Math 308 P Conceptual Problems #7

Due Friday, March 15

Please write your name and your quiz section (PA, PB, or PC) on your homework paper.

(1) Let A be a $n \times n$ matrix. Let $\mathbf{v}$ be an eigenvector for A with eigenvalue λ.
  (a) Show that $\mathbf{v}$ is also an eigenvector for A^2. What is the eigenvalue?
  (b) Show that if c is a real number, then $\mathbf{v}$ is also an eigenvector for $A + cI$. What is the eigenvalue?
  (c) Show that if r is a real number, then $\mathbf{v}$ is also an eigenvector for rA. What is the eigenvalue?
  (d) Suppose the matrix A satisfies $A^2 = A$. There are only two possible numbers for an eigenvalue of A. What are those numbers?
  (e) Suppose the matrix A satisfies $A^2 = I$. There are only two possible numbers for an eigenvalue of A. What are those numbers?

(2) This problem refers to Problem (1) of Conceptual Problems #5. You should go back and look at this problem (and also its solution).

Let A be the matrix

$$A = \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

  (a) Calculate the eigenvalues of A, and find a basis for each eigenspace. (Suggestion: First find the eigenvalues and eigenspaces of $3A$ and then use part 1(c) above, taking $r = \frac{1}{3}$.) **Answer for eigenvalues**: you should find that the eigenvalues of A are 1 and -1.
  (b) Remember from Problem (1) of Conceptual Problems #5 that the linear transformaion T corresponding to A is supposed to be reflection across a plane S. Explain what the eigenvalues and eigenvectors from (a) mean geometrically. Why does it make sense that the eigenvalues are 1 and -1? What is the geometric interpretation of the eigenspaces? What is their relationship to the plane S?

(3) (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates a vector counterclockwise by $90°$. What is the 2×2 matrix A corresponding to T?

We will now see another way to think about T and A in terms of complex numbers. Remember that a complex number z is of the form $z = x + iy$, where x and y are real numbers and $i = \sqrt{-1}$. So we can think of $\mathbb{R}^2$ as being the same as the set $\mathbb{C}$ of complex numbers, via the correspondence $\begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow z = x + iy$.

  (b) If $\begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow z = x + iy$, what vector in $\mathbb{R}^2$ corresponds to iz? What is $T(\begin{bmatrix} x \\ y \end{bmatrix})$? Conclude that when written in terms of vectors in $\mathbb{R}^2$, the transformation $z \to iz$ is the linear transformation T.
  (c) Now let $w = a + ib$, where a and b are any given real numbers (e.g. $a = 0$ and $b = 1$ gives $w = i$ in part (b)). Find the matrix corresponding to the linear transformation $z \to wz$ when z and wz are written in terms of vectors in $\mathbb{R}^2$. (The textbook calls the resulting 2×2 matrix a rotation-dilation matrix because it rotates every vector by some angle and scales its length by some factor.)