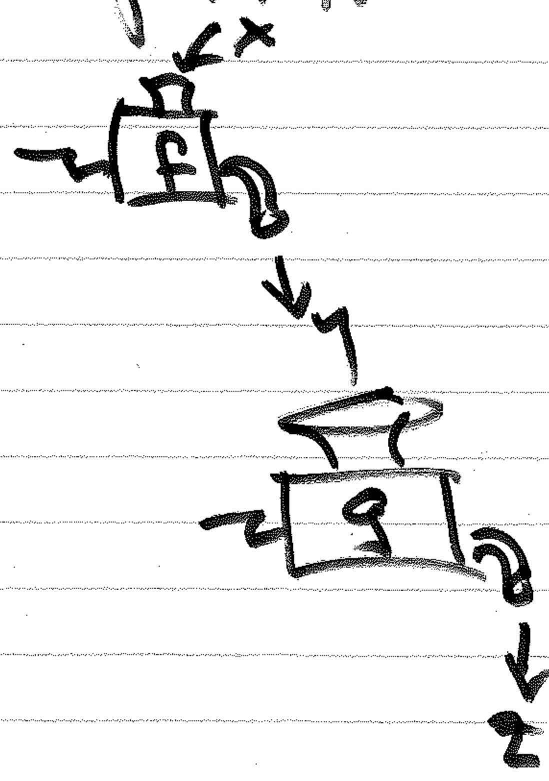
Chain Rule

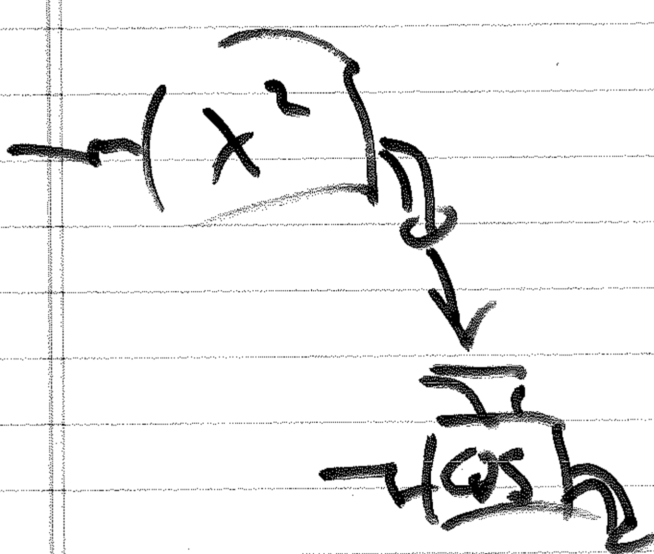
functions

$f+g \qquad f/g$

composition



Eg: $f(x) = \cos(x^2)$



$f(g(x)) \qquad f \circ g(x)$

$\cos^2 x \neq \cos x^2$

$(\cos x)^2$

$$\frac{d}{dx} f(g(x)) \Big|_{x=a}$$

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x-a} = (*)$$

theorem: Let f, g be <u>diff</u> at a

and f <u>diff</u> at g(a)

let $b = g(a)$

$u = g(x)$

$$\lim_{x\to a} \underbrace{\left[\frac{f(g(x)) - f(g(a))}{g(x)-g(a)}\right]}_{\to f'(b)} \cdot \underbrace{\frac{g(x)-g(a)}{x-a}}_{\to g'(a)}$$

linearization

$$f(u) \approx f(b) + f'(b)(u-b)$$

near $u=b$

x near a

$u = g(x)$ is near $b = g(a)$

need to compute

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}$$

$$\lim_{x \to a} \frac{f(u) - f(b)}{u - b}$$

$$\lim_{x \to a} \frac{\cancel{f(b)} + f'(b)\cancel{(u-b)} - \cancel{f(b)}}{\cancel{u-b}}$$

$$\lim_{x \to a} f'(b) = f'(b)$$

$$\frac{d}{dx} \quad \boxed{f'(b) \cdot g'(a)}$$

$$\frac{d}{dx} f(g(x)) =$$

$$f'(g(x)) \cdot g'(x)$$

$\cancel{y = f(x)}$ $\qquad u = g(x)$

$$y = f(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \cos^2 x = \frac{d}{dx} (\cos x)^2$$

