$ or $\mathrm{Gap}(\omega_l) g_l^{-2} \geq L$. Then

$$\frac{\mathbb{E}[V(\omega_{l+1})|\omega_l = \omega]}{V(\omega)} \leq \max\left(s^{-L(1-\lambda^{-2})}, 0.95^L, \frac{1}{q} \right)$$

and this last expression can be made to be smaller than λ^{-20} by taking L large. This proves (7.10).

To prove the last inequality (7.11), we will show that

$$\mathbb{E}[V(\omega_\rho)] \le \mathbb{E}[v_{H_\rho\uparrow H_{\rho+1}}(S)]^{1/2} \lesssim (2+(n/g_{\rho-1}^2)^{1/4})^{1/2} \le C_\lambda r^{-1/4}, \tag{7.15}$$

where the last inequality is clear from the definition of ρ.

For the first inequality, let $V'(\omega_l) = V(\omega_l)\prod_{j=1}^{k_{l-1}} q^{-\mathbb{1}(S|_{U_j}\notin \mathrm{good}_{1,2,3})}$ and notice that $V'(\omega_l)$ is $\mathbf{b}(\omega_l)$-measurable. Recall that $\mathbf{b}(\omega_l)$ is the vector of lengths of the $[H_{l+1}\uparrow H_{l+2}]$-excursion intervals. Thus

$$\mathbb{E}V(\omega_\rho) = \mathbb{E}\left[\mathbb{E}\left[V(\omega_\rho)|\mathbf{b}(\omega_{\rho-1})\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{V(\omega_\rho)}{V'(\omega_{\rho-1})}\middle|\mathbf{b}(\omega_{\rho-1})\right]V'(\omega_{\rho-1})\right] \le \mathbb{E}[V'(\omega_{\rho-1})],$$

where the inequality is from (7.14), using the fact that conditioning on $\omega_{\rho-1}$ is the same as conditioning on $\mathbf{b}(\omega_{\rho-1})$, and the fact that $s>1$. This last expectation is, by (7.6), equal to

$$\begin{aligned}\mathbb{E}[V'(\omega_{\rho-1})] &= \mathbb{E}\left[s^{(a_1+\cdots+a_{k+1})g_{\rho-1}^{-2}}\prod_{i=k+2}^{2k+1} q^{\mathbb{1}(a_i g_{\rho-1}^{-2}\le W_0)}(2+(a_i g_{\rho-1}^{-2})^{1/4})^{1/2}\right]\\ &= \mathbb{E}v_{H_\rho\uparrow H_{\rho+1}}(S)^{1/2} \le \mathbb{E}[v_{H_\rho\uparrow H_{\rho+1}}(S)]^{1/2},\end{aligned}$$

where S is a uniformly random element of $\mathbf{E}_n(0)$ and the $a_1,\dots,a_{2k+1}$ are the lengths in the $[0\uparrow H_\rho\uparrow H_{\rho+1}]$-decomposition.

For the second inequality of (7.15), apply Lemma 7.9 with our choices of q and W_0, and $\mu = \frac{H_{\rho+1}}{H_\rho}-1 \in [\lambda_0,\lambda_0+1]$ to obtain

$$\mathbb{E}[v_{H_\rho\uparrow H_{\rho+1}}(S)|\max S\ge H_\rho] \le \frac{7}{8}(2+(ng_{\rho-1}^{-2})^{1/4}).$$

On the other hand, we have by Proposition B.5 that $\mathbb{P}(\max S < H_\rho) \lesssim \exp(-c_0\frac{n}{H_\rho^2})$, while $\mathbb{E}[v_{H_\rho\uparrow H_{\rho+1}}(S)|\max S < H_\rho] = s^{2ng_{\rho-1}^{-2}}$. Thus by our choice of s, (7.13), we are done. $\square$

Now that we have proved that our test function V satisfies the hypotheses of Theorem A.3, we can apply the theorem and prove that most of the scales are good.

Proof of Proposition 7.5. Choose the constants L,Λ,s,λ,W_0 so that the conclusion of Lemma 7.8 holds. Applying Theorem A.3 (with $\epsilon=\frac{1}{4}$) to the Markov chain $(\omega_l)_{l\ge\rho}$ and the test function $u(\omega_l)=V(\omega_l)$ yields

$$\mathbb{P}\left(\frac{|\{l=\rho,\dots,N : S\in \mathrm{Good}_{1,2,3}(H_l,H_{l+1}) \text{ and } S\in \mathrm{Good}_{4,5}(H_{l+1},H_{l+2})\}|}{N-\rho}<\frac{3}{4}\right)$$
$$\lesssim r^{-1/4}(\lambda^{-5})^{(N-\rho+1)} \le r^{-1/4}(\lambda^{-1}r^{-1/2})^{-5} \le \lambda^5 r^{2.25}.$$

Here we have used (7.4).

To improve the Good_5 above into $\mathrm{Good}_{\tilde{5}}$, we use the union bound together with the following observation. If $k_l=0$ for some $l\le N$ then $\max S \le H_{N+2}$. Thus if $\max S > H_{N+2}$ then $\mathrm{Good}_5(H_l,H_{l+1})$ implies $\mathrm{Good}_{\tilde{5}}(H_l,H_{l+1})$. Now $H_{N+2}<\lambda^4 r^{1/2}\sqrt{n}$, and by Proposition B.5, $\mathbb{P}_{\mathbf{E}_n(0)}(\max S \le \lambda^4 r^{1/2}\sqrt{n}) \le \frac{3}{2}\exp\left(-c_0\frac{1}{\lambda^8 r}\right)$ which is bounded by $r^{2.25}$ for r sufficiently small.

We have shown that the proportion of scales $l=\rho,\dots,N$ that do not satisfy $\mathrm{Good}_{1,2,3}(H_l,H_{l+1})$ is bounded by $1/4$, and likewise the proportion of scales that do not satisfy $\mathrm{Good}_{4,\tilde{5}}(H_l,H_{l+1})$ is bounded by $1/4+1/(N-\rho)$. The statement of the proposition follows. □

7.6 Proof of bound for v

Recall the definition of v in (7.8).

Lemma 7.9. *There exists μ_0 such that for g,h integer with $\mu := h/g-1\ge\mu_0$ there exists $W_0>0$ such that for all $1<s<s(\mu)$, we have for all $w>0$ even and $g\ge 1$ integer,*

$$\frac{\mathbb{E}v_{g\uparrow h}(S)}{2+(wg^{-2})^{1/4}} \le \frac{7}{8}$$

whenever $q\le(\mu+1)^{20}$. Here the expectation is with respect to S being a uniformly random element of $\mathbf{E}_w(0,\max\ge g)$.

Proof. First we assume that $wg^{-2}\le 1$. In this case, it is likely that $k=0$, and it suffices to use the bounds $(2+(wg^{-2})^{1/4})\le 3$ and $(a_1+\cdots+a_{k+1})g^{-2}\le 1$ and $q^{\mathbb{1}(ag^{-2}\le W_0)}\le q$. We get

$$\begin{aligned}
\frac{\mathbb{E}(v_{g\uparrow(\mu+1)g}(\mathbf{a}_i))}{2+(wg^{-2})^{1/4}} &\le \frac{s^{2\mu^{-2}}\mathbb{E}\left(3^k q^{2k}\right)}{2}\\
&\le \frac{s^{2\mu^{-2}}}{2}\left(1+p\sum_{k=1}^{\infty}\tilde{p}^{k-1}3^k q^{2k}\right)\\
&= \frac{s^{2\mu^{-2}}}{2}\left(1+p\frac{3q^2}{1-3\tilde{p}q^2}\right)
\end{aligned}$$

In the second inequality we have used Lemma 7.6, where p and $\tilde{p}$ are the probabilities in the conclusion of that lemma. Recall that they converge to 0 exponentially in $(\mu-1)^2$ as $\mu \to \infty$ (when wg^{-2} is bounded). Thus for any $s > 1$, if q is polynomial in μ, this expression is bounded by $\frac{7}{8}$ as long as μ is sufficiently large.

