1. (12 total points) Compute the derivatives of the following functions. Do not simplify your answers.

   (a) (4 points) $f(t) = \dfrac{\ln(t^3+2)}{t^2+5t-1}$

   $$f'(t) = \frac{\frac{3t^2}{t^3+2} \cdot \left(t^2+5t-1\right) - (2t+5) \cdot \ln(t^3+2)}{\left(t^2+5t-1\right)^2}$$

   (b) (4 points) $y = (\cos x)^{\sin x}$

   We must use logarithmic differentiation.

   $$\begin{aligned} \ln y &= \ln(\cos x)^{\sin x} \\ &= \sin(x) \cdot \ln(\cos x) \\ \frac{1}{y} \cdot y' &= \cos(x) \cdot \ln(\cos x) - \sin(x) \cdot \frac{\sin x}{\cos x} \\ y' &= (\cos x)^{\sin x} \cdot \left[\cos(x) \cdot \ln(\cos x) - \sin(x) \cdot \tan(x)\right] \end{aligned}$$

   (c) (4 points) $g(y) = \tan^{-1}(ye^y)$

   $$g'(y) = \frac{1}{1+y^2e^{2y}} \cdot (e^y + ye^y)$$

2. (10 total points)Consider the curve given by the parametric equations

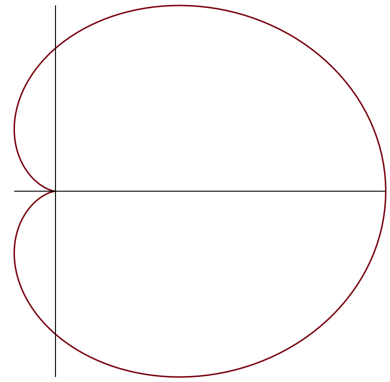
$$x = t^3 - 7t + 5,\ y = 2t^3 - 3t^2 + 3t$$

(a) (4 points) Find the equation of the tangent line to the curve when $t = -1$.

$$\frac{dy}{dt} = 6t^2 - 6t + 3$$
$$\frac{dx}{dt} = 3t^2 - 7$$
$$\left.\frac{dy}{dx}\right|_{t=-1} = \left.\frac{6t^2 - 6t + 3}{3t^2 - 7}\right|_{t=-1} = -\frac{15}{4}$$
$$x(-1) = 11, \quad y(-1) = -8$$
$$\textit{The line is} \quad y + 8 = -\frac{15}{4}(x - 11)$$

(b) (6 points) Find all times t when the tangent line has slope 3.

$$\begin{aligned} \frac{dy}{dx} &= 3 \\ \frac{6t^2 - 6t + 3}{3t^2 - 7} &= 3 \\ 6t^2 - 6t + 3 &= 9t^2 - 21 \\ 0 &= 3t^2 + 6t - 24 \\ &= 3(t-2)(t+4) \\ t = & \ -4, 2 \end{aligned}$$



3. (10 points) The curve with equation

$$x^2+y^2=\left(2x^2+2y^2-x\right)^2$$

is called the cardioid. Find the equation(s) of the tangent line(s) to the cardioid at the point(s) where $x=0$.

Plug in $x=0$ *to get* $y^2=\left(2y^2\right)^2$. *This gives* $4y^4-y^2=0$ *so* $y=-\frac{1}{2},0,\frac{1}{2}$.

But the curve is not differentiable at $(0,0)$. *(Look at the graph.)*

$$\begin{aligned} x^2+y^2 &= \left(2x^2+2y^2-x\right)^2 \\ 2x+2yy' &= 2\left(2x^2+2y^2-x\right)\cdot(4x+4yy'-1) \qquad \textit{plug in } x=0 \\ 2yy' &= 2\left(2y^2\right)\cdot(4yy'-1) \\ y' &= \frac{2y}{8y^2-1} \end{aligned}$$

The slope of the tangent line at $(0,\frac{1}{2})$ *is* 1.

The slope of the tangent line at $(0,-\frac{1}{2})$ *is* -1.

The lines are $y-\frac{1}{2}=x$ *and* $y+\frac{1}{2}=-x$

4. (6 points) Find the linearization of the function $f(x)=\sqrt{x^3+1}$ at $a=2$. Use it to approximate $f(1.98)$.

$f(2)=3$

$f'(x)=\dfrac{3x^2}{2\sqrt{x^3+1}}$

$f'(2)=\frac{12}{6}=2$

Thus, near $x=2$, $f(x)\approx 3+2(x-2)$.

$f(1.98)\approx 3+2(1.98-2)=2.96$

5. (12 total points) The length of a rectangle increases by 3 feet per minute while the width decreases by 2 feet per minute. When the length is 15 feet and the width is 8 feet, what is the rate at which the following changes. Make sure to state whether the rate is increasing or decreasing and include units.

   (a) (4 points) The area.

   Let x be the length and y be the width of the rectangle.
   $\frac{dx}{dt} = 3\,ft/min$ *and* $\frac{dy}{dt} = -2\,ft/min.$
   At the time of interest, $x = 15\,ft$ *and* $y = 8\,ft.$

$$\begin{aligned} A &= xy \\ \frac{dA}{dt} &= \frac{dx}{dt}\cdot y + x\cdot\frac{dy}{dt} \\ &= 3\cdot 8 - 15\cdot 2 \\ &= -6 \end{aligned}$$

   The area is decreasing at a rate of $6\,ft^2/min$.

   (b) (4 points) The perimeter.

$$\begin{aligned} P &= 2x + 2y \\ \frac{dP}{dt} &= 2\frac{dx}{dt} + 2\frac{dy}{dt} \\ &= 2\cdot 3 - 2\cdot 2 \\ &= 2 \end{aligned}$$

   The perimeter is increasing at a rate of $2\,ft/min$.

   (c) (4 points) The length of the diagonal.

$$\begin{aligned} \ell &= \sqrt{x^2 + y^2} \\ \frac{d\ell}{dt} &= \frac{2x\cdot\frac{dx}{dt} + 2y\cdot\frac{dy}{dt}}{2\sqrt{x^2+y^2}} \\ &= \frac{2\cdot 15\cdot 3 - 2\cdot 8\cdot 2}{2\sqrt{15^2+8^2}} \\ &= \frac{29}{17} \\ &\approx 1.706 \end{aligned}$$

   The length of the diagonal is increasing at a rate of $\frac{29}{17}$ *ft/min.*