Eg: $y = \sin(x^3)$

$$\frac{dy}{dx} = \cos(x^3) \cdot 3x^2$$

$$z = \cos^2 x + 2\cos x + 1$$

$$z = u^2 + 2u + 1$$

$$u = \cos x$$

$$\frac{dz}{dx} = 2\cos x \cdot {}^{-\sin x} - 2\sin x + 0$$

$$= -2\sin x \cos x - 2\sin x$$

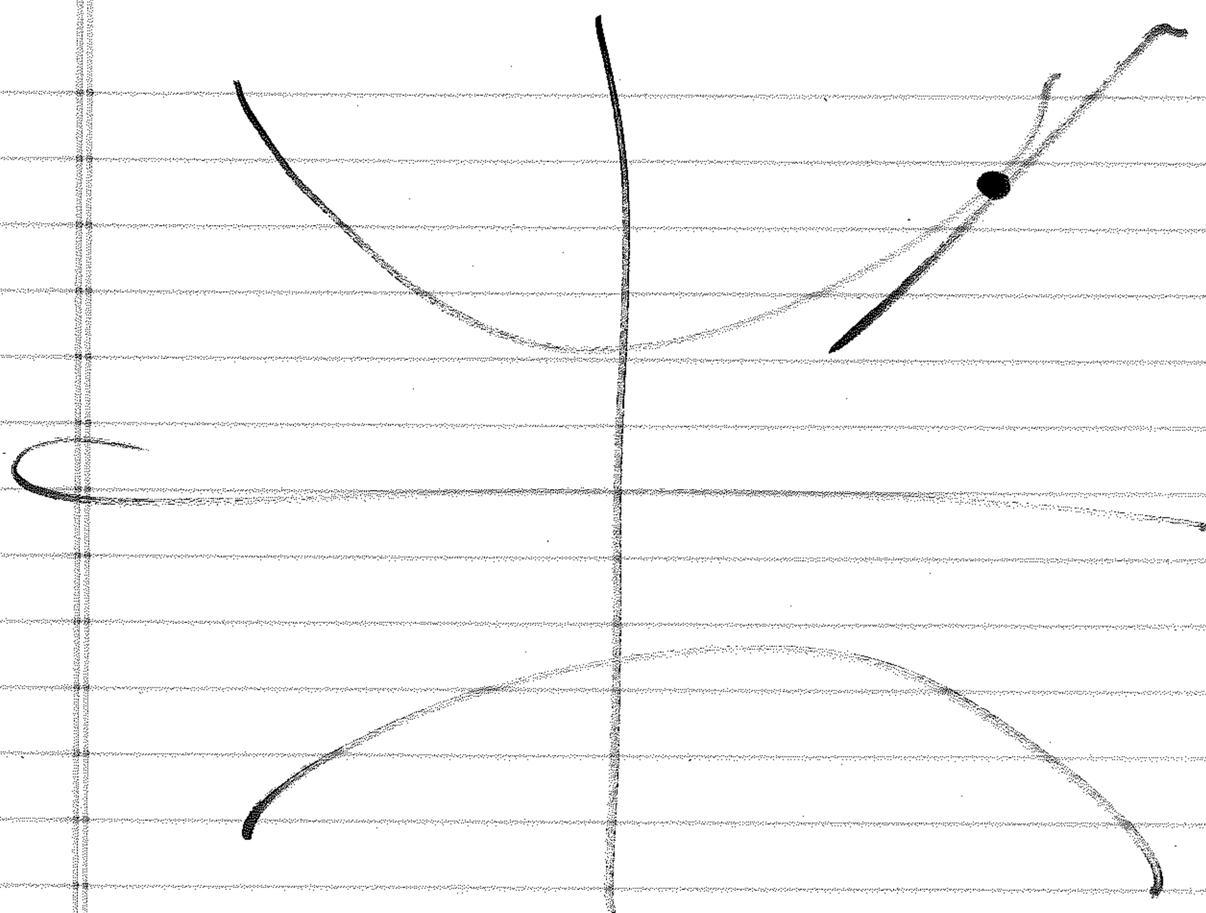
$$\cos x = \sin(\pi/2 - x)$$

$$\frac{d}{dx}\cos x = \cos\left(\frac{\pi}{2} - x\right) \cdot -1$$

$$= -\sin x$$

Eg: $y^2 - x^2 = 9$

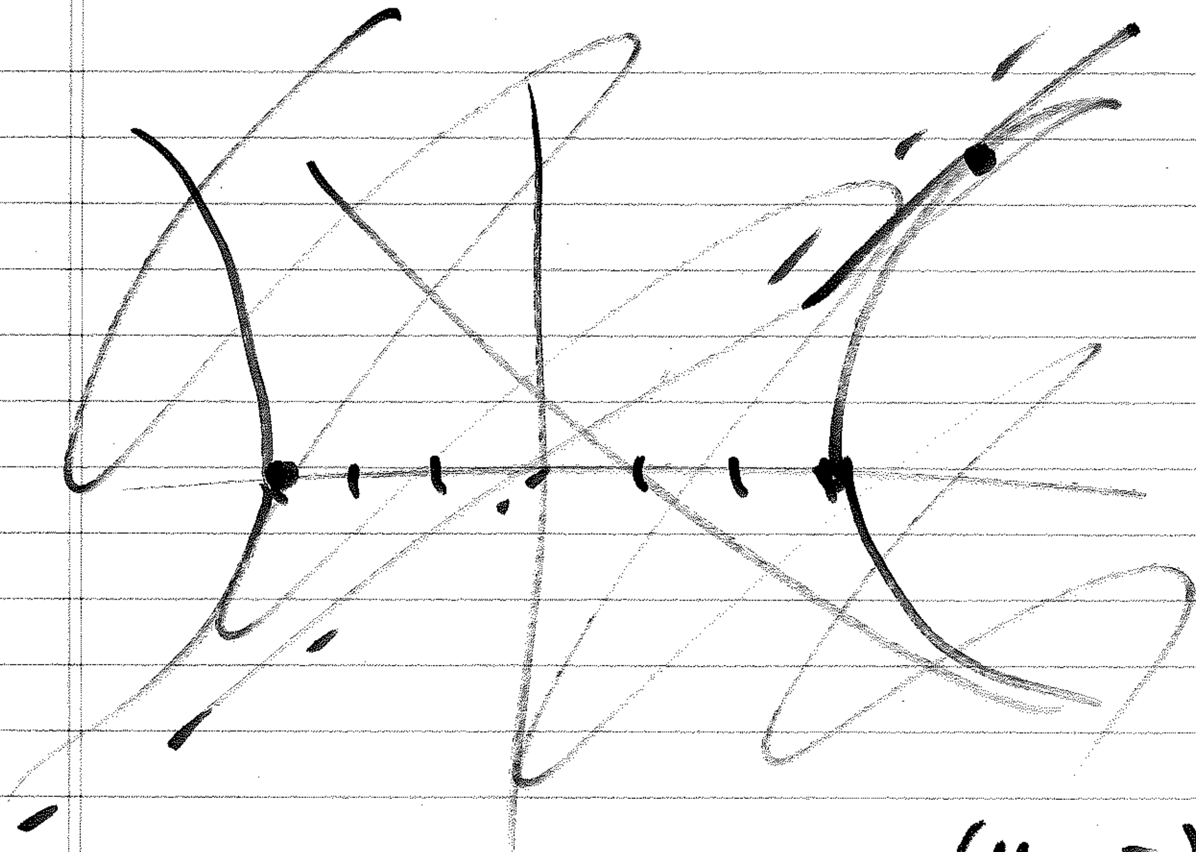
$(4, 5)$



$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+9}} \cdot 2x$$

$$= \frac{2x}{2\sqrt{x^2+9}} = \frac{x}{\sqrt{x^2+9}}$$

$$= \frac{x}{y}$$



$(4, 5)$

$$y^2 = x^2 + 9$$

$$y = \sqrt{x^2 + 9}$$

$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{4}{5}$$

Eg: $f(t) = 7 + 3\sin\left[\frac{\pi}{10}(t+3)\right]$

$$f'(t) = 3\cos\left[\frac{\pi}{10}(t+3)\right] \cdot \frac{\pi}{10}$$

Eg: $y = \sec\left(3x + \sqrt{x^2+1}\right)$

$$\frac{dy}{dx} = \sec\left(3x+\sqrt{\cdots}\right) \cdot \tan\left(3x+\sqrt{\cdots}\right) \cdot \left[3 + \frac{1}{2\sqrt{x^2+1}} \cdot 2x\right]$$

Eg: $\frac{d}{dx}\left(\frac{u}{v}\right)$

$= \frac{d}{dx}\left(u \cdot v^{-1}\right)$

$= u' \cdot v^{-1} + u \cdot \left(v^{-1}\right)'$

$-v^{-2} \cdot v'$

$\frac{d}{dx} x^{-1} = -x^{-2}$

$= \frac{u'}{v} - \frac{uv'}{v^2}$

Eg: $\frac{d}{dx} e^{\sqrt{\tan 3x}}$

$$= e^{\sqrt{\tan 3x}} \cdot \frac{1}{2\sqrt{\tan 3x}} \cdot \sec^2 3x \cdot 3$$