Socles of Buchsbaum modules, complexes and posets.

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Abstract

The socle of a graded Buchsbaum module is studied and is related to its local cohomology modules. This algebraic result is then applied to face enumeration of Buchsbaum simplicial complexes and posets. In particular, new necessary conditions on face numbers and Betti numbers of such complexes and posets are established. These conditions are used to settle in the affirmative Kühnel's conjecture for the maximum value of the Euler characteristic of a $2k$-dimensional simplicial manifold on n vertices as well as Kalai's conjecture providing a lower bound on the number of edges of a simplicial manifold in terms of its dimension, number of vertices, and the first Betti number.

1 Introduction

In this paper we prove several long standing conjectures on the face numbers of simplicial manifolds, and more generally Buchsbaum complexes. This is done via studying socles of graded Buchsbaum modules and relating them to local cohomology modules.

A basic invariant of a simplicial complex Δ (or a simplicial poset, see Section 6) is its f-vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$, where $d-1$ is the dimension of Δ and f_i is the number of its i-dimensional faces. One of the fundamental problems in geometric combinatorics is to characterize, or at least to obtain significant new necessary conditions, on the f-vectors of various classes of complexes. Here we study this problem for the class of

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Buchsbaum simplicial complexes and posets, and especially its subclass of complexes and posets representing manifolds. We start by discussing the history of the problem and describing our main results. All definitions are deferred to later sections.

Thirty years ago the pioneering work of Stanley and Hochster (see Chapter 2 of [37]) made the study of combinatorics of simplicial complexes inseparable from the study of monomial ideals and graded algebras. Their insight was to associate with every simplicial complex a certain graded ring, known today as the face ring or the Stanley-Reisner ring, and to read various combinatorial and topological invariants/properties of the complex off of the algebraic invariants of that ring.

Call a simplicial complex Δ Cohen-Macaulay, resp. Buchsbaum, if its Stanley-Reisner ring is Cohen-Macaulay, resp. Buchsbaum. Reisner [31], building on (then unpublished) work of Hochster, gave a purely combinatorial-topological characterization of Cohen-Macaulay complexes, while Stanley worked out a complete characterization of f-vectors of Cohen-Macaulay complexes [35], and later of f-vectors of Cohen-Macaulay simplicial posets [36]. In [33], Schenzel analyzed general Buchsbaum rings and modules and used these algebraic results to generalize both Reisner's result to a combinatorial-topological characterization of Buchsbaum complexes and the necessity portion of Stanley's result to certain necessary conditions on the f-vectors and Betti numbers of Buchsbaum complexes.

One motivation for the study of f-vectors of Cohen-Macaulay, resp. Buchsbaum, complexes came from the desire to extend McMullen's upper bound theorem [24] (UBT, for short) which provided sharp upper bounds on the face numbers of polytopes in terms of their dimension and the number of vertices, to the class of simplicial spheres and, more generally, Eulerian simplicial manifolds. That such an extension does hold was conjectured by Klee [16], and proved by Stanley [34] for the case of spheres, and then by Novik [28] for several classes of simplicial manifolds including all Eulerian ones. Novik's proof relied on Schenzel's results and on the method of algebraic shifting introduced by Kalai, see e.g. [13]. The main ingredient of the proof was a certain strengthening of Schenzel's conditions on the f-vectors and Betti numbers of Buchsbaum complexes.

In this paper we strengthen these conditions even further – see Theorems 3.4 and 4.3, verifying in the affirmative a part of Kalai's conjecture on the dimensions of certain kernels in the Stanley-Reisner rings of simplicial manifolds [13, Conjecture 36]. To derive these conditions we establish a new commutative algebra result, see Theorems 2.2 and 2.4, that relates the socle of a general Buchsbaum module to its local cohomology modules, a result that we hope will be of interest in its own right. The same algebraic theorem is then used to show that the lower bound part of our conditions on the f-vectors and Betti numbers also applies to all Buchsbaum simplicial posets, (Theorem 6.4). Based on the situation in dimensions up to four, we believe that these lower bounds provide a complete characterization of the f-vectors of Buchsbaum simplicial posets with prescribed Betti numbers.

Related to the UBT is a conjecture by Kühnel [18, Conjecture B] for the maximum value of the Euler characteristic of a $2k$-dimensional simplicial manifold on n vertices. This conjecture was previously known to hold only for manifolds with at least $4k+3$ or at most $3k+3$ vertices [28, 29]. Here, in Theorem 4.4 we prove it for all values of n.

In [12], Kalai conjectured a lower bound for the number of edges of a simplicial manifold in terms of its dimension, number of vertices, and first Betti number. This conjecture was verified by Swartz [40] for manifolds whose first Betti number is one, as well as for orientable manifolds of dimension at least four with vanishing second Betti number. In Section 5 we prove this conjecture for all manifolds.

The structure of the paper is as follows. In Section 2, after providing the necessary background on Buchsbaum modules, we state and prove our main algebraic result, Theorem 2.2, on which all other theorems of this paper are based. Section 3 contains an overview of simplicial complexes and their Stanley-Reisner rings as well as a combinatorial-topological translation of Theorem 2.2 for the case of Buchsbaum simplicial complexes. Section 4 is devoted to deriving new upper bounds on the face numbers of Buchsbaum simplicial complexes and in particular includes the proof of Kühnel's conjecture. In Section 5 we prove Kalai's lower bound conjecture. In Section 6 we study f-vectors of Buchsbaum simplicial posets. Finally, in Section 7 we discuss several examples and open problems.

2 Socles in terms of local cohomology

In this section we state and prove our main algebraic result concerning the socle of a Buchsbaum module. This theorem is the key to all the combinatorial applications discussed in the rest of the paper.

We start by reviewing necessary background material. For all undefined terminology as well as for more details the reader is referred to [38]. Let $\mathbf{k}$ be an **infinite** field of an arbitrary characteristic and let $S := \mathbf{k}[x_1, \ldots, x_n]$ be a polynomial ring. We denote by $\mathfrak{M}$ the irrelevant ideal of S, and by $\mathfrak{M}_j$ the jth homogeneous component of $\mathfrak{M}$. All modules considered in this paper are Noetherian ($\mathbb{Z}$-)graded modules over S.

Let M be a Noetherian graded S-module of Krull dimension $d \geq 0$. A *homogeneous system of parameters* of M, abbreviated h.s.o.p, is a sequence $\theta_1, \cdots, \theta_d$ of homogeneous elements of $\mathfrak{M}$ such that $\dim M/(\theta_1, \cdots, \theta_d)M = 0$ (equivalently, $M/(\theta_1, \cdots, \theta_d)M$ is a finite-dimensional vector space over $\mathbf{k}$). A h.s.o.p. all of whose elements belong to $\mathfrak{M}_1$ is called a *linear system of parameters*, l.s.o.p. for short. It follows from the Noether Normalization Lemma that a l.s.o.p. always exists. A sequence of elements $\theta_1, \cdots, \theta_r \in \mathfrak{M}$ is a *weak M-sequence* if for each $i = 1, \ldots, r$

$$(\theta_1, \cdots, \theta_{i-1})M : \theta_i = (\theta_1, \cdots, \theta_{i-1})M : \mathfrak{M}.$$

Our main object of study is the class of *Buchsbaum modules*. Following Definition 3.1 on page 95 of [38] combined with Theorem 3.7 on page 97, we say that a Noetherian graded S-module M is *Buchsbaum* if every h.s.o.p. of M is a weak M-sequence. Since any regular M-sequence is also a weak M-sequence, all Cohen-Macaulay modules are Buchsbaum. A large family of Buchsbaum modules most of which are not Cohen-Macaulay is given by the face rings of triangulated manifolds — see Section 3.

The following lemma summarizes several basic properties of Buchsbaum modules we will rely on frequently. Here $H^i(M)$ denotes the ith local cohomology of M with respect to $\mathfrak{M}$. In particular,

$$H^0(M) = 0 : \mathfrak{M}^\infty = \{y \in M \mid \mathfrak{M}^k y = 0 \text{ for some } k > 0\}$$

is a submodule of M. The modules $H^i(M)$ are graded provided M is.

Lemma 2.1 *Let M be a graded Noetherian S-module of Krull dimension $d \geq 0$. If M is Buchsbaum and $\theta_1, \ldots, \theta_r$ is a part of a h.s.o.p. for M, then*

1. *$M/(\theta_1, \ldots, \theta_r)M$ is a Buchsbaum module of Krull dimension $d - r$,*
2. *$(\theta_1, \ldots, \theta_{r-1})M : \mathfrak{M} = (\theta_1, \ldots, \theta_{r-1})M : \theta_r = (\theta_1, \ldots, \theta_{r-1})M : \theta_r^2$, and*
3. *$\mathfrak{M} \cdot H^i(M) = 0$ for all $0 \leq i < d$.*

All parts of the lemma can be found in [38]: for (1) see Corollary 1.11 on page 65, for (2) see Proposition 1.10 on pages 64-65, and for (3) see Corollary 2.4 on page 75.

Recall that the *socle* of a module M is

$$\operatorname{Soc} M := 0 : \mathfrak{M} = \{y \in M \mid \mathfrak{M} \cdot y = 0\}.$$

We are now in a position to state our main theorem relating the socle to the local cohomology modules. We denote by M_k the kth homogeneous component of a graded module M, and by rM the direct sum of r copies of M.

Theorem 2.2 *Let M be a Noetherian graded S-module of Krull dimension d, and let $\theta_1, \ldots, \theta_d$ be a l.s.o.p. If M is Buchsbaum, then for all integers k,*

$$(\operatorname{Soc} M/(\theta_1, \ldots, \theta_d)M)_k \cong \left(\bigoplus_{j=0}^{d-1} \binom{d}{j} H^j(M)_{k-j} \right) \bigoplus \mathcal{SB}_{k-d},$$

where $\mathcal{SB}$ is a graded submodule of $\operatorname{Soc} H^d(M)$.

We begin the proof with the following lemma. For a graded module M, $M(-a)$ is the same module, but with grading $M(-a)_k = M_{k-a}$.

