1. (6 points) Let $f(x) = x^2 + 13x - 7^x$. Suppose we take the graph of $y = f(x)$ and do three things:

   - First, expand it horizontally by a factor of 5.
   - Then, shift it 7 units to the right.
   - Then, shift it 3 units down.

   Write a function $g(x)$ for this new transformed graph.

   First do the expansion $\quad f\left(\frac{1}{5}x\right)$.

   Then shift this function to the right $\quad f\left(\frac{1}{5}(x-7)\right)$.

   Finally, shift down $\quad f\left(\frac{1}{5}(x-7)\right) - 3$.

   Thus $\quad g(x) = \left(\frac{1}{5}(x-7)\right)^2 + 13\left(\frac{1}{5}(x-7)\right) - 7^{\frac{1}{5}(x-7)} - 3$

2. (6 points) Let $f(x) = 2x - 5$ and $h(x) = x^2 - 8$. Find the fixed points of $f(h(x))$.

$$\begin{aligned} f(h(x)) &= 2\left(x^2 - 8\right) - 5 \\ &= 2x^2 - 21 \end{aligned}$$

   A number a is a fixed point of $f(h(x))$ if $f(h(a)) = a$.

$$\begin{aligned} 2a^2 - 21 &= a \\ 2a^2 - a - 21 &= 0 \\ (2a-7)(a+3) &= 0 \end{aligned}$$

   Thus $\quad a = -3,\ \frac{7}{2}$ $\quad$ *are the two fixed points of $f(h(x))$.*

3. (13 total points) Clovis is selling tickets to a concert by his rock band "Spent". From past experience he knows he can sell 30 tickets if he charges \$12 a ticket. If he charges \$10 he can sell 40 tickets. Renting the hall and printing the tickets and posters costs him \$150.

   (a) (5 points) Give a linear function $n = f(t)$ relating the number of tickets sold n to the price of a ticket t. How much money will he take in if he prices the tickets at \$8?

   We compute the line through $(12,30)$ *and* $(10,40)$
   $\Delta t = -2$ *and* $\Delta n = 10$ *so* $m = -5$.
   $n - 40 = -5(t - 10)$
   $n = f(t) = -5t + 90$

   If he takes $t = 8$, *he will sell* $n = f(8) = -5 \cdot 8 + 90 = 50$ *tickets.*
   Thus he will take in $8 \cdot 50 = 400$ *dollars.*

   (b) (4 points) Give a function $p = g(t)$ relating Clovis's profit p to the price of a ticket t. Remember to subtract his costs.

$$\begin{aligned} p = g(t) &= t \cdot f(t) - 150 \\ &= -5t^2 + 90t - 150 \end{aligned}$$

   (c) (4 points) What is the greatest profit he can make?

   First we use the vertex formula on $p = g(t) = -5t^2 + 90t - 150$ *to find the ticket price that maximizes his profit.*

$$\begin{aligned} t_{\text{max}} &= -b/2a \\ &= -90/2(-5) \\ &= 9 \end{aligned}$$

   The maximum profit is $p = g(9) = 255$ *dollars.*

4. (12 total points) Isobel bought a house in 2011 for $\$450,000$. In 2016, it was assessed (valued) at $\$630,000$. Assume the value of her house grows according to an exponential model.

   (a) (5 points) Find a formula for the value $H(x)$ of the house in year $2011+x$.

   We use the model $H(x) = A \cdot b^x$.
   Take the units of $\$10,000$ *on the value of the house.*
   The model must pass through the points $(0,45)$ *and* $(5,63)$.
   $45 = H(0) = A \cdot b^0 = A$

   $63 = H(5) = 45 \cdot b^5$ *so* $b = \left(\dfrac{63}{45}\right)^{1/5} = \left(\dfrac{7}{5}\right)^{1/5}$.

   Thus $H(x) = 45 \cdot \left(\dfrac{7}{5}\right)^{x/5}$

   (b) (2 points) Isobel wants to sell her house in 2019. What will the house be worth according to this model?

   $H(8) \approx 77.093$ *so the house should sell for about* $\$770,000$.

   (c) (5 points) When will Isobel's house be worth twice what she paid for it?

   We solve the equation $H(x) = 2 \cdot 45 = 90$

$$
\begin{aligned}
90 &= 45 \cdot \left(\frac{7}{5}\right)^{x/5} \\
2 &= \left(\frac{7}{5}\right)^{x/5} \\
\ln(2) &= \ln\left[\left(\frac{7}{5}\right)^{x/5}\right] \\
&= (x/5) \cdot \ln\left(\frac{7}{5}\right) \\
x &= \frac{5 \cdot \ln(2)}{\ln\left(\frac{7}{5}\right)} \\
&\approx 10.3
\end{aligned}
$$

   $x \approx 10$ *so the year is 2021.*

5. (13 total points) $f(x)$ is a **linear-to-linear rational function** such that the curve $y = f(x)$ passes through the points $(-1,-5)$ and $(7,3)$ and has vertical asymptote $x = -5$.

   (a) (8 points) Write a formula for $f(x)$.

   We use the model $f(x) = \dfrac{ax+b}{x+c}$.
   The vertical asymptote occurs when the denominator is 0. *If* $x+c=0$ *then* $x=-c$. *Thus* $-5=-c$ *so* $c=5$.
   Our model now looks like $f(x) = \dfrac{ax+b}{x+5}$.
   We must have $f(-1) = -5$ *and* $f(7) = 3$.

$$\begin{aligned} -5 &= f(-1) \\ &= \frac{-a+b}{-1+5} \\ a-b &= 20 \quad (\mathbf{1}) \\ 3 &= f(7) \\ &= \frac{7a+b}{7+5} \\ 7a+b &= 36 \quad (\mathbf{2}) \end{aligned}$$

   Add equations **(1)** *and* **(2)** *to get* $8a = 56$, *or* $a = 7$.
   Then **(1)** *says* $20 = a - b = 7 - b$, *so* $b = -13$.
   Thus $f(x) = \dfrac{7x-13}{x+5}$.

   (b) (5 points) Write a formula for its inverse function, $f^{-1}(x)$.

   First solve $y = f(x)$ *for x.*

$$\begin{aligned} y &= \frac{7x-13}{x+5} \\ y\cdot(x+5) &= 7x-13 \\ xy+5y &= 7x-13 \\ 5y+13 &= 7x-xy \\ &= x\cdot(7-y) \\ x &= \frac{5y+13}{7-y} \end{aligned}$$

   Thus $f^{-1}(x) = \dfrac{5x+13}{7-y}$.