Now we turn to the case $wg^{-2} > 1$. By (7.2), we have

$$\frac{\mathbb{E}v_{g\uparrow(\mu+1)g}(S)}{2+(wg^{-2})^{1/4}} = \frac{|\mathbf{E}_w(0, \max \geq g, \max < (\mu+1)g)|}{|\mathbf{E}_w(0, \max \geq g)|} \frac{s^{2wg^{-2}\mu^{-2}}}{2+(wg^{-2})^{1/4}} + I \tag{7.16}$$

where

$$\begin{aligned} I =& \frac{1}{2+(wg^{-2})^{1/4}} \frac{1}{|\mathbf{E}_w(0, \max \geq g)|} \sum_{k=1}^{\infty} \sum_{a_1+\cdots+a_{2k+1}=w} \\ & s^{2a_1 g^{-2}\mu^{-2}} |\mathbf{Z}_{a_1}(0 \uparrow g, \max < (\mu+1)g)| \cdot s^{2a_2 g^{-2}\mu^{-2}} |\mathbf{Z}_{a_2}(g \downarrow 0, \max < (\mu+1)g)| \\ & \times \prod_{i=3}^{k+1} s^{2a_i g^{-2}\mu^{-2}} |\mathbf{W}^{\uparrow}_{a_i}(g \to g, 0 \leq \min < \max < (\mu+1)g)| \\ & \times \prod_{i=k+2}^{2k+1} q^{2\mathbb{1}(a_i\mu^{-2}g^{-2} \leq W_0)} |\mathbf{E}_{a_i}(g, \max \geq (\mu+1)g)| \cdot (2 + (a_i\mu^{-2}g^{-2}\mu^{-2})^{1/4}). \end{aligned}$$

Multiplying each term in the sum by

$$1 = \mu^{-k} \frac{1}{g2^{-w}} 2^{-a_1} 2^{-a_2} \prod_{i=3}^{k+1} g^{-1} 2^{-a_i} \prod_{i=k+1}^{2k+1} 2^{-a_i} \mu g,$$

we can write $I = \sum_{k=1}^{\infty} A_k$ where

$$A_k = \frac{\mu^{-k} \sum_{a_1+\cdots+a_{2k+1}=w} F_{\mathbf{Z}}(a_1) F_{\mathbf{Z}}(a_2) \prod_{i=3}^{k+1} F_{\mathbf{B}}(a_i) \prod_{i=k+2}^{2k+1} F_{\mathbf{E}}(a_i)}{(2+(wg^{-2})^{1/4}) g |\mathbf{E}_w(0, \max \geq g)| 2^{-w}}$$

where

$$\begin{aligned} F_{\mathbf{Z}}(a) &= s^{2ag^{-2}\mu^{-2}} |\mathbf{Z}_a(0 \uparrow g, \max < (\mu+1)g)| \cdot 2^{-a} \\ F_{\mathbf{B}}(a) &= g^{-1} s^{2ag^{-2}\mu^{-2}} |\mathbf{W}^{\uparrow}_a(0 \to 0, -g \leq \min < \max < \mu g)| \cdot 2^{-a} \\ F_{\mathbf{E}}(a) &= \mu g q^{2\mathbb{1}(a\mu^{-2}g^{-2} \leq W_0)} |\mathbf{E}_a(0, \max \geq \mu g)| \cdot 2^{-a} (2 + (ag^{-2}\mu^{-2})^{1/4}). \end{aligned} \tag{7.17}$$

We bound A_k using the following observation, valid for any positive functions $F_1, \ldots, F_{2k+1}$:

$$\frac{1}{Z} \sum_{a_1+\cdots+a_{2k+1}=w} \prod_{i=1}^{2k+1} F_i(a_i) \leq \frac{1}{Z} \sum_{j=1}^{2k+1} \sum_{\substack{a_1+\cdots+a_{2k+1}=w \\ a_j \geq w(2k+1)^{-1}}} \prod_{i=1}^{2k+1} F_i(a_i)$$
$$\leq \sum_{j=1}^{2k+1} \sup_{w(2k+1)^{-1} \leq a \leq w} \frac{F_j(a)}{Z} \prod_{\substack{i=1 \\ i \neq j}}^{2k+1} \sum_{a=0}^{\infty} F_i(a_i).$$

Now Lemmas 7.10, 7.11 and 7.12 below show that, given ϵ, if $\mu > 1$ is sufficiently large and $s > 1$ is sufficiently small, then for all $q > 1$ there exists L large and W_0 small such that

$$\begin{aligned} A_k &\leq \mu^{-k} \cdot [2 \cdot C_{\mathbf{Z}}(2k+1)^{3/2} \Sigma_{\mathbf{Z}} \Sigma_{\mathbf{W}}^{k-1} \Sigma_{\mathbf{E}}^{k} + (k-1) \cdot C_{\mathbf{W}}(2k+1)^{3/2} \Sigma_{\mathbf{Z}}^2 \Sigma_{\mathbf{W}}^{k-2} \Sigma_{\mathbf{E}}^{k} \\ &\qquad + k \cdot \mu(2k+1)^{3/2} \epsilon \Sigma_{\mathbf{Z}}^2 \Sigma_{\mathbf{W}}^{k-1} \Sigma_{\mathbf{E}}^{k-1}] \\ &\leq C^{2k+1} \mu^{-k} (2k+1)^{3/2} (k+1) + C^{2k} \mu^{-k+1} (2k+1)^{5/2} k\epsilon, \end{aligned} \tag{7.18}$$

where in the last line we have absorbed all the constants $C_{\ldots}, \Sigma_{\ldots}$ into a single constant C.

Therefore

$$I \leq \sum_{k=1}^{\infty} C^{2k+1} \mu^{-k} (2k+1)^{3/2} + C^{2k} \mu^{-k+1} (2k+1)^{5/2} k\epsilon.$$

Taking ϵ small and μ large gives $I \leq 1/8$. Turning back to the rest of (7.16), we have

$$\frac{|\mathbf{E}_w(0, \max \geq g, \max < (\mu+1)g)|}{|\mathbf{E}_w(0, \max \geq g)|} \frac{s^{2wg^{-2}\mu^{-2}}}{2} \leq \mathbb{P}_{\mathbf{E}_w(0)}(\max S < (\mu+1)g) \frac{s^{2wg^{-2}\mu^{-2}}}{2}$$
$$\leq \frac{3}{2} e^{-c_0 w(\mu+1)^{-2} g^{-2}} \frac{s^{2wg^{-2}\mu^{-2}}}{2}.$$

Here we used Proposition B.5 for the second inequality. For sufficiently small $s > 1$, this is bounded above by $\frac{3}{4}$.

Together with our bound $I \leq 1/8$, this proves the desired estimate when $wg^{-2} > 1$. □

7.6.1 6 inequalities

In this subsection we prove the inequalities needed in the proof of Lemma 7.9. For the definitions of $F_{\mathbf{Z}}, F_{\mathbf{E}}$ and $F_{\mathbf{B}}$, see (7.17), and let

$$Z = g|\mathbf{E}_w(0, \max \geq g)| 2^{-w} \cdot (2 + (wg^{-2})^{1/4}).$$

Lemma 7.10. *There exists $\Sigma_{\mathbf{E}} > 0$ such that for $\mu \geq 2$ sufficiently large, all $q > 1$ and sufficiently small W_0 we have*

$$\sum_{a=1}^{\infty} F_{\mathbf{E}}(a) \leq \Sigma_{\mathbf{E}} \tag{7.19}$$

uniformly in g. Furthermore, for every $\epsilon > 0$,

$$\sup_{w \geq a \geq w(2k+1)^{-1}} \frac{F_{\mathbf{E}}(a)}{Z} \leq \mu(2k+1)^{3/2}\epsilon \tag{7.20}$$

if μ is large and W_0 small enough.

