Math 308 P Conceptual Problems #3

Due Wednesday, January 23

Please write your name and your quiz section (PA, PB, or PC) on your homework paper.

(1) When Jake works from $\vec{h}$ome, he typically spends 40 minutes of each hour on research, and 10 on teaching, and drinks half a cup of coffee. (The remaining time is spent on the internet.) For each hour he works in the math $\vec{d}$epartment, he spends around 20 minutes on research and 30 on teaching, and doesn't drink any coffee. Lastly, if he works at a $\vec{c}$offeeshop for an hour, he spends 25 minutes each on research and teaching, and drinks a cup of coffee.

(**Note:** be careful about units of minutes versus hours.)

(a) Last week, Jake spent 10 hours working from home, 15 hours working in his office in Padelford Hall, and 2 hours working at Cafe Allegro. Compute what was accomplished, and express the result as a vector equation.

(b) This week, Jake has 15 hours of research to work on and 10 hours of work related to teaching. He also wants 11 cups of coffee, because... of... very important reasons. How much time should he spend working from home, from his office, and from the coffeeshop?

(c) Describe the situation in part (b) as a vector equation and a matrix equation $A\vec{t} = \vec{w}$. What do the vectors $\vec{t}$ and $\vec{w}$ mean in this context? For which other vectors $\vec{w}$ does the equation $A\vec{t} = \vec{w}$ have a solution?

(d) Jake tries working in the math department $\vec{l}$ounge for an hour, and gets 30 minutes of research and 20 minutes of teaching work done, while having time to drink $\frac{1}{3}$ of a cup of coffee. Not bad. But Jake's colleague Vasu claims that there's no need to work in the lounge – the other options already give enough flexibility. Is he right? Explain mathematically.

(2) (a) The set

$$P = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 2x_1 - x_2 + 4x_3 = 0 \right\}$$

is a plane in $\mathbb{R}^3$. Find two vectors $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbb{R}^3$ so that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = P$. Explain your answer. Geometric reasoning is ok (preferred!).

(b) Consider the three vectors $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$. Use Gaussian elimination to show that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a plane in $\mathbb{R}^3$, and find an equation of the plane in the form $ax_1 + bx_2 + cx_3 = 0$.

(3) (a) Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} t \\ -3 \\ -7 \end{bmatrix}$. Find all values of t for which there is a unique solution to $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ for every vector $\mathbf{b}$ in $\mathbb{R}^3$. Explain your answer.

(b) Are the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ from part (a) linearly independent? Explain your answer.

(c) Let $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$ be as in (a). Let $\mathbf{a_4} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$. Without doing any further calculations, find all values of t for which there will be a unique solution to $y_1\mathbf{a_1} + y_2\mathbf{a_2} + y_3\mathbf{a_3} + y_4\mathbf{a_4} = \mathbf{c}$ for every vector $\mathbf{c}$ in $\mathbb{R}^3$. Explain your answer.

(4) Answer each of the questions below. **Give an example** if the answer is yes, or **explain why not** if the answer is no.

(a) Is there a set of vectors that does not span $\mathbb{R}^3$, but after adding one more vector, the set does span $\mathbb{R}^3$?

(b) Is there a set of vectors that is linearly dependent, but after adding one more vector, the set becomes linearly independent?

(c) Is there a set of vectors in $\mathbb{R}^3$ with the following properties? (Four different questions):

spans $\mathbb{R}^3$, linearly independent	spans $\mathbb{R}^3$, linearly dependent
doesn't span $\mathbb{R}^3$, linearly independent	doesn't span $\mathbb{R}^3$, linearly dependent

For each case that is possible, how many vectors could be in the set? (State any constraints, as in "there must be at least..." or "at most...")

(e) Is there a system of equations with a unique solution, but after adding another equation to the system, the new system has infinitely many solutions?

(f) Is there a system of equations without any solutions, but after deleting an equation, the system has infinitely many solutions?