Lemma 2.3 *If M is a Buchsbaum S-module and θ is a part of a l.s.o.p. for M, then*

1. *$H^i(\theta M) \cong H^i(M(-1))$ for all $i > 0$, and*
2. *the map $\iota^* : H^i(\theta M) \to H^i(M)$ induced by inclusion $\iota : \theta M \hookrightarrow M$ is the zero map for all $0 \leq i < \dim M$.*

Proof: We verify both claims simultaneously. Let $N = M/H^0(M)$. Consider the following diagram and the induced local cohomology diagram:

$$\begin{array}{ccc} \theta M & \xrightarrow{\iota} & M \\ f\big\downarrow & & \big\downarrow q \\ N(-1) & \xrightarrow{\cdot\theta} & N \end{array} \qquad\qquad \begin{array}{ccc} H^i(\theta M) & \xrightarrow{\iota^*} & H^i(M) \\ f^*\big\downarrow & & \big\downarrow q^* \\ H^i(N(-1)) & \xrightarrow{\cdot\theta} & H^i(N). \end{array}$$

Here ι is the inclusion map, $q : x \mapsto x + H^0(M)$ is the quotient map, and f is the map defined by $\theta x \mapsto x + H^0(M)$. To see that f is well-defined, suppose that $\theta x = \theta y$. Then $\theta(x - y) = 0$, and so, by the definition of a Buchsbaum module, $\mathfrak{M} \cdot (x - y) = 0$. Hence $x - y \in H^0(M)$. We claim that f (and hence also f^*) is an isomorphism. It is surjective since for any $x \notin H^0(M)$, $\theta x \neq 0$. To show injectivity, assume $x \in H^0(M)$. Then $\mathfrak{M}^l \cdot x = 0$ for some $l > 0$. In particular, $\theta^l x = 0$, which, by Part (2) of Lemma 2.1, implies that $\theta x = 0$.

The map f was chosen to make the diagram on the left commute. By naturality of local cohomology, the induced diagram also commutes. Now, since $H^0(M)$ has Krull dimension 0, $H^i(H^0(M)) = 0$ for all $i > 0$, and so q^* is an isomorphism for $i > 0$. If also $i < \dim M$, then Part (3) of Lemma 2.1 implies that the bottom horizontal map in the induced diagram is the zero map, and we conclude that $\iota^* = 0$ for all $0 < i < \dim M$. For $i = 0$, another application of Part (2) of Lemma 2.1 shows that $H^0(\theta M) = 0$, and so $\iota^* = 0$ in this case as well. This completes the proof of the second claim, while the string of isomorphisms

$$H^i(\theta M) \xrightarrow{f^*} H^i(N(-1)) \xrightarrow{(q^*)^{-1}} H^i(M(-1)) \quad \text{for } i > 0$$

implies the first claim. $\square$

It is convenient to introduce the following notation: for subsets A, C of $\{1, 2, \ldots, d\} = [d]$, write θ^C to denote $\prod_{i \in C} \theta_i$, and write $M(A)$ to denote $M/(\theta_i : i \notin A)M$. In particular, $M(\emptyset) = M/(\theta_1, \ldots, \theta_d)M$ and $M([d]) = M$. By repeated application of Lemma 2.3(1), to prove Theorem 2.2 it is enough to verify the following. (We distinguish between strict and non-strict inclusions by using symbols '$\subset$' and '$\subseteq$', respectively.)

Theorem 2.4 *Let M be a Noetherian graded S-module of Krull dimension d, and let $\theta_1, \ldots, \theta_d$ be a l.s.o.p. If M is Buchsbaum, then*

$$\mathrm{Soc}\,(M(\emptyset)) \cong \left(\bigoplus_{C \subset [d]} H^{|C|}(\theta^C M) \right) \bigoplus \mathcal{SB},$$

where $\mathcal{SB}$ is a graded submodule of $\mathrm{Soc}\, H^d(\theta^{[d]} M)$.

The proof of Theorem 2.4 involves "chasing" a few commutative diagrams. Our starting point is the short exact sequence

$$0 \to \theta_s \cdot \theta^C M(A) \xrightarrow{\iota} \theta^C M(A) \xrightarrow{\pi_s} \theta^C M(A \setminus s) \to 0,$$

where $A \subseteq [d], C \subset A$, $s \in A \backslash C$, ι is the inclusion map, and π_s is the projection map. (The subscript s indicates that π_s maps a module to its quotient by the submodule generated by θ_s.) We refer to such a sequence as an (A, C, s)-sequence. It gives rise to the long exact local cohomology sequence, where we denote by ϕ^*_s the connecting homomorphism:

$$\cdots \to H^i(\theta^{C\cup s}M(A)) \stackrel{\iota^*}{\to} H^i(\theta^C M(A)) \stackrel{\pi^*_s}{\to} H^i(\theta^C M(A \setminus s)) \stackrel{\phi^*_s}{\to} H^{i+1}(\theta^{C\cup s}M(A)) \stackrel{\iota^*}{\to} \cdots .$$

If M is Buchsbaum, then, as follows from Lemma 2.1(1), $M(A)$ is also Buchsbaum and has Krull dimension $|A|$. Thus, by Lemma 2.3, $\iota^* : H^i(\theta^{C\cup s}M(A)) \to H^i(\theta^C M(A))$ is the zero map provided $i < |A|$. The above long exact sequence then breaks into the following short exact sequences:

$$0 \to H^i(\theta^C M(A)) \stackrel{\pi^*_s}{\hookrightarrow} H^i(\theta^C M(A \setminus s)) \stackrel{\phi^*_s}{\to} H^{i+1}(\theta^{C\cup s}M(A)) \to 0, \quad \text{for all } i < |A| - 1 \quad (1)$$

$$0 \to H^{|A|-1}(\theta^C M(A)) \stackrel{\pi^*_s}{\hookrightarrow} H^{|A|-1}(\theta^C M(A \setminus s)) \stackrel{\phi^*_s}{\to} H^{|A|}(\theta^{C\cup s}M(A)) \stackrel{\iota^*}{\to} H^{|A|}(\theta^C M(A)). \quad (2)$$

For $A = \{1\}$, $s = 1$, and $C = \emptyset$, (2) becomes

$$0 \to H^0(M(\{1\})) \stackrel{\pi^*_1}{\hookrightarrow} H^0(M(\emptyset)) \stackrel{\phi^*_1}{\to} H^1(\theta_1 M(\{1\})) \stackrel{\iota^*}{\to} H^1(M(\{1\})). \quad (3)$$

Since the Krull dimension of $M(\{1\})$ is one, the image of π^*_1 is contained in the socle of $H^0(M(\emptyset)) = M/(\theta_1, \ldots, \theta_d)M$. This submodule of the socle can be analyzed using (1).

Lemma 2.5 *Let M be a Buchsbaum module of Krull dimension d. If $C \subset A \subseteq [d]$, and $i < |A|$, then*

$$H^i(\theta^C M(A)) \cong \bigoplus_{D \subseteq [d] \setminus A} H^{i+|D|}(\theta^{C\cup D}M). \quad (4)$$

Proof: The proof is by induction on $d - |A|$. If $d - |A| = 0$, then $A = [d]$, and the lemma is equivalent to $H^i(\theta^C M) \cong H^i(\theta^C M)$. For the induction step, the short exact sequence (1) implies that for $s \in [d] \setminus A$

$$H^i(\theta^C M(A)) \cong H^i(\theta^C M(A \cup s)) \oplus H^{i+1}(\theta^{C\cup s}M(A \cup s)).$$

The induction hypothesis applied to the two terms on the right-hand side finishes the proof. □

Corollary 2.6 *Let M be a Buchsbaum module of Krull dimension d. Then*

$$H^0(M(\{1\}) \cong \bigoplus_{C \subseteq [d] \setminus \{1\}} H^{|C|}(\theta^C M).$$

In view of (3), we have accounted for those terms of the direct sum in Theorem 2.4 such that $1 \notin C$. To finish the proof we examine the image of the socle of $H^{|A|-1}(M(A \setminus s))$ under ϕ_s^* in (2), then specialize to the case $A = \{1\}$ and $s = 1$.

If $r \in A \setminus (C \cup s)$, then the (A, C, s)-sequence and the $(A \setminus r, C, s)$-sequence can be combined together to form the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \theta^{C \cup s} M(A \setminus r) & \xrightarrow{\iota} & \theta^C M(A \setminus r) & \xrightarrow{\pi_s} & \theta^C M(A \setminus \{r, s\}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \pi_r} & & \uparrow{\scriptstyle \pi_r} & & \uparrow{\scriptstyle \pi_r} & & \\
0 & \longrightarrow & \theta^{C \cup s} M(A) & \xrightarrow{\iota} & \theta^C M(A) & \xrightarrow{\pi_s} & \theta^C M(A \setminus s) & \longrightarrow & 0.
\end{array}$$

Naturality of local cohomology then implies that the diagram whose rows consist of the corresponding long exact local cohomology sequences and all of whose vertical maps are induced by π_r also commutes. This observation together with equations (1) and (2) yields the following.

Lemma 2.7 *Let M be a Buchsbaum module of Krull dimension d. If $C \subset A \subseteq [d]$, $s \in A \setminus C$, and $r \in A \setminus (C \cup s)$, then for all $i < |A| - 1$ we have the following commutative diagram whose rows are exact*

$$\begin{array}{ccccccc}
H^i(\theta^C M(A \setminus r)) & \xhookrightarrow{\pi_s^*} & H^i(\theta^C M(A \setminus \{r, s\})) & \xrightarrow{\phi_s^*} & H^{i+1}(\theta^{C \cup s} M(A \setminus r)) & & \\
 & & \uparrow{\scriptstyle \pi_r^*}\ \text{hook} & & \uparrow{\scriptstyle \pi_r^*}\ \text{hook} & & \\
 & & H^i(\theta^C M(A \setminus s)) & \xrightarrow{\phi_s^*} \!\!\!\!\rightarrow & H^{i+1}(\theta^{C \cup s} M(A)) & \longrightarrow & 0
\end{array}$$

Recall that our goal is to compute the image of $\operatorname{Soc} H^{|A|-1}(M(A \setminus s))$ under ϕ_s^* in (2). We do this by induction with the following lemma allowing the inductive step.