Proof. (7.19) is equivalent to: For all $\mu \geq 2$ sufficiently large, for all $q > 1$, for sufficiently small W_0,

$$\sum_{a=1}^{\infty} \mu g q^{\mathbb{1}(ag^{-2} \leq W_0\mu^2)} |\mathbf{E}_a(0, \max \geq \mu g| \cdot 2^{-a} \cdot (2 + (ag^{-2}\mu^{-2})^{1/4}) \leq \Sigma_{\mathbf{E}}.$$

The left hand side only depends on μg, so we may replace μg with g. We will split the sum into two parts: $ag^{-2} \leq W_0$ and $ag^{-2} > W_0$. We have, by Stirling's approximation and Proposition B.5,

$$\begin{aligned} &\sum_{a > g^2 W_0}^{\infty} g q^{\mathbb{1}(ag^{-2} < W_0)} |\mathbf{E}_a(0, \max \geq g)| \cdot 2^{-a} \cdot (2 + (ag^{-2})^{1/4}) \\ &\lesssim \sum_{a=1}^{\infty} g \exp\left(-c_0 \frac{g^2}{a}\right) a^{-3/2}(2 + (ag^{-2})^{1/4}) = \sum_{a=1}^{\infty} \frac{1}{g^2} \exp\left(-c_0 \frac{g^2}{a}\right) (ag^{-2})^{-3/2}(2 + (ag^{-2})^{1/4}) \\ &\lesssim \int_0^{\infty} x^{-3/2} \exp\left(-c_0 \frac{1}{x}\right) (2 + x^{1/4}) dx. \end{aligned} \tag{7.21}$$

On the other hand, a similar sequence of computations shows that

$$\sum_{a \leq g^2 W_0} qg|\mathbf{E}_a(0, \max \geq g)| \cdot 2^{-a} \cdot (2 + (ag^{-2})^{1/4}) \lesssim q \int_0^{W_0} x^{-3/2} \exp\left(-c_0 \frac{1}{x}\right) (2 + x^{1/4}) dx.$$

Combining this with (7.21) proves (7.19).

Now we turn to the second inequality of the lemma. Choose $\mu_1 = \mu_1(\epsilon)$ large enough that

$$\sup_{w \geq a \geq w(2k+1)^{-1}} \frac{2 + (ag^{-2}\mu^{-2})^{1/4}}{2 + (wg^{-2})^{1/4}} < \epsilon \tag{7.22}$$

whenever $wg^{-2} > \mu_1^2$ and $\mu > \mu_1$. By Proposition B.5 (parts a and c), there exists $\mu_2 > 1$ such that if $\mu > \mu_2$ then

$$\frac{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq \mu g)}{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq g)} \lesssim \exp\left(-c_0\mu^2 g^2/a\right) \leq \epsilon, \text{ whenever } ag^{-2} \leq \mu_1^2. \tag{7.23}$$

Choose $\mu > \max(\mu_1, \mu_2)$. The same proposition also shows that if $0 < W_0 < W_0(\mu, q, \epsilon)$ is sufficiently small, then

$$\frac{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq \mu g)}{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq g)} \leq \epsilon q^{-1}, \text{ whenever } ag^{-2} < W_0. \tag{7.24}$$

By Stirling's approximation and the monotonicity Lemma B.10,

$$\begin{aligned}\frac{|\mathbf{E}_a(0, \max \geq \mu g)|2^{-a}}{|\mathbf{E}_w(0, \max \geq g)|2^{-w}} &= \frac{|\mathbf{E}_a(0)|2^{-a} \cdot \mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq \mu g)}{|\mathbf{E}_w(0)|2^{-w} \cdot \mathbb{P}_{\mathbf{E}_w(0)}(\max S \geq g)} \\ &\leq C_0 \left(\frac{a}{w}\right)^{-3/2} \cdot \frac{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq \mu g)}{\mathbb{P}_{\mathbf{E}_a(0\to 0)}(\max S \geq g)}.\end{aligned} \tag{7.25}$$

From this we see that to prove (7.20) it suffices to show that

$$\sup_{w \geq a \geq w(2k+1)^{-1}} q^{2\mathbb{1}(a\mu^{-2}g^{-2} \leq W_0)} \frac{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq \mu g)}{\mathbb{P}_{\mathbf{E}_a(0\to 0)}(\max S \geq g)} \cdot \frac{2 + (ag^{-2}\mu^{-2})^{1/4}}{2 + (wg^{-2})^{1/4}} \leq \epsilon.$$

The case $W_0 \leq ag^{-2} \leq \mu_1^2$ follows from (7.23), whereas (7.24) takes care of the case $ag^{-2} < W_0$, and the case $ag^{-2} > \mu_1^2$ follows from (7.22) and (7.25). □

Lemma 7.11. *There exist constants $\Sigma_{\mathbf{W}}, C_{\mathbf{W}} < \infty$ so that the following holds. For $\mu \geq 2$, there exists $s > 1$ such that*

$$\sum_{a=1}^{\infty} F_{\mathbf{B}}(a) \leq \Sigma_{\mathbf{W}} \tag{7.26}$$

uniformly in g, and if $wg^{-2} \geq 1$, then for all $k > 1$,

$$\sup_{w(2k+1)^{-1} \leq a \leq w} \frac{F_{\mathbf{B}}(a)}{Z} < C_{\mathbf{W}}(2k+1)^{3/2}. \tag{7.27}$$

Proof. Let $p = \mathbb{P}^{\mathbf{W}_a^{\uparrow}(0\to 0)}(-g \leq \min S < \max S < \mu g)$. Then by (B.7),

$$p \leq \mathbb{P}^{\mathbf{W}_a^{\uparrow}(0\to 0)}(-g \leq \min S) \leq 4(ag^{-2})^{-1}.$$

Using this together with Proposition B.5c) gives

$$p \leq \min(3/2e^{-c_0\mu^{-2}ag^{-2}}, 4a^{-1}g^2).$$

We have, by Stirling's approximation (Lemma B.4)

$$|\mathbf{W}_a^{\uparrow}(0 \to 0, -g \le \min < \max < \mu g)| \cdot 2^{-a} = |\mathbf{W}_a^{\uparrow}(0 \to 0)| \cdot 2^{-a} p \\ \le C_{\mathrm{stir}} a^{-1/2} \min(3/2 e^{-c_0 \mu^{-2} a g^{-2}}, 4a^{-1} g^2).$$

Now choose T large enough and s small enough that $C_0 e^{-c_0 \mu^{-2} x} s^{2\mu^{-2} x} \le 8x^{-1}$ for real numbers $x \ge T$. By making $s > 1$ smaller if necessary, we can assume that $s^{2x\mu^{-2}} \le 2$ for $x \in [0, T]$. Later in the proof we will also need to assume that $T \ge 1$.

By considering the cases $2ag^{-2} \ge T$ and $2ag^{-2} < T$ separately, we get

$$s^{ag^{-2}\mu^{-2}} |\mathbf{W}_a^{\uparrow}(0 \to 0, -g \le \min < \max < \mu g)| \cdot 2^{-a} \le 8 C_{\mathrm{stir}} a^{-3/2} g^2. \tag{7.28}$$

To prove (7.26), we split the sum into two parts and use (7.28) to bound the right sum:

$$\sum_{a=1}^{\infty} F_{\mathbf{B}}(a) \le \sum_{a \le g^2} g^{-1} \cdot s^{ag^{-2}\mu^{-2}} C_{\mathrm{stir}} a^{-1/2} + \sum_{a > g^2} g^{-1} 8 C_{\mathrm{stir}} a^{-3/2} g^2 \\ \le \sum_{a \le g^2} g^{-1} \cdot 2C_{\mathrm{stir}} \cdot a^{-1/2} + \sum_{a > g^2} g \cdot 8 C_{\mathrm{stir}} a^{-3/2},$$

and both sums are bounded by a constant $\Sigma_{\mathbf{B}}$ independent of g. This completes the proof of (7.26).

For the other statement (7.27), we have from Proposition B.5b) and Stirling's approximation (Lemma B.4) that there is a constant $c > 0$ such that

$$|\mathbf{E}_w(0, \max \ge g)| \cdot 2^{-w} \ge C_{\mathrm{stir}}^{-1} a^{-3/2} c \qquad \text{whenever } wg^{-2} \ge 1. \tag{7.29}$$

Together with (7.28), this immediately implies (7.27). □

Lemma 7.12. *There exist constants $\Sigma_{\mathbf{Z}}, C_{\mathbf{Z}} < \infty$ such that the following holds. For $\mu \ge 2$, for sufficiently small $s > 1$*

$$\sum_{a=1}^{\infty} F_{\mathbf{Z}}(a) \le \Sigma_{\mathbf{Z}} \tag{7.30}$$

uniformly in g, and if $wg^{-2} \ge 1$, then for all $k \ge 1$,

$$\sup_{w \ge a \ge w(2k+1)^{-1}} \frac{F_{\mathbf{Z}}(a)}{Z} < C_{\mathbf{Z}} (2k+1)^{3/2}. \tag{7.31}$$

Proof. We have by Lemma B.9 and Proposition B.5c),

$$\mathbb{P}^{\mathbf{Z}_a(0 \uparrow g)}(\max S \le (\mu+1)g) \le 3/2 e^{-c_0 (\mu+1)^{-2} a g^{-2}} \tag{7.32}$$

for some universal constant $c_0 > 0$. Substituting (B.3) and (7.32) into the sum (7.30) gives

$$\sum_{a=1}^{\infty} F_{\mathbf{Z}}(a) \leq \sum_{a=2}^{\infty} 2C_{\text{stir}} g^{-2}(a/g^2)^{-3/2} \cdot e^{-\frac{g^2}{3a}} e^{-c_0(\mu+1)^{-2}ag^{-2}} s^{2ag^{-2}\mu^{-2}}$$
$$\leq 2C_{\text{stir}} \sum_{a=2}^{\infty} g^{-2}(a/g^2)^{-3/2} e^{-\frac{g^2}{3a}}$$
$$\lesssim \int_0^{\infty} x^{-3/2} \cdot e^{-1/x} dx =: \Sigma_{\mathbf{Z}}.$$

where the second last inequality holds as long as $s^{2\mu^{-2}} \leq e^{c_0(\mu+1)^{-2}}$. This proves the first statement of the Lemma.

