

1. (6 points) Let $f(x) = x^2 + 13x - 7^x$. Suppose we take the graph of $y = f(x)$ and do three things:

- First, expand it horizontally by a factor of 5.
- Then, shift it 7 units to the right.
- Then, shift it 3 units down.

Write a function $g(x)$ for this new transformed graph.

First do the expansion $f\left(\frac{1}{5}x\right)$.

Then shift this function to the right $f\left(\frac{1}{5}(x-7)\right)$.

Finally, shift down $f\left(\frac{1}{5}(x-7)\right) - 3$.

Thus $g(x) = \left(\frac{1}{5}(x-7)\right)^2 + 13\left(\frac{1}{5}(x-7)\right) - 7^{\frac{1}{5}(x-7)} - 3$

2. (6 points) Let $f(x) = 2x - 5$ and $h(x) = x^2 - 8$. Find the fixed points of $f(h(x))$.

$$\begin{aligned} f(h(x)) &= 2(x^2 - 8) - 5 \\ &= 2x^2 - 21 \end{aligned}$$

A number a is a fixed point of $f(h(x))$ if $f(h(a)) = a$.

$$\begin{aligned} 2a^2 - 21 &= a \\ 2a^2 - a - 21 &= 0 \\ (2a - 7)(a + 3) &= 0 \end{aligned}$$

Thus $a = -3, \frac{7}{2}$ are the two fixed points of $f(h(x))$.

3. (13 total points) Clovis is selling tickets to a concert by his rock band “Spent”. From past experience he knows he can sell 30 tickets if he charges \$12 a ticket. If he charges \$10 he can sell 40 tickets. Renting the hall and printing the tickets and posters costs him \$150.

(a) (5 points) Give a linear function $n = f(t)$ relating the number of tickets sold n to the price of a ticket t . How much money will he take in if he prices the tickets at \$8?

We compute the line through $(12, 30)$ and $(10, 40)$

$\Delta t = -2$ and $\Delta n = 10$ so $m = -5$.

$$n - 40 = -5(t - 10)$$

$$n = f(t) = -5t + 90$$

If he takes $t = 8$, he will sell $n = f(8) = -5 \cdot 8 + 90 = 50$ tickets.

Thus he will take in $8 \cdot 50 = 400$ dollars.

(b) (4 points) Give a function $p = g(t)$ relating Clovis’s profit p to the price of a ticket t . Remember to subtract his costs.

$$\begin{aligned} p = g(t) &= t \cdot f(t) - 150 \\ &= -5t^2 + 90t - 150 \end{aligned}$$

(c) (4 points) What is the greatest profit he can make?

First we use the vertex formula on $p = g(t) = -5t^2 + 90t - 150$ to find the ticket price that maximizes his profit.

$$\begin{aligned} t_{\max} &= -b/2a \\ &= -90/2(-5) \\ &= 9 \end{aligned}$$

The maximum profit is $p = g(9) = 255$ dollars.

4. (12 total points) Isobel bought a house in 2011 for \$450,000. In 2016, it was assessed (valued) at \$630,000. Assume the value of her house grows according to an exponential model.

(a) (5 points) Find a formula for the value $H(x)$ of the house in year $2011 + x$.

We use the model $H(x) = A \cdot b^x$.

Take the units of \$10,000 on the value of the house.

The model must pass through the points $(0, 45)$ and $(5, 63)$.

$$45 = H(0) = A \cdot b^0 = A$$

$$63 = H(5) = 45 \cdot b^5 \text{ so } b = \left(\frac{63}{45}\right)^{1/5} = \left(\frac{7}{5}\right)^{1/5}.$$

$$\text{Thus } H(x) = 45 \cdot \left(\frac{7}{5}\right)^{x/5}$$

(b) (2 points) Isobel wants to sell her house in 2019. What will the house be worth according to this model?

$$H(8) \approx 77.093 \text{ so the house should sell for about } \$770,000.$$

(c) (5 points) When will Isobel's house be worth twice what she paid for it?

We solve the equation $H(x) = 2 \cdot 45 = 90$

$$90 = 45 \cdot \left(\frac{7}{5}\right)^{x/5}$$

$$2 = \left(\frac{7}{5}\right)^{x/5}$$

$$\ln(2) = \ln \left[\left(\frac{7}{5}\right)^{x/5} \right]$$

$$= (x/5) \cdot \ln \left(\frac{7}{5}\right)$$

$$x = \frac{5 \cdot \ln(2)}{\ln \left(\frac{7}{5}\right)}$$

$$\approx 10.3$$

$x \approx 10$ so the year is 2021.

5. (13 total points) $f(x)$ is a **linear-to-linear rational function** such that the curve $y = f(x)$ passes through the points $(-1, -5)$ and $(7, 3)$ and has vertical asymptote $x = -5$.

(a) (8 points) Write a formula for $f(x)$.

We use the model $f(x) = \frac{ax+b}{x+c}$.

The vertical asymptote occurs when the denominator is 0. If $x+c=0$ then $x=-c$. Thus $-5 = -c$ so $c = 5$.

Our model now looks like $f(x) = \frac{ax+b}{x+5}$.

We must have $f(-1) = -5$ and $f(7) = 3$.

$$\begin{aligned} -5 &= f(-1) \\ &= \frac{-a+b}{-1+5} \\ a-b &= 20 \quad (1) \\ 3 &= f(7) \\ &= \frac{7a+b}{7+5} \\ 7a+b &= 36 \quad (2) \end{aligned}$$

Add equations (1) and (2) to get $8a = 56$, or $a = 7$.

Then (1) says $20 = a - b = 7 - b$, so $b = -13$.

Thus $f(x) = \frac{7x-13}{x+5}$.

- (b) (5 points) Write a formula for its inverse function, $f^{-1}(x)$.

First solve $y = f(x)$ for x .

$$\begin{aligned} y &= \frac{7x-13}{x+5} \\ y \cdot (x+5) &= 7x-13 \\ xy+5y &= 7x-13 \\ 5y+13 &= 7x-xy \\ &= x \cdot (7-y) \\ x &= \frac{5y+13}{7-y} \end{aligned}$$

Thus $f^{-1}(x) = \frac{5x+13}{7-y}$.