Lemma 2.8 *With the assumptions of Lemma 2.7, for all $i < |A| - 1$, the preimage of $\pi_r^* \left(H^{i+1}(\theta^{C \cup s} M(A)) \right)$ under ϕ_s^* is contained in the socle of $H^i(\theta^C M(A \setminus \{r, s\}))$.*

Proof: Let $y \in H^{i+1}(\theta^{C \cup s} M(A))$. By surjectivity of ϕ_s^* (see the diagram of Lemma 2.7), there exists $x \in H^i(\theta^C M(A \setminus s))$ such that $\phi_s^*(x) = y$. Since $\dim(M(A \setminus s)) = |A| - 1 > i$ it follows from Lemma 2.1, that all elements of $H^i(\theta^C M(A \setminus s))$, including x, are in the socle of $H^i(\theta^C M(A \setminus s))$, and hence $\pi_r^*(x) \in \operatorname{Soc} H^i(\theta^C M(A \setminus \{r, s\}))$. But the diagram of Lemma 2.7 commutes, and so $\pi_r^*(x) \in (\phi_s^*)^{-1}(\pi_r^*(y))$. We have proved that each element $y \in H^{i+1}(\theta^{C \cup s} M(A))$ has a representative $\tilde{y} \in (\phi_s^*)^{-1}(\pi_r^*(y))$ that lies in $\operatorname{Soc} H^i(\theta^C M(A \setminus \{r, s\}))$. Now choose a $\mathbf{k}$-basis $B = \{y_1, \ldots, y_t\}$ for $H^{i+1}(\theta^{C \cup s} M(A))$ and let $\tilde{B} = \{\tilde{y_1}, \ldots, \tilde{y_t}\}$ be a set of their representatives in the pull-back that lie in the socle. By Lemma 2.7,

$$(\phi_s^*)^{-1} \pi_r^* \left(H^{i+1}(\theta^{C \cup s} M(A)) \right) = \pi_s^*(H^i(\theta^C M(A \setminus r))) \oplus \operatorname{Span}(\tilde{B}),$$

and the assertion follows, since $\operatorname{Span}(\tilde{B})$ is in the socle by the choice of $\tilde{B}$, and the first summand of the above decomposition is also in the socle by Lemma 2.1(3). □

Using a fixed r and varying values of s in Lemma 2.7, we can chain the corresponding commutative squares together to obtain that for subsets $T = T_k = \{s_1, \ldots, s_k\}$, A, and C of $[d]$ satisfying $T \subseteq A \setminus C$ and $r \in A \setminus T$, and for $i < |A| - k$, the following diagram with $\phi_T^* := \phi_{s_1}^* \circ \cdots \circ \phi_{s_k}^*$ commutes:

$$\begin{array}{ccccccc}
\ker \phi_T^* & \hookrightarrow & H^i(\theta^C M(A \setminus (T \cup r)) & \xrightarrow{\phi_T^*} & H^{i+k}(\theta^{C\cup T} M(A \setminus r)) & & \\
 & & \uparrow \pi_r^* & & \uparrow \pi_r^* & & \\
 & & H^i(\theta^C M(A \setminus T)) & \xrightarrow{\phi_T^*} & H^{i+k}(\theta^{C\cup T} M(A)) & \longrightarrow & 0
\end{array}$$

Here $\ker \phi_T^* = \oplus_{j=1}^k (\phi_{s_{j+1}}^* \circ \ldots \circ \phi_{s_k}^*)^{-1} \pi_{s_j}^* H^{i+k-j}(\theta^{C \cup \{s_{j+1},\ldots,s_k\}} M(A \setminus (T_{j-1} \cup r)))$, so the same argument as in Lemma 2.8 plus induction on k then implies

Lemma 2.9 *For all $i < |A| - |T|$, the preimage of $\pi_r^* H^{i+|T|}(\theta^{C\cup T} M(A))$ under ϕ_T^* is contained in the socle of $H^i(\theta^C M(A \setminus (T \cup r)))$.*

Now consider the following diagram.

$$\begin{array}{ccc}
 & & H^0(M(\emptyset)) \\
 & & \downarrow \phi_1^* \\
H^1(\theta^{[1]} M([2])) & \overset{\pi_2^*}{\hookrightarrow} & H^1(\theta^{[1]} M([1])) \\
 & & \downarrow \phi_2^* \\
H^2(\theta^{[2]} M([3])) & \overset{\pi_3^*}{\hookrightarrow} & H^2(\theta^{[2]} M([2])) \\
 & & \downarrow \phi_3^* \\
 & & \vdots \\
 & & \downarrow \phi_{d-1}^* \\
H^{d-1}(\theta^{[d-1]} M([d])) & \overset{\pi_d^*}{\hookrightarrow} & H^{d-1}(\theta^{[d-1]} M([d-1]) \\
 & & \downarrow \phi_d^* \\
 & & H^d(\theta^{[d]} M([d])).
\end{array}$$

Lemma 2.9 shows that for all $j,\ 1 \leq j \leq d-1$,

$$(\phi_j^* \circ \cdots \circ \phi_1^*)^{-1} \pi_{j+1}^*\ H^j(\theta^{[j]} M([j+1]))$$

lies in the socle of $H^0(M(\emptyset))$. Using Lemma 2.5 on each $H^j(\theta^{[j]} M([j+1]))$ accounts for all of the terms of the direct sum decomposition in Theorem 2.4 with $1 \in C$. Setting $\mathcal{SB} = \phi_d^* \circ \cdots \circ \phi_1^*(\mathrm{Soc}\, H^0(M(\emptyset))$ finishes the proof of Theorem 2.4. □

Remark 2.10 Instead of graded Buchsbaum S-modules, one can work in the generality of Buchsbaum modules over Noetherian local rings, see Definition 1.5 on page 63 in [38]. A proof identical to that of Theorem 2.4 then shows that if M is a Noetherian module of Krull dimension d over a local ring A, and $\theta_1, \ldots, \theta_d$ is a system of parameters of M, then

$$\mathrm{Soc}\,(M(\emptyset)) \cong \left(\bigoplus_{C \subset [d]} H^{|C|}(\theta^C M) \right) \bigoplus \mathcal{SB},$$

provided M is a Buchsbaum module. Here $\mathcal{SB}$ is a certain submodule of $\operatorname{Soc} H^d(\theta^{[d]}M)$.

3 Simplicial complexes and Stanley-Reisner rings

This section provides a short overview of several concepts and results related to simplicial complexes and their Stanley-Reisner rings. A comprehensive reference to this topic is Chapter 2 of [37]. The section concludes with a combinatorial-topological translation of Theorem 2.2 for the case of Buchsbaum complexes and resulting new lower bounds on their face numbers.

A *simplicial complex* Δ on the vertex set $[n]$ is a collection of subsets of $[n]$ that is closed under inclusion and contains all singletons $\{i\}$ for $i \in [n]$. The elements of Δ are called *faces* and the maximal faces (with respect to inclusion) are called *facets*. The *dimension of a face* $F \in \Delta$ is $\dim F := |F| - 1$. The *dimension of* Δ is then defined as the maximal dimension of its faces. A simplicial complex is called *pure* if all its facets have the same dimension.

If Δ is a simplicial complex and F is a face of Δ, then the *link* of F in Δ, $\operatorname{lk}(F)$, is the following subcomplex of Δ: $\operatorname{lk}_\Delta(F) := \{G \in \Delta \mid G \cap F = \emptyset \text{ and } G \cup F \in \Delta\}$. Thus the link of the empty face is the complex itself.

A basic combinatorial invariant of a simplicial complex Δ on the vertex set $[n]$ is its *f-vector*, $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$. Here $d - 1 = \dim \Delta$ and f_i is the number of i-dimensional faces of Δ. In particular, $f_{-1} = 1$ and $f_0 = n$. An invariant that contains the same information as the f-vector, but sometimes is more convenient to work with, is the *h-vector* of Δ, $h(\Delta) = (h_0, h_1, \ldots, h_d)$ whose entries are defined by

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}. \tag{5}$$

The Stanley-Reisner ring of a simplicial complex provides an important algebraic tool for studying f-numbers of simplicial complexes. If Δ is a simplicial complex on $[n]$, then its *Stanley-Reisner ring* (also called the *face ring*) is

$$\mathbf{k}[\Delta] = S/I_\Delta := \mathbf{k}[x_1, \ldots, x_n]/I_\Delta, \quad \text{where } I_\Delta = (x_{i_1}x_{i_2}\cdots x_{i_k} : \{i_1 < i_2 < \cdots < i_k\} \notin \Delta).$$

The ideal I_Δ is called the *Stanley-Reisner ideal of* Δ. (As in the previous section, here and throughout the paper, we assume that $\mathbf{k}$ is an infinite field of an arbitrary characteristic.) Since I_Δ is a monomial ideal, defining $\deg(x_i) = 1$ for all $1 \le i \le n$ makes $\mathbf{k}[\Delta]$ into a $\mathbb{Z}$-graded ring, while defining $\deg(x_i) = e_i$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{Z}^n$, makes $\mathbf{k}[\Delta]$ into a $\mathbb{Z}^n$-graded ring.

The utmost significance of Stanley-Reisner rings in the theory of f-numbers is explained by the fact that many combinatorial and topological properties of Δ translate to certain algebraic properties of $\mathbf{k}[\Delta]$ and vice versa. For instance, the Krull dimension of $\mathbf{k}[\Delta]$ (as a module over itself or over S) equals $\dim \Delta + 1$, while the *($\mathbb{Z}$-)Hilbert series of*

$\mathbf{k}[\Delta]$, $F(\mathbf{k}[\Delta],x) := \sum_{j=0}^{\infty} \dim_{\mathbf{k}} \mathbf{k}[\Delta]_j x^j$ can be expressed in terms of the h-vector of Δ:

$$F(\mathbf{k}[\Delta],x) = (1-x)^{-d} \sum_{i=0}^{d} h_i x^i, \quad \text{where } d = \dim \Delta + 1. \tag{6}$$

(Both results can be found in [34] or on pages 33, 54, and 58 of [37].) Moreover, the local cohomology of $\mathbf{k}[\Delta]$ (as a module over itself or over S) has a simple expression in terms of simplicial homology of the links of the faces of Δ. This result is known as *Hochster's formula*, see [37, Theorem II.4.1].