The second statement follows again from (B.3) and (7.29). $\square$

7.7 Conclusion of the proof of Theorem 1.4

We now have all ingredients to prove Theorems 1.4 and 1.3. Denote $f_S : \Delta \to \overline{\mathbb{C}} \setminus T_S$ the (unique) normalized conformal map that solves the welding problem associated with an excursion S of length $2n$, so that T_S is a balanced tree homeomorphic to the tree encoded by S via (1). The uniform measure on excursions pushes forward to a measure μ_n on the space $\mathcal{S}$ of conformal maps via $S \mapsto f_S$. Theorem 1.4 states that μ_n converges weakly to a measure μ on $\mathcal{S}$ equipped with the sup norm. It also claims that μ is supported on Hölder continuous conformal maps and that the law of the lamination $\mathcal{L}_f$ is that of the CRT. In particular, the CRT admits conformal welding, which is the statement of Theorem 1.3.

We first show that the sequence of measures μ_n is tight. Note that tightness in the weaker topology of compact convergence (equivalent to the sup-norm on a circle $\{|z| = R\}$ for any fixed $R > 1$) easily follows from compactness of the space of conformal maps in that topology, as observed in Proposition 1.5.1 of Joel Barnes' thesis [Bar14]. A main result of [Bar14] was the non-triviality of any subsequential limit μ, namely $\mu(\text{Identity}_{\mathbb{C}\setminus\overline{\mathbb{D}}}) = 0$. In the stronger topology of the sup-norm on $\mathbb{T}$, our space is not compact and we need an estimate for the modulus of continuity ω_f of f.

By the Arzela-Ascoli theorem it suffices to show that for every $\epsilon > 0$ there are $\delta > 0$

$$\mu_n(\omega_f(\delta) > \epsilon) < \epsilon$$

for all n. To this end, we first note that Brownian scaled excursions $\mathbb{e}_S$ are Hölder continuous with high probability: Indeed, the proof of Lemma B.4 can be adapted to show that for every Hölder exponent $\alpha < 1/2$ there is a constant $C = C_{\epsilon,\alpha}$ such that

$$\mathbb{P}(\mathbb{e}_S \text{ is } (C, \alpha)\text{-Hölder}) \geq 1 - \epsilon$$

independently of n. See Lemma 1.5.1 of [Bar14] for a different proof. Denoting $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}})$ the associated tree and $p : \mathbb{T} \to \mathcal{T}_{\mathbb{e}}$ the quotient map, it follows that

$$\mathbb{P}(p \text{ is } (2C, \alpha)\text{-Hölder}) \geq 1 - \epsilon.$$

Fix n and denote A the set of excursions of length $2n$ for which p is $(2C, \alpha)$-Hölder. Let $I \subset \mathbb{T}$ be an interval containing 0 and $S \in A$. Then $p(I)$ contains the root and has diameter $< 2C \operatorname{diam}(I)^\alpha$. Denote J the ball of radius $r = 2C \operatorname{diam}(I)^\alpha$ centered at the root. By Proposition 7.5 and Proposition 7.4 , $f(I)$ can be separated from a circle of fixed radius by a family of curves of modulus $M > \delta_0 \log_\lambda(1/r)/2$ with probability

$$\mathbb{P} > 1 - Cr^{T_0} = 1 - C' \operatorname{diam}(I)^{\alpha T_0}.$$

By Lemma 2.6, we have

$$\operatorname{diam}(f(I)) \lesssim \exp(-2\pi M) \leq r^{\pi\delta_0/\log\lambda} = C \operatorname{diam}(I)^{\alpha\pi\delta_0/\log\lambda}$$

on this event. By the rotational invariance of the uniform arc pairing lamination, we get the same estimate for all intervals $I \subset \mathbb{T}$, not only those containing 0, if we restrict to the excursions in A. Set

$$\beta = \alpha\pi\delta_0/\log\lambda$$

and consider dyadic intervals $I_{j,k}$ of size 2^{-k}. By the above estimate we can make

$$\sum_{(j,k):k\geq k_0} \mathbb{P}(\operatorname{diam}(f(I)) > C \operatorname{diam}(I)^\beta) < \epsilon$$

by choosing k_0 large, provided that $\alpha T_0 > 1$. Since every interval $I \subset \mathbb{T}$ can be covered by two adjacent dyadic intervals of lesser size, we have

$$\mathbb{P}(\operatorname{diam}(f(I)) > 2C \operatorname{diam}(I)^\beta \text{ for some } I \text{ with } \operatorname{diam}(I) < 2^{-k_0}) < \epsilon$$

and tightness follows at once. Note that this also shows Hölder continuity with exponent β on the support of any subsequential limit μ.

Let ν_n be the measure on pairs $(\mathbb{e}_n, f_n)$ of normalized excursions and conformal maps, so that the marginal ν'_n with respect to the first coordinate is the uniform measure on (Brownian normalized) excursions of length $2n$, and the marginal with respect to $\mathcal{S}$ is μ_n. Since μ_n is tight and ν'_n converges weakly to the Ito measure $\mathbf{n}$ on normalized excursions, ν_n is tight. Consider any weakly convergent sequence $\nu_{n_k} \to \nu$. By the Skorokhod representation theorem, we may assume that $\mathbb{e}_{n_k} \to \mathbb{e}$ and $f_n \to f$ almost surely with respect to ν. If we show that

$$\mathcal{L}_{\mathbb{e}} = \mathcal{L}_f \tag{7.33}$$

ν–a.s., then the Theorem follows at once: Indeed, since $\nu'_{n_k} \to \mathbf{n}$, $\mathcal{L}_{\mathbb{e}}$ is the Brownian lamination, and it follows that the Brownian lamination is a conformal lamination. Moreover, since the limit ν' of μ_{n_k} is supported on Hölder continuous conformal maps, the conformal map f associated with $\mathbb{e}$ is unique by the Jones-Smirnov theorem [JS00] regarding conformal removability of boundaries of Hölder domains. Consequently there is only one possible limit ν', and convergence of μ_n is established.

To prove (7.33), we first notice that if $\mathbb{e}$ has distinct local minima (so that the tree coded by $\mathbb{e}$ has degree ≤ 3), then $\mathbb{e}_{n_k} \to \mathbb{e}$ implies $\mathcal{L}_{\mathbb{e}_{n_k}} \to \mathcal{L}_{\mathbb{e}}$ in the Hausdorff metric, see the proof of Proposition 1.3.2 in [Bar14]. Next, since $\mathcal{L}_{\mathbb{e}_n} = \mathcal{L}_{f_n}$, it easily follows that $\mathcal{L}_{\mathbb{e}} \subset \mathcal{L}_f$: Indeed, if $(x, y) \in \mathcal{L}_{\mathbb{e}}$, then there are $(x_n, y_n) \in \mathcal{L}_{\mathbb{e}_n}$ and $f(x) = \lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} f_n(y_n) = f(y)$ so that $(x, y) \in \mathcal{L}_f$. To see that $\mathcal{L}_f$ cannot be strictly larger than $\mathcal{L}_{\mathbb{e}}$, observe that $\mathcal{L}_{\mathbb{e}}$ is maximal in the sense that every chord $(x, y) \notin \mathcal{L}_{\mathbb{e}}$ intersects infinitely many chords of $\mathcal{L}_{\mathbb{e}}$. But if f is $\alpha-$Hölder continuous, then every equivalence class of $\mathcal{L}_f$ has cardinality bounded by $2/\alpha$ as can easily be seen by Pfluger's theorem.Thus $\mathcal{L}_f \setminus \mathcal{L}_{\mathbb{e}} = \emptyset$ almost surely and we are done.

