1. (8 points) Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

$$x = e^{-t}\cos t,\ y = e^{-t}\sin t,\ z = e^{-t} \quad (1,0,1)$$

Let $\mathbf{r}(t) = \langle e^{-t}\cos t,\ e^{-t}\sin t,\ e^{-t}\rangle$.

Then $\mathbf{r}'(t) = \langle -e^{-t}\cos t - e^{-t}\sin t,\ -e^{-t}\sin t + e^{-t}\cos t,\ -e^{-t}\rangle$.

At $(1,0,1)$, *we have* $1 = z = e^{-t}$ *so* $t = 0$. *(Check that this also works for x and y.)*

The direction vector of the tangent line is $\mathbf{r}'(0) = \langle -1,1,-1\rangle$.

The vector equation of the tangent line is $\langle x,y,z\rangle = \langle 1,0,1\rangle + t\cdot\langle -1,1,-1\rangle$.

The parametric equations are

$$\begin{cases} x = 1-t \\ y = t \\ z = 1-t \end{cases}$$

2. (8 points) Find symmetric equations for the line through the point $(2,0,-5)$ that is parallel to the plane $x-2y+z=3$ and perpendicular to the line $x = 3-t,\ y = 2+t,\ z = t-5$.

$\mathbf{N} = \langle 1,-2,1\rangle$ *is the normal vector to the plane* $x-2y+z=3$.

$\mathbf{v} = \langle -1,1,1\rangle$ *is the direction vector to the given line.*

The direction vector of the line we want is orthogonal to both $\mathbf{v}$ *and* $\mathbf{N}$.

Thus it is parallel to $\mathbf{v}\times\mathbf{N} = \langle 3,2,1\rangle$.

The vector equation of the line is

$$\langle x,y,z\rangle = \langle 2,0,5\rangle + t\cdot\langle 3,2,1\rangle$$

The symmetric equations are

$$\frac{x-2}{3} = \frac{y}{2} = z-5$$

3. (9 points) Find the length of the curve.

$$\mathbf{r}(t) = \frac{4}{3}t^{3/2}\,\mathbf{i} + 2t\,\mathbf{j} - \frac{1}{2}t^2\,\mathbf{k}$$

from $t = 0$ to $t = 4$.

$$\begin{aligned}
\mathbf{r}'(t) &= 2t^{1/2}\,\mathbf{i} + 2\,\mathbf{j} - t\,\mathbf{k} \\
\left|\mathbf{r}'(t)\right| &= \sqrt{4t + 4 + t^2} \\
&= \sqrt{(t+2)^2} \\
&= t + 2 \quad \textit{(magnitude is positive)} \\
s &= \int_0^4 \left|\mathbf{r}'(t)\right|\,dt \\
&= \int_0^4 t + 2\,dt \\
&= \frac{1}{2}t^2 + 2t\,\bigg|_0^4 \\
&= 16
\end{aligned}$$

4. (8 points) Find the cosines of the angles between the curves at their points of intersection.

$$y = 2x \quad \text{and} \quad y = -x^2 + 3x + 2$$

Find the points of intersection:

$$\begin{aligned} 2x &= -x^2 + 3x + 2 \\ 0 &= -x^2 + x + 2 \\ &= -(x-2)(x+1) \end{aligned}$$

The points are $(2,4)$ *and* $(-1,-2)$.

The tangent vector to $y = 2x$ *is constant* $\langle 1,2 \rangle$.
The tangent vector to $y = -x^2 + 3x - 2$ *is* $\langle 1, -2x+3 \rangle$.
At $(2,4)$, $\quad \cos\theta = \dfrac{\langle 1,2\rangle \cdot \langle 1,-1\rangle}{\sqrt{5}\cdot\sqrt{2}} = -\dfrac{1}{\sqrt{10}}$
At $(-1,-2)$, $\quad \cos\theta = \dfrac{\langle 1,2\rangle \cdot \langle 1,5\rangle}{\sqrt{5}\cdot\sqrt{26}} = \dfrac{11}{\sqrt{130}}$

5. (8 points) Find a vector function, $\mathbf{r}(t)$ that represents the curve of intersection of the following two surfaces: the hyperboloid $\quad 4x^2 + y^2 - z^2 = 1 \quad$ and the plane $\quad 2x - y - z = 0$.

First eliminate z:

$$\begin{aligned} z &= 2x - y \\ 1 &= 4x^2 + y^2 - (2x-y)^2 \\ &= 4xy \\ y &= \frac{1}{4x} \end{aligned}$$

Set $\quad x = t$. *Then* $\quad y = \dfrac{1}{4t} \quad$ *and* $\quad z = 2x - y = 2t - \dfrac{1}{4t}$.

The vector function is $\quad \mathbf{r}(t) = \left\langle t,\ \dfrac{1}{4t},\ 2t - \dfrac{1}{4t} \right\rangle$.

6. (9 points) Find an equation for the surface consisting of all points P for which the distance from P to the point $(0,0,-1)$ is equal to the distance from P to the plane $z=2$. Identify the surface.

   Write $P=(x,y,z)$.

   The distance from P to $(0,0,-1)$ *is* $\sqrt{(x-0)^2+(y-0)^2+(z+1)^2}$.

   $\mathbf{N}=\langle 0,0,1\rangle$ *is the normal vector of the plane* $z=2$.

   Choose any point Q on the plane. I'll use $Q=(0,0,2)$.

   The distance from P to the plane equals $\operatorname{comp}_{\mathbf{N}}\overrightarrow{QP}$

$$\begin{aligned}\overrightarrow{QP} &= \langle x,y,z-2\rangle\\ \operatorname{comp}_{\mathbf{N}}\overrightarrow{QP} &= \frac{\overrightarrow{QP}\cdot\mathbf{N}}{|\mathbf{N}|}\\ &= z-2\end{aligned}$$

Set the two distances equal.

$$\begin{aligned}z-2 &= \sqrt{x^2+y^2+(z+1)^2}\\ z^2-4z+4 &= x^2+y^2+z^2+2z+1\\ -6z+3 &= x^2+y^2\end{aligned}$$

   This is an elliptic paraboloid.