Theorem 3.1 (Hochster) *For a simplicial complex Δ, $\alpha \in \mathbb{Z}^n$, $F = \{j \in [n] \mid \alpha_j \neq 0\}$, and $i \geq 0$,*

$$H^i(\mathbf{k}[\Delta])_\alpha \cong \begin{cases} 0, & \text{if } F \notin \Delta \text{ or } \alpha_j > 0 \text{ for some } j \in [n] \\ \widetilde{H}_{i-|F|-1}(\operatorname{lk} F; \mathbf{k}), & \text{otherwise,} \end{cases}$$

where $\widetilde{H}_i$ denotes the ith reduced simplicial homology with coefficients in $\mathbf{k}$.

Among the main objects of this paper are Cohen-Macaulay and Buchsbaum simplicial complexes. A simplicial complex Δ is called *Cohen-Macaulay* (over $\mathbf{k}$), if $\mathbf{k}[\Delta]$ is Cohen-Macaulay (considered as a module over itself or over S). Similarly, Δ is called *Buchsbaum* (over $\mathbf{k}$), if $\mathbf{k}[\Delta]$ is Buchsbaum.

Using Hochster's formula, Reisner [31] gave a purely combinatorial-topological characterization of Cohen-Macaulay complexes. His criterion was later generalized by Schenzel [33] to the class of Buchsbaum complexes. We combine both these results in the following theorem.

Theorem 3.2 *Let Δ be a simplicial $(d-1)$-dimensional complex. Then Δ is Cohen-Macaulay (over $\mathbf{k}$) if and only if $\widetilde{H}_i(\operatorname{lk} F; \mathbf{k}) = 0$ for all $F \in \Delta$, including $F = \emptyset$, and all $i < d - |F| - 1$. Δ is Buchsbaum (over $\mathbf{k}$) if and only if it is pure and the link of each vertex is Cohen-Macaulay (over $\mathbf{k}$).*

A simplicial $(d-1)$-dimensional complex Δ is a *($\mathbf{k}$-)homology sphere* if it is Cohen-Macaulay (over $\mathbf{k}$) and $\dim_{\mathbf{k}} \widetilde{H}_{d-|F|-1}(\operatorname{lk} F; \mathbf{k}) = 1$ for all $F \in \Delta$. The complex Δ is a ($\mathbf{k}$-)homology manifold if the links of all its vertices are $(d-2)$-dimensional homology spheres. Thus, all $\mathbf{k}$-homology manifolds are Buchsbaum over $\mathbf{k}$. The class of homology spheres includes all triangulations of topological spheres, which we refer to as simplicial spheres. Similarly, the class of homology manifolds includes all triangulations of topological manifolds — called simplicial manifolds.

Let Δ be a $(d-1)$-dimensional simplicial complex. For the rest of the paper we denote by (Θ) the ideal of $\mathbf{k}[\Delta]$ generated by the elements $\theta_1, \ldots, \theta_d$ of a l.s.o.p. for $\mathbf{k}[\Delta]$. What is the Hilbert series of $\mathbf{k}[\Delta]/(\Theta)$? For Cohen-Macaulay complexes the answer was given by Stanley. Schenzel [33] then generalized it to the Buchsbaum case.

To state Schenzel's result, for a $(d-1)$-dimensional complex Δ and $0 \le j \le d$, define

$$h'_j(\Delta) := h_j + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \beta_{i-1}(\Delta), \quad \text{where } \beta_{i-1}(\Delta) = \dim_{\mathbf{k}} \widetilde{H}_{i-1}(\Delta; \mathbf{k}). \tag{7}$$

Note that if Δ is Cohen-Macaulay, then $h'_j(\Delta) = h_j(\Delta)$.

Theorem 3.3 (Schenzel) *Let Δ be a $(d-1)$-dimensional Buchsbaum complex and let $\theta_1, \ldots, \theta_d$ be a l.s.o.p. for $\mathbf{k}[\Delta]$. Then*

$$\dim_{\mathbf{k}}(\mathbf{k}[\Delta]/(\Theta))_j = h'_j(\Delta), \quad \textit{for all } 0 \le j \le d.$$

Using Hochster's formula and Schenzel's theorem, we can now derive a combinatorial-topological version of our Theorem 2.2 for the case of Buchsbaum complexes as well as new lower bounds on their face numbers. This material concludes this section.

Theorem 3.4 *Let Δ be a $(d-1)$-dimensional Buchsbaum complex and let $\theta_1, \ldots, \theta_d$ be a l.s.o.p. for $\mathbf{k}[\Delta]$. Then for all $0 \le j \le d$,*

$$\dim_{\mathbf{k}} (\operatorname{Soc} \mathbf{k}[\Delta]/(\Theta))_j \ge \binom{d}{j} \beta_{j-1}(\Delta).$$

In particular, $h'_j(\Delta) \ge \binom{d}{j}\beta_{j-1}(\Delta)$, or, equivalently, $h_j \ge \binom{d}{j} \sum_{i=1}^{j} (-1)^{j-i} \beta_{i-1}(\Delta)$.

Proof: For $1 \le j \le d$, we have

$$\dim_{\mathbf{k}}(\operatorname{Soc} \mathbf{k}[\Delta]/(\Theta))_j \ge \binom{d}{j} \dim H^j(\mathbf{k}[\Delta])_0 = \binom{d}{j} \beta_{j-1}(\Delta),$$

where the first step follows from Theorem 2.2 and the second one from Hochster's formula. Since, $\operatorname{Soc} \mathbf{k}[\Delta]/(\Theta) \subseteq \mathbf{k}[\Delta]/(\Theta)$, Theorem 3.3 completes the proof. □

Corollary 3.5 *Let Δ be a $(d-1)$-dimensional Buchsbaum complex, let $\theta_1, \ldots, \theta_d$ be a l.s.o.p. for $\mathbf{k}[\Delta]$, and let $\omega \in \mathfrak{M}_1$ be a linear form. Then for all $0 < j \le d$,*

$$\dim_{\mathbf{k}} \left(\ker \left(\mathbf{k}[\Delta]/(\Theta)_j \xrightarrow{\cdot \omega} \mathbf{k}[\Delta]/(\Theta)_{j+1} \right) \right) \ge \binom{d}{j} \beta_{j-1}(\Delta). \tag{8}$$

Proof: Use Theorem 3.4 and the fact that $\ker(\cdot\omega) \supseteq \operatorname{Soc} \mathbf{k}[\Delta]/(\Theta)$. □

Corollary 3.5 settles in the affirmative a part of [13, Conjecture 36] — the conjecture that served as main motivation and starting point for this paper.

4 Upper bounds on Buchsbaum complexes

In this section we use Theorem 3.4 to derive new upper bounds on the face numbers of Buchsbaum simplicial complexes. As an application, we prove Kühnel's conjecture on the Euler characteristic of even-dimensional manifolds.

Somewhat surprisingly, to describe the upper bounds on the f-numbers of simplicial complexes, one needs the notion of a multicomplex. A *multicomplex* $\mathcal{M}$ is a subset of monomials, say in variables $x_1, \ldots, x_{n-d}$, that is closed under division, i.e. if $\mu \in \mathcal{M}$ and $\nu|\mu$, then also $\nu \in \mathcal{M}$. For a multicomplex $\mathcal{M}$, we denote by $\mathcal{M}_j$ the set of its elements of degree j, and by $F_j = F_j(\mathcal{M})$ the cardinality of $\mathcal{M}_j$. We refer to $F(\mathcal{M}) := (F_0, F_1, \ldots)$ as the F-vector of $\mathcal{M}$.

The F-vectors of multicomplexes were completely characterized by Macaulay [23] (see also [37, Theorem II.2.2]). Given two positive integers l and j there exists a unique expression of l in the form

$$l = \binom{n_j}{j} + \binom{n_{j-1}}{j-1} + \cdots + \binom{n_s}{s}, \qquad \text{where } n_j > n_{j-1} > \cdots > n_s \geq s \geq 1. \tag{9}$$

Define

$$l^{<j>} := \binom{n_j+1}{j+1} + \binom{n_{j-1}+1}{j} + \cdots + \binom{n_s+1}{s+1}.$$

We say that R is a *standard graded* **k**-*algebra* if it is a $\mathbb{Z}$-graded **k**-algebra with $R_i = 0$ for $i < 0$, $R_0 \cong \mathbf{k}$ and is generated as an algebra by R_1 with $\dim_{\mathbf{k}} R_1 < \infty$. Equivalently, as a **k**-algebra, $R \cong \mathbf{k}[x_1, \ldots, x_m]/I$ for some homogeneous ideal I. The Hilbert function of such an R is the sequence $(\dim_{\mathbf{k}} R_0, \dim_{\mathbf{k}} R_1, \ldots)$.

Theorem 4.1 (Macaulay) *Let $\mathcal{F} = (F_0, F_1, \ldots)$ be a sequence of nonnegative integers. The following are equivalent:*

- *$\mathcal{F}$ is the F-vector of a multicomplex.*
- *$F_0 = 1$ and $0 \leq F_{j+1} \leq F_j^{<j>}$ for $j \geq 1$.*
- *$\mathcal{F}$ is the Hilbert function of a standard graded* **k**-*algebra.*

Using Theorems 3.3 and 4.1, Stanley [35, Theorem 6] characterized all possible h-vectors of Cohen-Macaulay simplicial complexes.