A Large Deviations upper bounds for Markov chains

The results of this section can be found in [DV75]. For the reader's convenience, we give a self contained presentation.

Let $\omega_1, \omega_2, \ldots$ be a Markov chain on a state space Ω with transition kernels $\pi(x, dy)$. Let $u : \Omega \to \mathbb{R}$ be a function. For each $x, y \in \Omega \times \Omega$, let $f(x, y) = \frac{u(y)}{\int u(z)\pi(x,dz)}$.

Lemma A.1. *For each $n \geq 1$ and any choice of ω_1, we have*

$$\mathbb{E}\left(f(\omega_1, \omega_2)f(\omega_2, \omega_3)\cdots f(\omega_{n-1}, \omega_n)\right) = 1.$$

Proof. We have, by the tower property and the Markov property,

$$\begin{aligned}
\mathbb{E}\left(f(\omega_1, \omega_2)f(\omega_2, \omega_3)\cdots f(\omega_{n-1}, \omega_n)\right) &= \mathbb{E}[\mathbb{E}[f(\omega_1, \omega_2)f(\omega_2, \omega_3)\cdots f(\omega_{n-1}, \omega_n)|(\omega_1, \omega_2)]] \\
&= \mathbb{E}[f(\omega_1, \omega_2)\mathbb{E}[f(\omega_2, \omega_3)\cdots f(\omega_{n-1}, \omega_n)|\omega_2]] \\
&= \mathbb{E}[f(\omega_1, \omega_2)] \\
&= 1
\end{aligned}$$

where the second last equality is by induction on n. $\square$

Now let Γ be a set of probability measures on Ω. For $u : \Omega \to (0, \infty)$, let $\lambda_u : \Omega \to \mathbb{R}$ be the function

$$\lambda_u(x) = \log\left(\frac{u(x)}{\int u(y)\pi(x, dy)}\right).$$

For $x \in \Omega$, let δ_x denote the Dirac mass at x.

Theorem A.2. *For any $u : \Omega \to (0, \infty)$, we have*

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n}\delta_{\omega_k} \in \Gamma\right) \le \frac{\mathbb{E}u(\omega_1)}{\inf u} \exp\left(-n \inf_{\mu\in\Gamma}\int \lambda_u d\mu\right). \tag{A.1}$$

Proof. Observe that

$$\mathbb{1}\left\{\frac{1}{n}\sum_{k=1}^{n}\delta_{\omega_k} \in \Gamma\right\} \le \exp\left(\frac{1}{n}\sum_{k=1}^{n}\lambda_u(\omega_k) - \inf_{\mu\in\Gamma}\int \lambda_u d\mu\right)$$

because the term in the parentheses is positive whenever the expression on the left is equal to 1. Taking expectations of the $n-$th power of both sides yields

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n}\delta_{\omega_k} \in \Gamma\right) \le \exp\left(-n \inf_{\mu\in\Gamma}\int \lambda_u d\mu\right) \mathbb{E}\exp\left(\sum_{k=1}^{n}\lambda_u(\omega_k)\right). \tag{A.2}$$

We have

$$\begin{aligned}
\exp\left(\sum_{k=1}^{n}\lambda_u(\omega_k)\right) &= \prod_{k=1}^{n}\frac{u(\omega_k)}{\int u(y)\pi(\omega_k, dy)} \\
&= \frac{u(\omega_1)}{\int u(y)\pi(\omega_n, dy)}\prod_{k=1}^{n-1}\frac{u(\omega_{k+1})}{\int u(y)\pi(\omega_k, dy)}.
\end{aligned}$$

Thus

$$\mathbb{E}\exp\left(\sum_{k=1}^{n}\lambda_u(\omega_k)\right) \le \mathbb{E}\frac{u(\omega_1)}{\inf u}\mathbb{E}\prod_{k=1}^{n-1}\frac{u(\omega_{k+1})}{\int u(y)\pi(\omega_k, dy)}.$$

The expectation of the product is equal to 1, by Lemma A.1. Substituting this into (A.2) yields the result. $\square$

Theorem A.3. *Let $A \subset \Omega$ be a subset. Suppose $u : \Omega \to [1, \infty)$ is a function with $\lambda_u \ge 0$. Then for each $\epsilon > 0$,*

$$\mathbb{P}\left(\frac{1}{n}\left|\{k : \omega_k \in A\}\right| \ge \epsilon\right) \le \mathbb{E}u(\omega_1)\exp\left(-n\epsilon \inf_{\omega\in A}\lambda_u(\omega)\right). \tag{A.3}$$

Proof. We apply Theorem A.2 with $\Gamma = \{\mu : \mu(A) \ge \epsilon\}$. We have $\inf_{\mu\in\Gamma}\int\lambda_u d\mu \ge \inf_{\mu\in\Gamma}\int_A \lambda_u d\mu \ge \mu(A)\inf_{\omega\in A}\lambda_u(\omega) \ge \epsilon\inf_{\omega\in A}\lambda_u(\omega)$ because $\lambda_u \ge 0$. $\square$

B Random Walk estimates

The following lemma will be used to convert probabilistic statements about the maximum and modulus of continuity of bridge walks to corresponding statements about walks conditioned to be positive. Note that the statement is meaningful only when $w - g$ is even and $g \geq 1$. The $g = 1$ case is special because then the left hand side can be identified with $\{0, \dots, w - 1\} \times \mathbf{E}_{w-1}(0)$.

Lemma B.1. *There is a bijection*

$$\varphi : \{0, \dots, w-1\} \times \mathbf{Z}_w(0 \uparrow g) \to \{1, \dots, g\} \times \mathbf{W}_w(0 \to g) \tag{B.1}$$

Moreover, the mapping preserves the maximum and modulus of continuity in the following sense: if $\varphi(t, S) = (y, \tilde{S})$, then

1. *If $\mathbb{e}_{\tilde{S}}$ is $(L, 1/3)$-Hölder continuous the $\mathbb{e}_S$ is $(2L, 1/3)$-Hölder continuous.*
2. $\frac{1}{3} \max |S| \leq \max |\tilde{S}| \leq 3 \max |S|$.

Recall that $\mathbb{e}_S : [0,1] \to \mathbb{R}$ is the Brownian rescaling of S, $\mathbb{e}_S(t) = w^{-1/2} S(wt)$.

The existence of the bijection is known as the Dvoretzky-Motzkin cycle lemma [DM47], but the other two properties are usually not stated in the literature so we present the proof here.

Proof. First observe that there is a natural action of the cyclic group $\mathbb{Z}_w = \{0, \dots, w-1\}$ on $\mathbf{W}_w(0 \to g)$, defined by cyclically permuting the increments of the walks. More formally, if $t_0 \in \mathbb{Z}_w$ and $S \in \mathbf{W}_w(0 \to g)$, define

$$\mathcal{C}_{t_0} S(t) = \begin{cases} S(t_0 + t) - S(t_0) & \text{if } 0 \leq t \leq w - t_0 \\ S(t + t_0 - w) + g - S(t_0) & \text{if } w - t_0 \leq t \leq w. \end{cases} \tag{B.2}$$

Now we describe the map φ. Suppose $(t_0, S) \in \{0, \dots, w-1\} \times \mathbf{Z}(0 \uparrow g)$. Let $y_0 = g - \min S|_{[t_0, w]}$. Since for $S \in \mathbf{Z}(0 \uparrow g)$, the last step is always going up from $g - 1$ to g, we have $\min S|_{[t_0,w]}| \leq g - 1$ and hence $y_0 \in \{1, \dots, g\}$. Then $\varphi(t_0, S) = (y_0, \mathcal{C}_{t_0} S)$ is the desired mapping.

The inverse mapping is described as follows. Suppose $(y_0, \tilde{S}) \in \{1, \dots, g\} \times \mathbf{W}_w(0 \to g)$. Let $h_0 = \min \tilde{S} + y_0$. Let $s_0 = \max\{s : \tilde{S}(s) = h_0 - 1\} + 1$. Then $\tilde{S}(s_0) = h_0$ and $\varphi^{-1}(y_0, \tilde{S}) = (w - s_0, \mathcal{C}_{s_0} \tilde{S})$ is the desired inverse mapping, and the bijection is proved.

