1. Determine if the following limits exist. If they exist, compute them. Justify your answers.

   (a) (4 points) $\displaystyle\lim_{x\to 2}\frac{\sqrt{3x^2-8}-2}{x-2}$

   This is a $\frac{0}{0}$-type limit. We remove the singularity.

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{3x^2-8}-2}{x-2} &= \lim_{x\to 2}\frac{\sqrt{3x^2-8}-2}{x-2}\cdot\frac{\sqrt{3x^2-8}+2}{\sqrt{3x^2-8}+2}\\ &= \lim_{x\to 2}\frac{(3x^2-8)-4}{(x-2)\cdot\left(\sqrt{3x^2-8}+2\right)}\\ &= \lim_{x\to 2}\frac{3(x+2)}{\sqrt{3x^2-8}+2}\\ &= 3\end{aligned}$$

   (b) (4 points) $\displaystyle\lim_{t\to\infty}\tan^{-1}\left(\frac{t^2+1}{1+3t-5t^2}\right)$

$$\begin{aligned}\lim_{t\to\infty}\frac{t^2+1}{1+3t-5t^2} &= \lim_{t\to\infty}\frac{t^2+1}{1+3t-5t^2}\cdot\frac{1/t^2}{1/t^2}\\ &= \lim_{t\to\infty}\frac{1+\frac{1}{t^2}}{\frac{1}{t^2}+\frac{3}{t}-5}\\ &= -\frac{1}{5}\end{aligned}$$

   So $\displaystyle\lim_{t\to\infty}\tan^{-1}\left(\frac{t^2+1}{1+3t-5t^2}\right)=\tan^{-1}(-1/5)\approx -0.1974.$

   (c) (4 points) $\displaystyle\lim_{x\to 0}\frac{\cos(x)}{10x^2-x}$

   This is a $\frac{1}{0}$-type limit. We must determine if it is $+\infty$, $-\infty$ or DNE.
   Near $x=0$ the numerator is positive.
   $0=10x^2-x=x(10x-1)$, *so the denominator is 0 at $x=0$ and $x=1/10$.*
   Checking values, say $x=-1$ and $x=1/20$, shows that the denominator is positive when $x<0$ and negative when $0<x<1/10$.
   Thus $\displaystyle\lim_{x\to 0^-}\frac{\cos(x)}{10x^2-x}=\infty$ *and* $\displaystyle\lim_{x\to 0^+}\frac{\cos(x)}{10x^2-x}=-\infty.$
   The limit does not exist.

2. (8 points) Calculate the equation of the tangent line to $g(x) = \left|x^2 - 4x\right|$ at $x = 3$.

   Calculate $g(3) = |-3| = 3$.

   Write $g(x) = \begin{cases} -x^2 + 4x & \text{if } 0 < x < 4; \\ x^2 - 4x & \text{otherwise.} \end{cases}$

   Then $g'(x) = \begin{cases} -2x + 4 & \text{if } 0 < x < 4; \\ 2x - 4 & \text{if } x < 0 \text{ or } x > 4. \end{cases}$

   (Note that the derivative is undefined at $x = 0, 4$.)

   Thus $g'(3) = -2 \cdot 3 + 4 = -2$

   The equation of the line is $y - 3 = -2(x - 3)$.

3. (8 points) Find all the points (a, b) on the curve $y = \dfrac{e^x}{x^2 - 15}$ where the tangent line is horizontal.

   We must find all points on the curve where $\dfrac{dy}{dx} = 0$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x^2-15)\cdot e^x - (2x)\cdot e^x}{(x^2-15)^2} \\ 0 &= \frac{(x^2-15)\cdot e^x - (2x)\cdot e^x}{(x^2-15)^2} \\ 0 &= (x^2-15)e^x - (2x)e^x \\ &= (x^2 - 2x - 15)e^x \\ 0 &= x^2 - 2x - 15 \\ &= (x-5)(x+3) \end{aligned}$$

   The solutions are $x = -3, 5$

   The points are $\left(-3, -\frac{1}{6e^3}\right)$ *and* $\left(5, \frac{e^5}{10}\right)$.

   These can also be approximated as $(-3, -0.0083)$ *and* $(5, 14.84)$.

4. (10 points) Find the equations of all the tangent lines to the curve $y = x^2 + 3x$ that pass through the point $(2,1)$.

First find the points (a,b) on the curve where this happens.

Note that $\dfrac{dy}{dx} = 2x + 3.$

Since the point is on the curve, we have $b = a^2 + 3a.$

Since the line is a tangent line, we have $m = 2a + 3$

$$\begin{aligned} y - b &= m(x-a) \\ y - (a^2+3a) &= (2a+3)(x-a) \\ 1 - (a^2+3a) &= (2a+3)(2-a) \\ a^2 - 4a - 5 &= 0 \\ (a-5)(a+1) &= 0 \end{aligned}$$

The solutions are $a = -1, 5$

The points are $(-1,-2)$ *and* $(5,40)$.

The lines are $y + 2 = 1 \cdot (x+1)$ *and* $y - 40 = 13(x-5)$.

5. A bug is travelling along the x-axis so that its x-coordinate is given by the formula $x = \dfrac{1-t}{t+2}$. Here x is in feet and t is in seconds. Assume $t \geq 0$.

   (a) (4 points) Calculate the bug's average velocity between $t = 3$ and $t = 3.1$ seconds.

   Note that $x(3) = -\frac{2}{5} = -0.4$ *and* $x(3.1) = -\frac{7}{17} \approx -0.41176$

$$\begin{aligned} v_{\text{av}} &= \frac{x(3.1) - x(3)}{3.1 - 3} \\ &= \frac{-\frac{7}{17} + \frac{2}{5}}{0.1} \\ &\approx -0.1176\, \textit{feet/sec} \end{aligned}$$

   (b) (8 points) Find the bug's instantaneous velocity at time $t = 3$. Do not use any differentiation formulas in this problem. Use the limit definition of the derivative.

$$\begin{aligned} v(3) &= \lim_{h \to 0} \frac{x(3+h) - x(3)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1-3-h}{3+h+2} + \frac{2}{5}}{h} \\ &= \lim_{h \to 0} \frac{\frac{-2-h}{5+h} + \frac{2}{5}}{h} \cdot \frac{5(5+h)}{5(5+h)} \\ &= \lim_{h \to 0} \frac{5(-2-h) + 2(5+h)}{5h(5+h)} \\ &= \lim_{h \to 0} \frac{-3h}{5h(5+h)} \\ &= \lim_{h \to 0} -\frac{3}{5(5+h)} \\ &= -\frac{3}{25} \end{aligned}$$

The velocity is -0.12 *feet/sec.*