Theorem 4.2 (Stanley) *A vector $h = (h_0, h_1, \ldots, h_d) \in \mathbb{Z}^{d+1}$ is the h-vector of a $(d-1)$-dimensional Cohen-Macaulay complex on n vertices if and only if $h_0 = 1$, $h_1 = n-d$, and $0 \leq h_{j+1} \leq h_j^{<j>}$ for $1 \leq j \leq d-1$.*

A generalization of the necessity portion of Theorem 4.2 for Buchsbaum complexes was given in [28, Theorem 1.7], where it was shown that if Δ is a $(d-1)$-dimensional Buchsbaum complex, then its h'-vector, $(h'_0, h'_1, \ldots, h'_d)$, (as defined in (7)) satisfies

$$h'_{j+1} \leq \left(h_j - \binom{d-1}{j}\beta_{j-1}(\Delta)\right)^{<j>}, \qquad \text{for } 1 \leq j \leq d-1. \tag{10}$$

The first result of this section is to use Theorem 3.4 to strengthen the above inequalities.

Theorem 4.3 *Let Δ be a $(d-1)$-dimensional Buchsbaum complex on n vertices. Then $h'_0 = 1$, $h'_1 = n-d$, and*

$$h'_{j+1} \leq \left(h_j - \binom{d}{j}\beta_{j-1}(\Delta)\right)^{<j>}, \quad \text{for } 1 \leq j \leq d-1.$$

Proof: Let $I = \operatorname{Soc} \mathbf{k}[\Delta]/(\Theta)$. Since I is killed by $\mathfrak{M}$ (or in other words, the S-module structure on I is trivial), any vector subspace of I is an ideal of $\mathbf{k}[\Delta]/(\Theta)$. In particular, I_j is an ideal, so $(\mathbf{k}[\Delta]/(\Theta))/I_j$ is a standard graded $\mathbf{k}$-algebra. Let $(F_0, F_1, \dots, F_d, 0, \dots)$ be its Hilbert function. By Theorem 3.3 and Theorem 3.4, $F_j \leq h'_j - \binom{d}{j}\beta_{j-1}(\Delta)$ and $F_{j+1} = h'_{j+1}$. Macaulay's theorem finishes the proof. $\square$

The inequalities (10) served in [28] as a key to extending the Upper Bound Theorem for polytopes and spheres (UBT, for short) to several classes of orientable homology manifolds (among them, the class of all odd-dimensional homology manifolds and the class of all even-dimensional homology manifolds of Euler characteristic 2). This theorem, originally proved by McMullen [24] for polytopes and later extended by Stanley [34] to homology spheres, asserts that among all d-dimensional polytopes on n vertices, the cyclic polytope, $C_d(n)$, has componentwise maximal f-vector.

A conjecture related to the UBT was proposed by Kühnel [18, Conjecture B]. It asserts that if a simplicial complex Δ is a (combinatorial) $2k$-dimensional manifold (without boundary) on n vertices, then its Euler characteristic, $\chi(\Delta) := \sum_{j=0}^{2k}(-1)^j f_j = 1 + \sum_{j=0}^{2k}(-1)^j \beta_j(\Delta)$, satisfies

$$(-1)^k \binom{2k+1}{k}(\chi(\Delta) - 2) \leq \binom{n-k-2}{k+1}. \tag{11}$$

Moreover, equality happens if and only if Δ is $(k+1)$-*neighborly*, that is, every $k+1$ vertices of Δ form the vertex set of a face of Δ.

While inequalities (10) (together with Klee's extension of the Dehn-Sommerville relations - Theorem 5.1 [15]) were strong enough to imply the UBT for several classes of homology manifolds, they were insufficient to completely prove the Kühnel conjecture, which was verified in [28] and [29] only for $2k$-dimensional orientable $\mathbf{k}$-homology manifolds with at least $4k+3$ or at most $3k+3$ vertices. (Paper [28] treated the case of $\operatorname{char} \mathbf{k} = 0$, while [29] dealt with a field of an arbitrary characteristic.) However, it was observed in [28] (see proof of Theorem 7.6 there) that if the inequalities of Theorem 4.3 were true, they would imply Kühnel's conjecture for all n. Thus we now have

Theorem 4.4 *Kühnel's conjecture holds for all orientable $2k$-dimensional $\mathbf{k}$-homology manifolds. In particular, Kühnel's conjecture holds for all simplicial $2k$-manifolds.*

Proof: For completeness we include here a sketch of the proof for $n \geq 3k+4$ case. We set $d = 2k+1$ and let $N_r := \binom{n-d+r-1}{r}$. A weaker version of Macaulay's theorem asserts that

$$\text{if } a \leq N_r, \text{ then } a^{<r>} \leq \frac{N_{r+1}}{N_r}\, a, \tag{12}$$

and equality is attained if and only if $a = N_r$.

If Theorem 4.3 is applied j times and inequality (12) is also used j times, then we obtain that for a $2k$-dimensional $\mathbf{k}$-homology manifold Δ and $1 \le j \le d-1$,

$$h'_{j+1} \le N_{j+1} - \sum_{i=1}^{j} \frac{N_{j+1}}{N_i} \binom{d}{i} \beta_{i-1}(\Delta). \tag{13}$$

Moreover, equality is attained if and only if $\beta_{i-1}(\Delta) = 0$ for all $i \le j$ and $h'_{j+1} = N_{j+1}$. This, in turn, is equivalent to $h_{j+1} = N_{j+1}$, which happens if and only if Δ is $(j+1)$-neighborly.

Now let $z = N_k - h'_k$. Then (13) yields that

$$z \ge \sum_{i=1}^{k-1} \frac{N_k}{N_i} \binom{d}{i} \beta_{i-1}(\Delta), \tag{14}$$

which together with Theorem 4.3 and (12) implies

$$h'_{k+1} \le N_{k+1} - \frac{N_{k+1}}{N_k} z - \frac{N_{k+1}}{N_k} \binom{d}{k} \beta_{k-1}(\Delta),$$

and equality is attained if and only if Δ is $(k+1)$-neighborly. Subtracting equation $h'_k = N_k - z$ from the above inequality, and using the fact that for $2k$-dimensional orientable $\mathbf{k}$-homology manifolds, $h'_{k+1} - h'_k = \binom{d}{k}(\beta_k(\Delta) - \beta_{k-1}(\Delta))$ (see [28, Lemma 5.1]), we infer that

$$\binom{d}{k}(\beta_k(\Delta) - \beta_{k-1}(\Delta)) \le (N_{k+1} - N_k) - \left(\frac{N_{k+1}}{N_k} - 1\right) z - \frac{N_{k+1}}{N_k} \binom{d}{k} \beta_{k-1}(\Delta).$$

This inequality combined with inequality (14) reduces to

$$\beta_k(\Delta) + \left(\frac{N_{k+1}}{N_k} - 1\right) \beta_{k-1}(\Delta) + \sum_{i=1}^{k-1} \frac{N_{k+1} - N_k}{N_i} \cdot \frac{\binom{d}{i}}{\binom{d}{k}} \beta_{i-1}(\Delta) \le \frac{N_{k+1} - N_k}{\binom{d}{k}}. \tag{15}$$

A straightforward computation now shows that for $n \ge 3k+4$, the coefficient of $\beta_{k-1}(\Delta)$ on the left-hand-side of (15) is non-negative, while the coefficients of all lower Betti numbers are ≥ 2. Thus, by Poincaré duality, the left-hand-side of (15) is at least as large as $(-1)^k(\chi(\Delta) - 2)$. On the other hand, the right-hand-side of (15) equals $\binom{n-k-2}{k+1}/\binom{2k+1}{k}$, and Kühnel's inequality follows. Moreover, from the above discussion, equality is attained if and only if Δ is $(k+1)$-neighborly.

Finally, the 'in-particular'-part follows from the fact that every simplicial manifold is orientable over a field of characteristic two. $\square$

Problem 4.5 *Is there a less computational and more conceptual proof of Kühnel's conjecture?*

We remark that there are only a few known triangulations of $2k$-manifolds which are also $(k+1)$-neighborly. For surfaces there are the 2-neighborly triangulations in [11] and [32]. Other examples include $\mathbb{C}P^2$ [19], $K3$-surfaces [3], $S^3 \times S^3$ [21], and $\mathbb{H}P^2$ [2], where $\mathbb{H}P^2$ is a manifold whose cohomology ring is isomorphic to the cohomology ring of the quaternionic projective plane.

Buchsbaum complexes with symmetries

Using Theorem 3.4 and techniques developed in [29], the inequalities of Theorem 4.3 can be significantly strengthened for the family of *centrally symmetric* Buchsbaum complexes, i.e. complexes with a free $\mathbb{Z}/2\mathbb{Z}$-action. Combinatorially, these inequalities can be described as follows.

Theorem 4.6 *Let Δ be a $(d-1)$-dimensional centrally symmetric Buchsbaum complex with $n = 2m$ vertices. Then for every $1 \leq j \leq d-1$, there exists a multicomplex $\mathcal{M} = \mathcal{M}(j)$ on $2m-d$ variables $x_1, \ldots, x_{2m-d}$ all of whose elements are* **squarefree** *in the first m variables and such that*

$$F_{j+1}(\mathcal{M}) = h'_{j+1}(\Delta), \quad \textit{while} \quad F_j(\mathcal{M}) \leq h'_j(\Delta) - \binom{d}{j}\beta_{j-1}(\Delta).$$

Proof: Label the vertices of Δ so that for every $1 \leq i \leq m$, x_i and x_{m+i} are antipodal (i.e., x_i, x_{m+i} form an orbit under the given $\mathbb{Z}/2\mathbb{Z}$-action). Consider $u \in \mathrm{GL}_n(\mathbf{k})$ of the form

$$u = \begin{bmatrix} I_m & I_m \\ O & Y^{-1} \end{bmatrix}. \quad \text{Equivalently,} \quad u^{-1} = \begin{bmatrix} I_m & -Y \\ O & Y \end{bmatrix}.$$

Here I_m denotes the $m \times m$ identity matrix, O stands for the $m \times m$ zero matrix, and $Y \in \mathrm{GL}_m(\mathbf{k})$ satisfies the condition that all of its $d \times d$-minors supported on the last d columns of Y are non-singular. Since $\mathbf{k}$ is infinite, such Y exists.