Statement 1) follows from the fact that if $\mathbb{e}_S$ is (L, α)-Hölder continuous then for any t_0, $\mathbb{e}_{\mathcal{C}_{t_0} S}$ is $(2L, \alpha)$-Hölder continuous. Statement 2) follows from the fact that the maximum of any cyclic permutation of any walk $S \in \mathbf{W}_w(0 \to g)$ is bounded by $(g - \min S) + \max S$ which is in turn bounded by $3 \max |S|$. $\square$

We will need the following asymptotics for the number of $\mathbf{Z}$ walks in Lemma 7.12.

Corollary B.2. *We have, for integers $a, g > 0$*

$$|\mathbf{Z}_a(0 \uparrow g)| = \frac{g}{a}\binom{a}{\frac{a}{2}+\frac{g}{2}},$$

and

$$|\mathbf{Z}_a(0 \uparrow g)| \cdot 2^{-a} \le C_{\mathrm{stir}} g^{-2}(a/g^2)^{-3/2} e^{-\frac{g^2}{3a}} \tag{B.3}$$

for some constant C_{stir}.

Proof. The equality follows immediately from (B.1). For the inequality, we use the simple consequence of Stirling's formula

$$C_{\mathrm{stir}}^{-1} w^{-1/2} \le \binom{w}{w/2} 2^{-w} \le C_{\mathrm{stir}} w^{-1/2} \tag{B.4}$$

and obtain

$$\begin{aligned}|\mathbf{Z}_a(0 \uparrow g)| \cdot 2^{-a} &= \frac{g}{a}\binom{a}{\frac{a}{2}+\frac{g}{2}} 2^{-a} = \frac{g}{a}\binom{a}{\frac{a}{2}} 2^{-a} \cdot \frac{(a/2)(a/2-1)\cdots(a/2-g/2+1)}{(a/2+1)(a/2+2)\cdots(a/2+g/2)} \\ &\le C_{\mathrm{stir}} g a^{-3/2}\left(\frac{1}{1+\frac{g}{a}}\right)^{g/2} \le C_{\mathrm{stir}} g^{-2}(a/g^2)^{-3/2} \cdot \exp\left(-\frac{g^2}{3a}\right),\end{aligned}$$

where in the last equality we have used the fact that $y \log(1+x) \ge \frac{2}{3}xy$ when $x \in [0,1]$ and $y \ge 0$. $\square$

The next lemma allows us to convert probabilistic statements about walks to probabilistic statements about bridges, and vice versa.

Lemma B.3 (Local absolute continuity of bridges and walks)**.** *Fix $0 < u < w$ integer, suppose $|h| \le c_0 w^{1/2}$, and let A be a subset of $\mathbf{W}_u(0)$. Suppose $\frac{u}{w} \le \frac{3}{4}$. Then $\mathbb{P}^{\mathbf{W}_w(0\to h)}(S|_{[0,u]} \in A) \le C_0 \mathbb{P}^{\mathbf{W}_w(0)}(S|_{[0,u]} \in A)$, where the constant C_0 only depends on c_0.*

If, in addition, there exists $c_1 \le 1$ such that A only contains walks for which $S(u) \le c_1(w-u)^2$, then $\mathbb{P}^{\mathbf{W}_w(0)}(S|_{[0,u]} \in A) \le C_0 \mathbb{P}^{\mathbf{W}_w(0\to 0)}(S|_{[0,u]} \in A)$ where C_0 only depends on c_1.

Proof. The first statement is proved for $h = 0$ in [KM09, Lemma 3], and the statement for general h follows from the obvious modifications. It suffices to consider the case when A has only one element, $A = \{S'\}$ and then the relevant probabilities can be written down explicitly in terms of $S'(u)$. The second statement follows from the same proof. □

For example, the previous lemma allows us to deduce the Hölder continuity of the Brownian rescaling (7.3) of $\mathbf{Z}$-walks:

Lemma B.4. *For $\epsilon > 0$, $\Lambda, L_2 > 1$, there exists $L_3 > 0$ large such that the following holds. If $g \geq \Lambda$ with $g - w$ even and if $g' = \lfloor g/\Lambda \rfloor$ satisfies $wg'^{-2} \in [L_2^{-1}, L_2]$, then*

$$\mathbb{P}(\mathbb{e}_S \textit{ is } (L_3, 1/3) - \textit{Hölder continuous }) \geq 1 - \epsilon$$

where S is a uniform random walk of type $\mathbf{Z}_w(0 \uparrow g', \max < g)$.

Proof. The idea of the proof is to relate walks of this type to walks of type $\mathbf{B}$ using Lemma B.1. This relationship essentially preserves the modulus of continuity of the walk. Then we use the fact that walks of type $\mathbf{B}$ are locally absolutely continuous to the simple random walk. The Hölder continuity of the $\mathbf{Z}$ walks then follows from the Hölder continuity of the simple random walk. This sort of argument was used in [KM09] to get uniform bounds for the maximum of a Brownian excursion.

It suffices to prove the result when S is a uniform random walk of type $\mathbf{Z}_w(0 \uparrow g')$, because the uniform measure on $\mathbf{Z}_w(0 \uparrow g', \max \leq g')$ is absolutely continuous to the uniform measure on $\mathbf{Z}_w(0 \uparrow g')$, indeed by Proposition B.5 we have $\mathbb{P}_{\mathbf{Z}_w(0 \uparrow g')}(\max S < g) \geq c_0$ for some constant c_0 that only depends on Λ and L_2.

Lemma B.1 and its proof implies that we can sample a uniform random element of $\mathbf{Z}_w(0 \uparrow g')$ by choosing a uniform random element of $\mathbf{W}_w(0 \to g')$, and then applying a certain (random) cyclic permutation of the increments, and as observed in that lemma, this cyclic permutation preserves the modulus of continuity. Therefore it suffices to prove the result for uniform random walks of type $\mathbf{W}_w(0 \to g')$. Now observe from the triangle inequality that if a function is $(L/2, \alpha)$-Hölder continuous when restricted to $[0, 1/2]$ and $[1/2, 1]$ respectively, then it is (L, α)-Hölder continuous on $[0, 1]$. Therefore from symmetry and the union bound it suffices to find L_3 large enough that

$$\mathbb{P}_{\mathbf{W}_w(0 \to g)}(\mathbb{e}_S|_{[0,1/3]} \text{ is not } (L_3/2, 1/3) - \text{Hölder continuous}) \leq \epsilon/2.$$

By Lemma B.3 below, the law of $S|_{[0, \lceil w/2 \rceil]}$ under $\mathbb{P}_{\mathbf{W}_w(0 \to g')}$is absolutely continuous to the law of $S|_{[0, \lceil w/2 \rceil]}$ under $\mathbb{P}_{\mathbf{W}_w(0)}$ with a constant C_0 that only depends on L_2. So it suffices to find L_3 large enough that

$$\mathbb{P}_{\mathbf{W}_w(0)}(\mathbb{e}_S|_{[0,1/2]} \text{ is not } (L_3/2, 1/3) - \text{Hölder continuous}) \leq \epsilon/(2C_0).$$

This last statement follows from the proof of Kolmogorov's continuity criterion. □

B.1 Bounds on the extrema of a random walk

In this section we collect some bounds on the probability that a random walk of length w exceeds a given height g. The analogous bounds for Brownian motion are simpler to state and prove, but we need statements that are uniform in w and g.

