1. (7 points) Compute the curvature of $\mathbf{r}(t) = \left\langle t^3,\ -\frac{1}{t},\ 3t \right\rangle$ at the point $(-1,1,-3)$.

$$\begin{aligned}
t &= -1 \\
\mathbf{r}'(t) &= \left\langle 3t^2,\ \frac{1}{t^2},\ 3 \right\rangle \\
\mathbf{r}'(-1) &= \langle 3,\ 1,\ 3 \rangle \\
\mathbf{r}''(t) &= \left\langle 6t,\ -\frac{2}{t^3},\ 0 \right\rangle \\
\mathbf{r}''(-1) &= \langle -6,\ 2, 0 \rangle \\
\mathbf{r}'(-1) \times \mathbf{r}''(-1) &= \langle -6,\ 18,\ 12 \rangle \\
\kappa &= \frac{|\mathbf{r}'(-1) \times \mathbf{r}''(-1)|}{|\mathbf{r}'(-1)|^3} \\
&= \frac{6\sqrt{14}}{\sqrt{19^3}} \\
&\approx 0.27
\end{aligned}$$

2. (7 points) A force with magnitude 14 N acts directly downward from the *xy*-plane on an object with mass 5 kg. The object starts at the point $(0,0,1)$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Find its position at time $t = 2$.

$$\begin{aligned}
F &= ma \\
14 &= 5a \\
a &= 2.8
\end{aligned}$$

$$\begin{aligned}
\mathbf{a}(t) &= -2.8\,\mathbf{k} \\
\mathbf{v}(t) &= -2.8t\,\mathbf{k} + \mathbf{v}(0) \\
&= \mathbf{i} - \mathbf{j} + (2 - 2.8t)\,\mathbf{k} \\
\mathbf{r}(t) &= t\,\mathbf{i} - t\,\mathbf{j} + (2t - 1.4t^2)\,\mathbf{k} + \mathbf{r}(0) \\
&= t\,\mathbf{i} - t\,\mathbf{j} + (1 + 2t - 1.4t^2)\,\mathbf{k} \\
\mathbf{r}(2) &= 2\,\mathbf{i} - 2\,\mathbf{j} - 0.6\,\mathbf{k}
\end{aligned}$$

The object is at the point $(2,-2,-0.6)$ at $t = 2$.

3. (7 points) Compute all first partial derivatives of the function $w = 3y^{x+z}$.

$$\begin{aligned}
\frac{\partial w}{\partial y} &= 3(x+z)\, y^{x+z-1} \\
\ln(w) &= \ln 3 + (x+z)\ln y \\
\frac{1}{w}\cdot\frac{\partial w}{\partial x} &= \ln y \\
\frac{\partial w}{\partial x} &= 3y^{x+z}\cdot \ln y \\
\frac{1}{w}\cdot\frac{\partial w}{\partial z} &= \ln y \\
\frac{\partial w}{\partial z} &= 3y^{x+z}\cdot \ln y
\end{aligned}$$

4. (7 points) Consider the surface given implicitly by $2xz^3 + yz + 2x^2y = 0$.
Compute the tangent plane to this surface at the point $(1,2,-1)$. Use this tangent plane to approximate the z-coordinate of a point on the surface where $x = 0.8$ and $y = 2.01$.

$$\begin{aligned}
0 &= 2z^3 + 6xz^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} + 4xy \\
0 &= -2 + 6\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial x} + 8 \\
\frac{\partial z}{\partial x} &= -\frac{3}{4} \\
0 &= 6xz^2\frac{\partial z}{\partial y} + z + y\frac{\partial z}{\partial y} + 2x^2 \\
0 &= 6\frac{\partial z}{\partial y} - 1 + 2\frac{\partial z}{\partial y} + 2 \\
\frac{\partial z}{\partial y} &= -\frac{1}{8} \\
0 &= -\frac{3}{4}(x-1) - \frac{1}{8}(y-2) - (z+1) \\
z &\approx -1 - \frac{3}{4}(0.8-1) - \frac{1}{8}(2.01-2) \\
&= -0.85125
\end{aligned}$$

5. (10 points) Find the points on the surface $y^2 = xz + 3x + 12$ that are closest to the origin. Justify that your answer is a minimum.

We wish to minimize $D = x^2 + y^2 + z^2$, subject to the constraint that $y^2 = xz + 3x + 12$.

Use the constraint to eliminate y from our objective D.

$$\begin{aligned} D &= x^2 + y^2 + z^2 \\ &= x^2 + xz + 3x + 12 + z^2 \end{aligned}$$

$$\begin{aligned} D_x &= 2x + z + 3 \\ D_y &= x + 2z \end{aligned}$$

$$\begin{aligned} 2x + z &= -3 \\ x + 2z &= 0 \end{aligned}$$

$$\begin{aligned} z &= 1 \\ x &= -2 \end{aligned}$$

$$\begin{aligned} y^2 &= (-2)(1) + 3(-2) + 12 = 4 \\ y &= -2, 2 \end{aligned}$$

There are two critical points: $(-2, 2, 1)$ and $(-2, -2, 1)$. Note that they are both 3 units away from the origin.

We use the Hessian to check that this is a minimum.

$$\begin{aligned} D_{xx} &= 2 \\ D_{xz} = D_{zx} &= 1 \\ D_{zz} &= 2 \end{aligned}$$

$$H = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

Since $H > 0$ and $D_{xx} > 0$, we conclude that the distance is a local minumum.

6. (12 total points) Evaluate the double integrals.

   (a) (6 points) $\displaystyle\int_0^1\int_{\pi y}^{\pi} \cos\left(x^2\right)\,dx\,dy$

$$\begin{aligned}
\int_0^1\int_{\pi y}^{\pi} \cos\left(x^2\right)\,dx\,dy &= \int_0^{\pi}\int_0^{\frac{1}{\pi}x} \cos\left(x^2\right)\,dy\,dx \\
&= \int_0^{\pi} \frac{1}{\pi} x\cos\left(x^2\right)\,dx \\
&= \frac{1}{2\pi}\sin\left(x^2\right)\,\Big|_0^{\pi} \\
&= \frac{1}{2\pi}\sin\left(\pi^2\right)
\end{aligned}$$

   (b) (6 points) $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

$$\begin{aligned}
\int_0^2\int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx &= \int_0^{\pi/2}\int_0^2 r\cdot r\,dr\,d\theta \\
&= \int_0^{\pi/2} d\theta\cdot\int_0^2 r^2\,dr \\
&= \frac{\pi}{2}\cdot\frac{8}{3} \\
&= \frac{4\pi}{3}
\end{aligned}$$