Note that u defines a graded automorphism of S via $u(x_j) = \sum_{i=1}^n u_{ij}x_i$, and so uI_Δ is a homogeneous ideal of S. Let $I = uI_\Delta + (x_{n-d+1}, \ldots, x_n)$, let $\mathrm{Soc}\, I = I : \mathfrak{M}$ be the socle of I, and let $J = I + (\mathrm{Soc}\, I)_j$. Since for every element $y \in (\mathrm{Soc}\, I)_j$, $\mathfrak{M} \cdot y \subset I$, it follows that J is an ideal of S. As no face of Δ contains two antipodal points, the structure of u^{-1} and [37, Lemma III.2.4] imply that $x_{n-d+1}, \ldots, x_n$ is a l.s.o.p. for S/uI_Δ. Hence, by Theorem 3.3 and Theorem 3.4,

$$\dim_{\mathbf{k}}(S/J)_{j+1} = h'_{j+1} \quad \text{and} \quad \dim_{\mathbf{k}}(S/J)_j \leq h'_j - \binom{d}{j}\beta_{j-1}(\Delta). \tag{16}$$

To construct a required multicomplex, fix the reverse lexicographic order $\succ$ on the set of all monomials of $S = \mathbf{k}[x_1, \ldots, x_n]$ that refines the partial order by degree and satisfies $x_1 \succ x_2 \succ \ldots \succ x_n$ (e.g. $x_1^2 \succ x_1x_2 \succ x_2^2 \succ x_1x_3 \succ x_2x_3 \succ x_3^2 \succ \cdots$). Consider $\mathrm{In}\, J$ — the reverse lexicographic initial ideal of J (see [8, Section 15.2]), and define $\mathcal{M}$ to be the collection of all monomials that are not in $\mathrm{In}\, J$. Since $\mathrm{In}\, J$ is a monomial ideal that contains $x_{n-d+1}, \dots, x_n$, $\mathcal{M}$ is a multicomplex on $n-d$ variables. Moreover, $\mathcal{M}$ has "correct" F-numbers. This follows from Eq. (16) and the fact that $\mathcal{M}$ is a **k**-basis for S/J (see [8, Theorem 15.3]). Finally, the structure of u and that $\{x_i, x_{i+m}\}$ is not a face of Δ imply that $\operatorname{In} J \ni \operatorname{In} u(x_i x_{i+m}) = x_i^2$ for all $1 \leq i \leq m$, and hence that all elements of $\mathcal{M}$ are squarefree in the first m variables. □

A complete characterization of F-vectors of multicomplexes that are squarefree in the first m variables was worked out by Clements and Lindsröm [5]. Their theorem provides an explicit sharp upper bound on F_{j+1} of such a multicomplex in terms of its F_j and j. (Compare to Macaulay's theorem that characterizes F-vectors of multicomplexes without any restrictions on degrees.) Thus using Clements-Lindsröm theorem, one can restate Theorem 4.6 in purely numerical terms.

Remark 4.7 The same proof as in Theorem 4.6 but with matrix u chosen as in [29, Theorem 3.3] allows to extend Theorem 4.6 to all Buchsbaum simplcial complexes with a proper $\mathbb{Z}/p\mathbb{Z}$-action, where p is a prime number, thus proving Conjecture 6.1 of [29]. So far we have been unable to settle Conjecture 6.2 of [29] — an analog of Kühnel's conjecture for manifolds with symmetry. The statement in [29] that [29, Conjecture 6.1] would imply [29, Conjecture 6.2] at least for all centrally symmetric manifolds is erroneous.

5 Lower bounds

The Dehn-Sommerville relations for simplicial polytopes states that $h_i = h_{d-i}$. Klee proved an analogous formula for semi-Eulerian complexes. A pure complex is *semi-Eulerian* if the Euler characteristic of the link of every nonempty face is the same as the Euler characteristic of a sphere of the same dimension. A prototypical example is an arbitrary triangulation of a homology manifold without boundary.

Theorem 5.1 (Klee's formula [15]) *Let Δ be a semi-Eulerian $(d-1)$-dimensional complex. Then*

$$h_{d-i} - h_i = (-1)^i \binom{d}{i} [\chi(\Delta) - \chi(S^{d-1})].$$

An immediate consequence of Klee's formula is that for semi-Eulerian complexes knowledge of the g-vector is sufficient to recover the f-vector. The *g-vector* of Δ is $(g_0, \dots, g_{\lfloor d/2 \rfloor})$, where $g_i = h_i - h_{i-1}$. Of particular interest in this section is $g_2 = h_2 - h_1 = f_1 - df_0 + \binom{d+1}{2}$.

In [12] Kalai conjectured that if Δ is a triangulation of a closed manifold with $d \geq 4$, then $g_2 \geq \binom{d+1}{2}\beta_1(\Delta; \mathbb{Q})$. This bound is sharp for triangulations in $\mathcal{H}^d$. A complex Δ is in $\mathcal{H}^d$ if it can be obtained from the boundary of the d-simplex by a sequence of the following three operations:

- Subdivide a facet with one new vertex in the interior of the facet.

- Form the connected sum of $\Delta_1, \Delta_2 \in \mathcal{H}^d$ by identifying a pair of facets, one from each complex, and then removing the interior of the identified facet.

- Form a handle by identifying a pair of facets in $\Delta \in \mathcal{H}^d$ and removing the interior of the identified facet in such a way that the resulting complex is still a simplicial complex. Equivalently, the distance in the 1-skeleton between every pair of identified vertices must be at least three.

If the only type of operation used is the first one (subdividing a facet), then the resulting space is a *stacked sphere.* Another characterization of $\mathcal{H}^d$, due to Walkup in dimension three [43] and Kalai in higher dimensions [12], is as those triangulations all of whose vertex links are stacked spheres.

Kalai's conjecture was verified for $\beta_1 = 1$ and for orientable manifolds when $d \geq 5$ and $\beta_2 = 0$ in [40]. In the latter case, if $g_2 = \binom{d+1}{2}\beta_1(\Delta; \mathbb{Q})$, then $\Delta \in \mathcal{H}^d$. This last result was then used to determine all possible pairs (f_0, f_1) for triangulations of spherical bundles over the circle [6]. We now settle Kalai's conjecture in its full generality. (Recall that a connected $\mathbf{k}$-homology $(d-1)$-dimensional manifold Δ is *orientable* if $\widetilde{H}_{d-1}(\Delta; \mathbf{k})$ is one-dimensional.)

Theorem 5.2 *Let Δ be a connected triangulation of an orientable $\mathbf{k}$-homology $(d-1)$-dimensional manifold with $d \geq 4$. Then*

$$g_2 \geq \binom{d+1}{2}\beta_1(\Delta; \mathbf{k}). \tag{17}$$

Furthermore, if $g_2 = \binom{d+1}{2}\beta_1(\Delta)$ and $d \geq 5$, then $\Delta \in \mathcal{H}^d$.

Proof: First we consider the situation when the characteristic of $\mathbf{k}$ is zero. By [28, Lemma 5.1], $h'_{d-2} - h'_2 = \binom{d}{2}(\beta_2(\Delta) - \beta_1(\Delta))$ and $h'_{d-1} - h'_1 = d(\beta_1(\Delta) - \beta_0(\Delta)) = d\beta_1(\Delta)$. For generic l.s.o.p. Θ and one-form ω, multiplication by ω induces a surjection $\omega : (\mathbf{k}[\Delta]/(\Theta))_{d-2} \to (\mathbf{k}[\Delta]/(\Theta))_{d-1}$ [40, Corollary 4.29]. Since the dimension of the socle of $(\mathbf{k}[\Delta]/(\Theta))_{d-2}$ is at least $\binom{d}{d-2}\beta_{d-3}(\Delta)$ (see Theorem 3.4),

$$h'_{d-2} - \binom{d}{d-2}\beta_{d-3}(\Delta) \geq h'_{d-1}.$$

Combining this with Poincaré duality

$$\begin{array}{rcl} h'_2 + \binom{d}{2}(\beta_2(\Delta) - \beta_1(\Delta)) - \binom{d}{2}\beta_2(\Delta) & \geq & d\beta_1(\Delta) + h'_1 \\ h'_2 - h'_1 & \geq & d\beta_1(\Delta) + \binom{d}{2}\beta_1(\Delta) \\ h_2 - h_1 & \geq & \binom{d+1}{2}\beta_1(\Delta), \end{array}$$

where the last line follows from Schenzel's formula (Theorem 3.3).

Suppose $g_2 = \binom{d+1}{2}\beta_1(\Delta)$ and $d \geq 5$. The previous computation shows that for generic ω, the kernel of multiplication by ω in degree $d-2$ equals the socle of $\mathbf{k}[\Delta]/(\Theta)$ in degree $d-2$. Consider the ideals generated by the variables (x_i). By [40, Proposition 4.24],

$(x_i) \subseteq \mathbf{k}[\Delta]/(\Theta)$ is isomorphic as an S-module to $\mathbf{k}[\mathrm{lk}\, i]/(\Theta')$ with a degree one shift for a suitably defined Θ'. Hence, if $\ker\omega \cap ((x_i))_{d-2} \neq 0$, then the socle of $(\mathbf{k}[\mathrm{lk}\, i]/(\Theta'))_{d-3}$ is also nonzero. This is impossible since the link, $\mathrm{lk}\, i$, is a homology sphere, hence $\mathbf{k}[\mathrm{lk}\, i]/(\Theta')$ is Gorenstein*, and so its socle vanishes in all degrees but the top one (see [37, page 50]). As multiplication by a generic one-form from $(\mathbf{k}[\mathrm{lk}\, i]/(\Theta'))_{d-3}$ surjects onto $(\mathbf{k}[\mathrm{lk}\, i]/(\Theta'))_{d-2}$, $h_{d-3}(\mathrm{lk}\, i) = h_{d-2}(\mathrm{lk}\, i)$. Equivalently, by the Dehn-Sommerville relations, $h_1(\mathrm{lk}\, i) = h_2(\mathrm{lk}\, i)$. The lower bound theorem [12, Theorem 1.1] shows that each $\mathrm{lk}\, i$ must be a stacked sphere.