Proposition B.5. *a) We have, for some $c_0 > 0$,*

$$\mathbb{P}_{\mathbf{W}_w(0)}(\max |S| \geq y) \lesssim \exp\left(-\frac{2y^2}{w}\right),$$

$$\mathbb{P}_{\mathbf{E}_w(0\to 0)}(\max |S| \geq y) \lesssim \exp\left(-c_0\frac{y^2}{w}\right),$$

and

$$\mathbb{P}_{\mathbf{W}_w(0\to 0)}(\max |S| \geq y) \lesssim \exp\left(-c_0\frac{y^2}{w}\right),$$

$$\mathbb{P}_{\mathbf{W}^{\uparrow}_w(0\to 0)}(\max |S| \geq y) \lesssim \exp\left(-c_0\frac{y^2}{w}\right).$$

b) For $\delta > 0$ there exists $c_\delta > 0$ such that if $w \geq \delta y^2$, then the conditional probabilities of a) are either zero or bounded below by c_δ,

$$\mathbb{P}_\cdot(\max S \geq y) \geq c_\delta.$$

c) Finally, there exists $c_0 > 0$ such that the conditional probabilities of a) satisfy

$$\mathbb{P}_\cdot(\max |S| \leq y) \leq \frac{3}{2}\exp\left(-c_0\frac{w}{y^2}\right).$$

Proof. By André's reflection principle [Fel68, page 72],

$$\mathbb{P}_{\mathbf{W}_w(0)}(\max S \geq y) = 2\mathbb{P}_{\mathbf{W}_w(0)}(S(w) > y) + \mathbb{P}_{\mathbf{W}_w(0)}(S(w) = y)$$

so that

$$2\mathbb{P}_{\mathbf{W}_w(0)}(S(w) > y) \leq \mathbb{P}_{\mathbf{W}_w(0)}(\max S \geq y) \leq 2\mathbb{P}_{\mathbf{W}_w(0)}(S(w) \geq y). \tag{B.5}$$

Now $\mathbb{P}_{\mathbf{W}_w(0)}(S(w) \geq y) \leq \exp(-2y^2/w)$ by Hoeffding's inequality [Hoe63, Theorem 2] and by the union bound we have proved the first claim $\mathbb{P}_{\mathbf{W}_w(0)}(\max |S| \geq y) \leq 4\exp(-2y^2/w)$.

On the other hand, by the central limit theorem we have that

$$\mathbb{P}_{\mathbf{W}_w(0)}(S(w) \geq \delta^{-1/2}w^{1/2}) \to c'_\delta > 0 \text{ as } w \to \infty \tag{B.6}$$

and claim b) for $\mathbf{W}_w(0)$ follows. Claim c) for $\mathbf{W}_w(0)$ now follows from the strong Markov property by decomposing the walk into subwalks of length proportional to y^2 and then using the result of part b) on each of these walks: if any subwalk varies more than $2y$ from its initial point, then the maximum absolute value of the walk must exceed y. This shows that the probability that the maximum is bounded by y is less than $C_0 \exp\left(-c_0 \frac{w}{y^2}\right)$. The constant C_0 may be taken to be $\frac{3}{2}$ by taking c_0 smaller if necessary.

A coupling argument similar to the proof of Lemma B.8 below shows that the absolute maximum of a bridge is stochastically dominated by the absolute maximum of a Bernoulli walk, so this proves a) for bridges $\mathbf{W}_w(0 \to 0)$. The exact same proof works for $\mathbf{W}^{\uparrow}_w(0 \to 0)$ and $\mathbf{W}_w(0 \to 1)$.

This latter statement can be used together with the cycle lemma, Lemma B.1, to prove part a) for $\mathbb{P}_{\mathbf{E}_w(0)}$.

Part b) for bridges and excursions is proved similarly to part b) for walks, and follows from the fact that the measures converge to the Brownian bridge and Brownian excursion respectively.

Part c) for bridges follows from part c) for walks together with Lemma B.3 below, which says that the initial part of a random bridge is almost indistinguishable from the initial part of a random walk. Part c) for excursions then follows from the cycle lemma. $\square$

The following bounds are useful when wg^{-2} large. In particular, the second bound does not degenerate even when $wg^{-2} \to \infty$.

Lemma B.6. *For integers $w, g > 0$,*

$$\mathbb{P}_{\mathbf{W}^{\uparrow}_w(0\to 0)}(\min S \geq -g) \leq 4w^{-1}g^2. \tag{B.7}$$

For $\epsilon > 0$, we have for sufficiently large μ that, for all w, g,

$$\mathbb{P}_{\mathbf{W}_w(\mu g \to \mu g)}(\min S > g | \min S \geq 0) \geq 1 - \epsilon.$$

Proof. We begin with the first inequality. The statement is vacuously true for $w^{-1}g^2 > 1/4$, so in what follows we can assume in particular that $\frac{g}{w/2} \leq 1/2$.

Recall that $\mathbf{W}^{\uparrow}_w(0 \to 0)$ maps bijectively onto $\mathbf{W}_{w-2}(0 \to 0)$, and this map can be realized by forgetting the first and last steps of the walk and translating the whole walk up one unit. Therefore

$$\mathbb{P}_{\mathbf{W}^{\uparrow}_w(g\to g)}(\min S \geq 0) = \mathbb{P}_{\mathbf{W}^{\uparrow}_w(0\to 0)}(\min S \geq -g) = \mathbb{P}_{\mathbf{W}_{w-2}(0\to 0)}(\min S \geq -g + 1). \tag{B.8}$$

We have $|\mathbf{W}_{w-2}(0 \to 0)| = \binom{w-2}{w/2-1}$, and by André's reflection principle, $|\mathbf{W}_{w-2}(0 \to 0, \min S < -g+1)| = |\mathbf{W}_{w-2}(0 \to -2g)| = \binom{w-2}{w/2+g-1}$. Therefore

$$\begin{aligned}\mathbb{P}_{\mathbf{W}_{w-2}(0\to 0)}(\min S < -g+1) &= \left(\frac{w/2-g}{w/2}\right)\left(\frac{w/2-g+1}{w/2+1}\right)\cdots\left(\frac{w/2-1}{w/2+g-1}\right) \\ &\geq \left(1-\frac{g}{w/2}\right)^g \\ &\geq e^{-4g^2/w} \\ &\geq 1-4g^2/w \end{aligned} \tag{B.9}$$

where in the last two inequalities we have used the facts that $1-x \geq e^{-2x}$ for $x \in [0,1/2]$, and $e^{-x} \geq 1-x$.

Together with (B.8), this proves the first inequality of the lemma.

Notice that the derivation leading up to (B.9) actually shows that for $\theta > 0$ there exists $M > 0$ such that

$$\mathbb{P}_{\mathbf{W}_{w-2}(0\to 0)}(\min S \geq -g+1) \geq e^{-\frac{(2+\theta)g^2}{w}} \qquad \text{whenever } \frac{g}{w/2} \leq \frac{2}{M+1}, \tag{B.10}$$

because for $\theta > 0$ there exists $M > 0$ such that $1-x \geq e^{-(1+\theta)x}$ for $x \in [0, 2/(M+1)]$.

We also have the upper bound

$$\begin{aligned}\mathbb{P}_{\mathbf{W}_{w-2}(0\to 0)}(\min S < -g+1) &= \left(\frac{w/2-g}{w/2}\right)\left(\frac{w/2-g+1}{w/2+1}\right)\cdots\left(\frac{w/2-1}{w/2+g-1}\right) \\ &\leq \left(1-\frac{g}{w/2+g-1}\right)^g \\ &\leq e^{-\frac{g^2}{w/2+g-1}}. \end{aligned} \tag{B.11}$$

For the second inequality, fix $\theta > 0$ small and $M > 1$ large such that $\frac{1}{2+\theta}\cdot\frac{1}{1/2+(M+1)^{-1}} \geq 1-\epsilon/2$ and such that (B.9) holds. First consider the case $w^{-1}g^2 > \frac{1}{2M\mu}$. Then by Proposition B.5, we have for sufficiently large μ,

$$\begin{aligned}\mathbb{P}_{\mathbf{W}_w(\mu g\to\mu g)}(\min S > g) &= \mathbb{P}_{\mathbf{W}_w(0\to 0)}(\min S > -(\mu-1)g) \\ &\geq 1-4e^{-\frac{(\mu-1)^2g^2}{2w}} \geq 1-4e^{-\frac{(\mu-1)^2}{4M\mu}} \geq 1-\epsilon.\end{aligned}$$

Now suppose $w^{-1}g^2 \leq \frac{1}{2M\mu}$ so that in particular $\frac{\mu g}{w/2} \leq 1/M$ and so $\frac{\mu g+1}{w/2+1} \leq 2/(M+1)$. We have

$$\begin{aligned}\mathbb{P}_{\mathbf{W}_w(\mu g\to\mu g)}(\min S > g | \min S \geq 0) &= \frac{\mathbb{P}_{\mathbf{W}_w(\mu g\to\mu g)}(\min S > g)}{\mathbb{P}_{\mathbf{W}_w(\mu g\to\mu g)}(\min S \geq 0)} \\ &= \frac{\mathbb{P}_{\mathbf{W}_w(0\to 0)}(\min S > -(\mu-1)g)}{\mathbb{P}_{\mathbf{W}_w(0\to 0)}(\min S \geq -\mu g)}.\end{aligned}$$

