Math 308 P Conceptual Problems #5

Due Friday, February 22

Please write your name and your quiz section (PA, PB, or PC) on your homework paper.

(1) Let S be a plane in $\mathbb{R}^3$ passing through the origin, so that S is a two-dimensional subspace of $\mathbb{R}^3$. There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ called the *reflection about* S which is defined as follows. If $\mathbf{v}$ is any vector in S, then $T(\mathbf{v}) = \mathbf{v}$. And if $\mathbf{n}$ is a vector which is perpendicular to S, then $T(\mathbf{n}) = -\mathbf{n}$.

Let T be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$, where A is the matrix

$$\frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

This linear transformation is the reflection about a plane S. Find an equation for S and find a basis for S.

Hint: Which matrix has null space equal to S? Which matrix has null space equal to $\operatorname{span}\{\mathbf{n}\}$?

(2) Let L be a line through the origin in $\mathbb{R}^2$, so that L is a one-dimensional subspace of $\mathbb{R}^2$. There is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ called the *projection onto* L which is defined as follows. If $\mathbf{v}$ is any vector in L, then $T(\mathbf{v}) = \mathbf{v}$. And if $\mathbf{n}$ is a vector which is perpendicular to L, then $T(\mathbf{n}) = 0$.

Let L be the line $\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : 3x_1 = x_2 \right\}$.

(a) Find a basis for $\ker(T)$ and find a basis for $\operatorname{range}(T)$.

(b) Find the matrix A of T, which satisfies $T(\mathbf{x}) = A\mathbf{x}$.

You can do this however you like, but here are some hints/steps that you might follow.

- Remember that $\operatorname{range}(T)$ is the span of the columns of A. So what must the columns of A look like?
- Pick a nonzero vector in $\ker(T)$. What does this tell you about A?
- Pick a nonzero vector $\mathbf{v}$ in L. What does the condition that $T(\mathbf{v}) = \mathbf{v}$ tell you about A?

(3) (a) Determine the dimension of the subspace S of $\mathbb{R}^4$ defined by

$$S = \{\mathbf{x} \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}.$$

(b) Add one or more vectors to the set $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ to obtain a basis for S.

Explain your reasoning.