What if the characteristic of $\mathbf{k}$ is not zero? The only part of the above which needs to be changed is the proof that for generic Θ and one-form ω, multiplication induces a surjection $\omega : (\mathbf{k}[\Delta]/(\Theta))_{d-2} \to (\mathbf{k}[\Delta]/(\Theta))_{d-1}$. The proof given in [40] depends on [20] and the generic rigidity of embeddings of two-dimensional spheres in $\mathbb{R}^3$. Hence this approach is only valid in characteristic zero. However, Murai's recent preprint [25, Corollary 3.5], combined with Whiteley's proof that two-dimensional spheres are strongly edge decomposable [44] (see [27] for the definition of strongly edge decomposable), provide an alternative proof which is valid in nonzero characteristics. □

Problem 5.3 *Suppose Δ is a $\mathbf{k}$-orientable 3-dimensional manifold without boundary and $g_2 = 10\beta_1(\Delta)$. Is $\Delta \in \mathcal{H}^4$?*

The answer to this problem is known to be yes when $\beta = 1$ [43] and $\beta = 2$ [22].

Absolute lower bounds

Under certain conditions, Theorem 3.4 can provide absolute lower bounds for h'-vectors (and hence f-vectors) of Buchbaum complexes of a fixed homological type. Suppose β_{i-1} is the only nontrivial Betti number of Δ. By Theorem 3.4, $h'_i \geq \binom{d}{i}\beta_{i-1}(\Delta)$. Furthermore, assume that $\binom{d}{i}\beta_{i-1}(\Delta) = \binom{m}{i}$ for some m. Macaulay's upper bound for Hilbert functions implies that for $j \leq i, h'_j \geq \binom{m-i+j}{j}$. Thus, if Δ satisfies all of these restrictions as equalities and $h'_j = 0$ for $j > i$, then Δ has the minimum possible h'-vector for a Buchbaum complex of this homological type. Terai and Yoshida examined precisely this situation in [42].

Theorem 5.4 [42, Theorem 2.3] *Let Δ be a $(d-1)$-dimensional Buchsbaum complex that is i-neighborly, but not $(i+1)$-neighborly. Set $\beta = \binom{n-d+i-1}{i}/\binom{d}{i}$. Then the following are equivalent.*

- $h(\Delta) = \left(1, n-d, \binom{n-d+1}{2}, \ldots, \binom{n-d+i-1}{i}, -\binom{d}{i+1}\beta, \binom{d}{i+2}\beta, \ldots, (-1)^{d-i}\beta\right)$.
- *For every vertex j, the link satisfies $h_m(\mathrm{lk}\, j) = 0$ if and only if $m > i-1$.*

As the previous paragraph shows, any space satisfying the above conditions has the minimum f-vector among all Buchsbaum complexes with the specified β_{i-1}. Terai and Yoshida also proved that Alexander duals of cyclic polytopes form an infinite family of examples of the above phenomenon with $\beta = 1$.

For examples with Betti numbers greater than one, let Δ be a $2k$-dimensional manifold which is also $(k+1)$-neighborly. Now consider Δ with one vertex, say n, and all of its incident faces removed and call this new complex Δ'. As Δ was $(k+1)$-neighborly, its only nonzero reduced Betti numbers are β_k and β_{2k}. The Mayer-Vietoris sequence for $\Delta = \Delta' \cup (n * \operatorname{lk} n)$ shows that the only nonzero reduced Betti number for Δ' is β_k. Since Δ' is a manifold with boundary it is Buchsbaum. The h-vector of the link of any vertex of Δ is given by $h_i = \binom{n-2k-2+i}{i}$, for $i \leq k$ and $h_i = h_{2k-i}$ for $k < i \leq 2k$. Similarly, for each vertex $j < n$ the h-vector of $\overline{\operatorname{st}}\, n \subset \operatorname{lk} j$, the closed star of n within the link of j, is specified by the same equation for $i < k, h_i = h_{2k-i-1}$ for $k \leq i \leq 2k-1$, and $h_{2k} = 0$. Using the same reasoning as in [4, Lemma 3],

$$h_i(\operatorname{lk}_{\Delta'} j) = h_i(\operatorname{lk}_{\Delta} j) - h_{2k-i}(\overline{\operatorname{st}}\, n \subset \operatorname{lk} j).$$

Hence Δ' satisfies the second condition of Theorem 5.4.

6 Buchsbaum simplicial posets

The goal of this section is to rework most of material of Section 3, including Theorem 3.4, in the generality of Buchsbaum simplicial posets. Simplicial posets (also sometimes referred to in the literature as Boolean cell complexes or pseudo-simplicial complexes) provide a certain generalization of simplicial complexes. We start by reviewing their definition and related notions as well as the corresponding algebraic background.

A *simplicial poset* is a (finite) poset P that has a unique minimal element, $\hat{0}$, and such that for every $\tau \in P$, the interval $[\hat{0}, \tau]$ is a Boolean algebra [36]. In particular, P is graded, and the face poset of any simplicial complex is a simplicial poset. As with simplicial complexes, one can think of simplicial posets geometrically: it follows from results of [1] that every simplicial poset P is the face poset of a certain regular CW-complex, $|P|$, all of whose closed cells are simplices. We call $|P|$ the *realization of* P, and refer to its elements as faces. It also follows from [1] that $|P|$ has a well-defined barycentric subdivision which is the simplicial complex isomorphic to the order complex $\Delta(\overline{P})$ of the poset $\overline{P} = P - \{\hat{0}\}$.

As in the case of simplicial complexes, we denote by $f_i = f_i(P)$ the number of i-dimensional faces of $|P|$ (equivalently, the number of rank $i+1$ elements of P), and by $f(P) = (f_{-1}, f_0, \ldots, f_{d-1})$ the *f-vector of* P, and we define the h-vector of P, $h(P) = (h_0, \ldots, h_d)$ according to Eq. (5). Here $d-1$ is the *dimension of* $|P|$, that is, the maximal dimension of faces of $|P|$. Equivalently, $d = \operatorname{rk} P$, the rank of P. From now on we refer to P and $|P|$ almost interchangeably.

As with simplicial complexes, we need a notion of a link: for an element τ of P, we define the *link of* τ *in* P, to be

$$\operatorname{lk} \tau = \operatorname{lk}_P(\tau) := \{\sigma \in P \mid \sigma \geq \tau\}.$$

It is easy to check that $\operatorname{lk} \tau$ is also a simplicial poset with its $\hat{0}$ element being τ, and that if $F = \{\tau_0 < \tau_1 < \ldots < \tau_r = \tau\}$ is a saturated chain in $(\hat{0}, \tau]$, then $\operatorname{lk}_{\Delta(\overline{P})}(F) \cong \Delta(\overline{\operatorname{lk}_P(\tau)})$.

Associated to a simplicial poset P is an algebra A_P [36], defined as follows. For each element τ of P, consider a variable x_τ. Let $\tilde{S}$ be the polynomial ring $\mathbf{k}[x_\tau \mid \tau \in P]$. We assume that the set of atoms of P (equivalently, the set of vertices of $|P|$) is $V(P) = [n]$, so that, $S = \mathbf{k}[x_1, \cdots, x_n]$ is a subring of $\tilde{S}$. The *face ring of* P, A_P, is then $\tilde{S}/I_P$, where I_P is the ideal of $\tilde{S}$ generated by the elements of the following form:

- $x_\tau x_\sigma$ for all pairs of elements $\tau, \sigma \in P$ that have no common upper bound in P.
- $\mathbf{x}_\tau x_\sigma - \mathbf{x}_{\tau\wedge\sigma} \sum \mathbf{x}_\rho$ for pairs of τ, σ incomparable in P, where the sum is over the set of all *minimal upper bounds of* τ *and* σ. Note that if τ and σ have an upper bound ρ, then $\tau \wedge \sigma$ is well-defined, as τ and σ are elements of $[\hat{0}, \rho]$, a Boolean algebra.
- $\hat{0} - 1$.

Defining $\deg x_\tau := \operatorname{rk} \tau$ makes A_P into a $\mathbb{Z}$-graded algebra. There is also a $\mathbb{Z}^n$-refinement of this grading on A_P given by $\deg \tau := \sum \{e_i \mid i \in [n], i \leq \tau\}$. Here $e_1, \ldots, e_n$ is the standard basis for $\mathbb{Z}^n$.

We cite from [36] a few basic properties of A_P:

- A_P is an algebra with straightening laws (this is [36, Lemma 3.4]).
- A_P is integral over S [36, Lemma 3.9]. Since A_P is also finitely-generated algebra over S, it follows that A_P is a (graded) Noetherian S-module.
- The Krull dimension of A_P is $\operatorname{rk} P = \dim P + 1 =: d$, and (as was the case for a simplicial complex) the $\mathbb{Z}$-graded Hilbert series of A_P is given by $F(A_P, x) = (1-x)^{-d} \sum_{i=0}^{d} h_i(P) x^i$ (see [36, Prop. 3.8]).

An analog of Hochster's formula for the local cohomology of A_P (as a module over S) was worked out by Duval in [7, Theorem 5.9].

Theorem 6.1 (Duval) *For a simplicial poset P with $V(P) = [n]$, the $\mathbb{Z}^n$-graded Hilbert series of the local cohomology modules of A_P as S-modules is*

$$F(H^i(A_P), \lambda) = \sum_{\tau \in P} \beta_{i - \operatorname{rk}(\tau) - 1}(\operatorname{lk} \tau) \prod_{j \in [n],\ j \leq \tau} \frac{\lambda_j^{-1}}{1 - \lambda_j^{-1}},$$

where $\beta_i(\operatorname{lk} \tau)$ is the ith reduced Betti number of the order complex $\Delta(\overline{\operatorname{lk} \tau})$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$.