Now, (B.10) implies that

$$\mathbb{P}_{\mathbf{W}_w(0\to 0)}(\min S \geq -\mu g) = \mathbb{P}_{\mathbf{W}_{(w+2)-2}(0\to 0)}(\min S \geq -(\mu g+1)+1) \leq 1 - e^{-(2+\theta)\frac{(\mu g+1)^2}{w+2}},$$

On the other hand, (B.11) gives

$$\begin{aligned}\mathbb{P}_{\mathbf{W}_w(0\to 0)}(\min S > -(\mu-1)g) &= 1 - \mathbb{P}_{\mathbf{W}_{(w+2)-2}(0\to 0)}(\min S \leq -((\mu-1)g-1)+1) \\ &\geq 1 - e^{-\frac{((\mu-1)g-1)^2}{w/2+(\mu-1)g-1}}\end{aligned}$$

and we get

$$\begin{aligned}\mathbb{P}_{\mathbf{W}_w(\mu g\to\mu g)}(\min S > g | \min S \geq 0) &\geq \frac{1 - e^{-\frac{((\mu-1)g-1)^2}{w/2+(\mu-1)g-1}}}{1 - e^{-(2+\theta)\frac{(\mu g+1)^2}{w+2}}} \\ &\geq \frac{((\mu-1)g-1)^2}{(\mu g+1)^2}\cdot\frac{1}{2+\theta}\cdot\frac{w+2}{w/2+(\mu-1)g-1} \\ &\geq \frac{((\mu-1)g-1)^2}{(\mu g+1)^2}\cdot\frac{1}{2+\theta}\cdot\frac{w+2}{w/2+(w+2)(M+1)^{-1}} \\ &\geq \frac{\mu^2}{(\mu+1)^2}\cdot\frac{1}{2+\theta}\cdot\frac{1}{1/2+(M+1)^{-1}} \\ &\geq \frac{\mu^2}{(\mu+1)^2}(1-\epsilon/2).\end{aligned}$$

where in the second inequality we have used the fact that $\frac{1-e^{-x}}{1-e^{-y}} \geq \frac{x}{y}$ when $x \leq y$, which in turn follows from the fact that $x \mapsto \frac{1-e^{-x}}{x}$ is decreasing. In the third inequality we have used the fact that $\frac{(\mu-1)g-1}{w/2+1} \leq 2(M+1)^{-1}$. The fourth inequality is true for $g \geq 1$ and can be seen be differentiating both sides with respect to g.

For sufficiently large μ, this last expression is greater than $1-\epsilon$, as desired. □

Finally, the next lemma says that the bounds of Proposition B.5 still hold even when we make the obvious conditionings.

Lemma B.7. *Let S be a uniformly random walk of type $\mathbf{W}_w(g \to g, \min \geq 0)$. Then* $\mathbb{P}(\max S \geq \mu g) \lesssim e^{-c_0(\mu-1)^2 g^2/w}$.

Similarly, $\mathbb{P}_{\mathbf{W}_w(\mu g\to\mu g)}(\min S \leq g | \min S \geq 0) \lesssim e^{-c_0(\mu-1)^2 g^2/w}$.

Proof. By Lemma B.11 below and Proposition B.5,

$$\mathbb{P}_{\mathbf{W}_w(g\to g)}(\max S \geq \mu g|\min S \geq 0) \leq \mathbb{P}_{\mathbf{E}_T(0\to 0)}(\max S \geq (\mu-1)g) \lesssim \exp\left(-c_0\frac{(\mu-1)^2g^2}{w}\right)$$

and this proves the first statement.

The second statement follows from a similar argument, using Lemma B.12 below and Proposition B.5. □

B.2 Monotonicity properties of conditioned random walks

The next few lemmas make precise some intuitively clear monotonicity relations between various types of walks.

Lemma B.8 (Strong Monotonicity)**.** *Suppose $w > 0$ is even and fix a partition of $[0,w]$ into almost disjoint closed intervals $A_1,\dots,A_m$ with endpoints that are even integers. Let S be a uniformly random walk of type $\mathbf{E}_w(0\to 0)$. For $i=1,\dots,m$ let $\tilde{S}_i$ be a uniformly random element of $\mathbf{E}_{A_i}(0\to 0)$. Let $\tilde{S}$ be the concatenation of the S_i, so that $\tilde{S}\in\mathbf{E}_w(0)$. Then $\max S \succ \max\tilde{S} = \max\tilde{S}_1\vee\cdots\vee\max\tilde{S}_m$.*

Proof. For $i=1,\dots,m$, let $I_i=[\tau_i^-,\tau_i^+]$ where τ_i^-,τ_i^+ is the first and last time respectively that S intersects S_i. In the case that these times do not exist, let $I_i=\emptyset$.

Conditioned on $\{I_i\}_{i=0}^m$ and the values of S at the endpoints of the I_i, the collection of walks $\{S|_{I_i}\}_i$ has the distribution of $m+1$ independent bridges (conditioned to be nonnegative). Likewise, $\{\tilde{S}|_{I_i}\}_i$ has the same distribution.

Thus if we define S' to be the excursion obtained by replacing the part of S on each I_i with the corresponding part of $\tilde{S}$, then S' has uniform distribution on $\mathbf{E}_w(0)$. On the other hand, $S|_{A_i\setminus I_i}\geq \tilde{S}|_{A_i\setminus I_i}$ pointwise, due to the boundary conditions of $\tilde{S}|_{A_i}$. It follows that $S'\geq\tilde{S}$ pointwise, and so $(S',\tilde{S})$ is a coupling which proves the desired result. □

Lemma B.9. *Let S be a uniform random element of $\mathbf{E}_w(0)$ and let $\tilde{S}$ be a uniform random element of $\mathbf{Z}_{w+1}(0\uparrow g)$, where $g\geq 1$. Then $\max\tilde{S}\succ\max S$.*

Proof. The proof is the similar to the proof of Lemma B.8. □

Lemma B.10 (Monotonicity)**.** *For all integer $g>0$ and $w_2\geq w_1$ even,*

$$\mathbb{P}_{\mathbf{E}_{w_2}(0\to 0)}(\max\geq g)\geq\mathbb{P}_{\mathbf{E}_{w_1}(0\to 0)}(\max\geq g).$$

Proof. Take $I_1=[0,w_1]$ and $I_2=[w_1,w_2]$ in Lemma B.8. □

Lemma B.11. *Fix $g \geq 0$. Let S be a uniformly random element of $\mathbf{E}_w(0)$ and let $\tilde{S}$ be a uniformly random element of $\mathbf{W}_w(0 \to 0, \min \geq -g)$. Then $\max S \succ \max \tilde{S}$.*

Proof. Let $A \subset [0, w]$ be the set of times for which (the linear interpolation of) $\tilde{S}$ is strictly negative. Conditioned on A, the distribution of $\tilde{S}|_{[0,w]\setminus\overline{A}}$ is that of independent excursions over each component of $[0, w]\setminus\overline{A}$. It follows from Lemma B.8 that $\max S \succ \max \tilde{S}$. □

Lemma B.12. *Fix $w > 0$ even, $g < 0$ and $h \geq g$. Let S be a uniform random walk in $\mathbf{W}_w(h \to 0)$ and let $\tilde{S}$ be the same thing but conditioned on $\min S \geq g$. Then $\tilde{S} \succ S$.*

Proof. We have

$$\mathbb{P}(\min S \geq g | S(1) = h + 1) \geq \mathbb{P}(\min S \geq g).$$

This can be proved by a coupling argument (run the walks independently until they meet, then make them equal each other). By Bayes' rule, we this implies

$$\mathbb{P}_{\mathbf{W}_w(h\to 0)}(S(1) = h + 1 | \min S \geq g) \geq \mathbb{P}_{\mathbf{W}_w(h\to 0)}(S(1) = h + 1). \tag{B.12}$$

Using this, we get a coupling of $\tilde{S}$ and S for which $\tilde{S} \geq S$: We ensure that $\tilde{S}(1) \geq S(1)$ using (B.12), then we let the walks evolve independently until they meet again, and then we use (B.12) again, and so on. □

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