Call a simplicial poset P a *Cohen-Macaulay poset* if its order complex, $\Delta(\overline{P})$, is a Cohen-Macaulay simplicial complex, as defined in Section 3. Similarly, call P a *Buchsbaum poset* if $\Delta(\overline{P})$ is a Buchsbaum simplicial complex. Stanley [36, Cor. 3.7] showed that if P is a Cohen-Macaulay simplicial poset, then its face ring, A_P, is Cohen-Macaulay as a module over itself or over S. Here we use Theorem 6.1 to prove a similar result about Buchsbaum posets.

Proposition 6.2 *If P is a Buchsbaum simplicial poset, then the ring A_P is Buchsbaum as an S-module.*

Proof: Since $\Delta(\overline{P})$ is a Buchsbaum simplicial complex, say, of dimension $d-1$, it follows from Theorem 3.2, that for $i < d$,

$$\beta_{i-\mathrm{rk}\,(\tau)-1}(\mathrm{lk}\,\tau) = 0 \quad \text{unless } \tau = \hat{0}. \tag{18}$$

Thus, by Theorem 6.1, for $i < d$, $F(H^i(A_P), \lambda) = \beta_{i-1}(\Delta(\overline{P}))$ is a number rather than a series, and hence for $i < d$, $H^i(A_P)$ is concentrated in degree 0. Therefore, $\mathfrak{M}\cdot H^i(A_P) = 0$. Also for $0 \leq i < j < d$, the only integer degrees p and q for which $(H^i(A_P))_p \neq 0$ and $(H^j(A_P))_q \neq 0$ are $p = q = 0$. In particular, $0 > i - j = (i+p) - (j+q)$, and so $(i+p) - (j+q) \neq 1$. Proposition 3.10 on page 98 of [38] then implies that A_P is a Buchsbaum module. $\square$

Stanley showed [36, Section 3] that Theorem 3.3 holds in the generality of Cohen-Macaulay simplicial posets, that is, if P is a Cohen-Macaulay poset of rank d and $\{\theta_1, \ldots, \theta_d\} \subset S$ is a l.s.o.p. for A_P, then $\dim_{\mathbf{k}}(A_P/(\Theta)A_P)_j = h_j$ for all $0 \leq j \leq d$. We next use Proposition 6.2 to verify that Schenzel's theorem, Theorem 3.3, also continues to hold in the generality of Buchsbaum simplicial posets. Our proof mostly mimics that of Schenzel and is included here only for completeness.

Proposition 6.3 *Let P be a rank d Buchsbaum simplicial poset, let*

$$h'_j(P) := h_j(P) + \binom{d}{j} \sum_{i=1}^{j-1} (-1)^{j-i-1} \beta_{i-1}(\Delta(\overline{P})) \qquad \text{for } 0 \leq j \leq d,$$

and let $\{\theta_1, \ldots, \theta_d\} \subset S$ be a l.s.o.p. for A_P. Then $\dim_{\mathbf{k}}(A_P/(\Theta)A_P)_j = h'_j$ for $0 \leq j \leq d$.

Proof: From the following exact sequence of graded S-modules:

$$0 \longrightarrow (0 :_{A_P} \theta_1)(-1) \longrightarrow A_P(-1) \xrightarrow{\cdot\theta_1} A_P \longrightarrow A_P/(\theta_1)A_P,$$

we obtain an expression for the Hilbert series:

$$(1-x)F(A_P, x) = F(A_P/(\theta_1)A_P, x) - x \cdot F((0 :_{A_P} \theta_1), x).$$

Iterating the above d times yields

$$(1-x)^d F(A_P, x) = F(A_P/(\theta_1, \ldots, \theta_d)A_P, x) - \sum_{i=0}^{d-1} x(1-x)^i \cdot F(L_i, x), \tag{19}$$

where $L_i := ((\theta_1, \ldots, \theta_{d-1-i})A_P : \theta_{d-i})/(\theta_1, \ldots, \theta_{d-1-i})A_P$.

Now, since A_P is a Buchsbaum module (see Proposition 6.2), we have

$$\begin{aligned} L_i &\cong H^0(A_P/(\theta_1,\dots,\theta_{d-1-i})A_P) && \text{(by [38, pp. 64-65])} \\ &\cong \bigoplus_{l=0}^{d-1-i} \binom{d-1-i}{l} H^l(A_P)(-l) && \text{(by [38, Lemma II.4.14'(b)])} \\ &\cong \bigoplus_{l=0}^{d-1-i} \mathbf{k}^{\binom{d-1-i}{l}\beta_{l-1}(\Delta(\overline{P}))}(-l) && \text{(by Theorem 6.1 and Eq. (18)),} \end{aligned}$$

and so

$$F(L_i,x) = \sum_{l=0}^{d-i-1} \binom{d-i-1}{l} \beta_{l-1}(\Delta(\overline{P}))\cdot x^l. \qquad (20)$$

Plugging (20) into (19), and using that $F(A_P,x) = (1-x)^{-d}\sum_{i=0}^{d} h_i(P)x^i$ (see properties of A_P listed above in this section), completes the proof. □

We are now in a position to derive the following poset-generalization of Theorem 3.4.

Theorem 6.4 *Let P be a rank d Buchsbaum simplicial poset and let $\theta_1,\dots,\theta_d$ be a l.s.o.p. for A_P. Then for all $0 \le j \le d$,*

$$\dim_{\mathbf{k}} (\operatorname{Soc} A_P/(\Theta)A_P)_j \ge \binom{d}{j}\beta_{j-1}(\Delta(\overline{P})).$$

Hence, $h'_j(P) \ge \binom{d}{j}\beta_{j-1}(\Delta(\overline{P}))$, or, equivalently, $h_j(P) \ge \binom{d}{j}\sum_{i=0}^{j}(-1)^{j-i}\beta_{i-1}(\Delta(\overline{P}))$.

Proof: The proof is the same as in Theorem 3.4, just use Theorem 6.1 instead of Theorem 3.1 and Proposition 6.3 instead of Theorem 3.3. □

7 Examples, concluding remarks, and open problems

7.1 Toward the g-conjecture

Perhaps the most important problem in the theory of f-vectors is the g-conjecture. The most optimistic version states that if Δ is a $(d-1)$-dimensional $\mathbf{k}$-homology sphere and Θ is a l.s.o.p. for $\mathbf{k}[\Delta]$, then for a generic one-form ω and $i \le d/2$, multiplication

$$\omega^{d-2i} : \mathbf{k}[\Delta]/(\Theta)_i \to \mathbf{k}[\Delta]/(\Theta)_{d-i}$$

is an isomorphism. Kalai has suggested a far-reaching generalization of this to homology manifolds [28].

Suppose Δ is a connected simplicial complex which is homeomorphic to a $\mathbf{k}$-orientable homology manifold. Define

$$h''_i = h'_i - \binom{d}{i}\beta_{i-1}(\Delta).$$

As pointed out in [28], $h''_{d-i} = h''_i$ for $1 \le i \le d-1$. Let

$$I = \bigoplus_{j=1}^{d-1} \mathrm{Soc}\,(\mathbf{k}[\Delta]/(\Theta))_j.$$

Since I is a vector subspace of the socle it is also an ideal of $\mathbf{k}[\Delta]/(\Theta)$. Now set $\overline{\mathbf{k}[\Delta]} = (\mathbf{k}[\Delta]/\Theta)/I$. By Theorem 3.4 the dimension of $\overline{\mathbf{k}[\Delta]}_i$ is at most h''_i for $1 \le i \le d-1$.

Conjecture 7.1 *[28] For generic $\omega \in \mathbf{k}[\Delta]_1$ and $1 \le i \le d/2$,*

- $\dim_{\mathbf{k}} \overline{\mathbf{k}[\Delta]}_i = h''_i$.
- *Multiplication $\omega^{d-2i} : \overline{\mathbf{k}[\Delta]}_i \to \overline{\mathbf{k}[\Delta]}_{d-i}$ is an isomorphism.*

Consider the special case of Δ a homology sphere. The first part of the above conjecture holds since $\mathbf{k}[\Delta]$ is Gorenstein*. The second part is the g-conjecture. This suggests the following conjecture.

Conjecture 7.2 *Let Δ be a $(d-1)$-dimensional Buchsbaum complex over $\mathbf{k}$. Let $\mathcal{SB}$ be given by Theorem 2.2, with $M = \mathbf{k}[\Delta]$. Then $\dim_{\mathbf{k}} \mathcal{SB} = \dim_{\mathbf{k}} \mathcal{SB}_0 = 1$ if and only if Δ is a connected orientable $\mathbf{k}$-homology manifold without boundary.*

A closely related, but potentially weaker conjecture is the following.

Conjecture 7.3 *If Δ is a connected simplicial complex homeomorphic to a $(d-1)$-dimensional $\mathbf{k}$-homology manifold, then $\overline{\mathbf{k}[\Delta]}$ is a Gorenstein ring.*

Remark Since this paper was originally written, the authors have verified the first part of Conjecture 7.1, established one direction of Conjecture 7.2 and proved Conjecture 7.3 [30].

7.2 How tight are the bounds?

Theorem 6.4 together with a complete characterization of the h-numbers of Cohen-Macaulay simplicial posets, [36, Theorem 3.10], naturally leads to the following question.

Question 7.4 *Do the bounds $h'_j \ge \binom{d}{j}\beta_{j-1}$ for $j = 1, 2, \cdots, d-1$ together with $h'_0 = 1$ and $h'_d = \beta_{d-1}$ generate the complete set of sufficient conditions for the h-numbers of Buchsbaum simplicial posets with prescribed Betti numbers?*

We believe that the answer is yes, and hence that this set of conditions gives a complete characterization of the possible pairs (h, β) for Buchsbaum simplicial posets. The following result provides partial evidence.

Proposition 7.5 *Let $b_1, \ldots, b_{d-1}, h'_1, \ldots, h'_{d-1}$ be nonnegative integers. Assume $d \le 5$ or $b_2 = \cdots = b_{d-3} = 0$. Then there exists a Buchsbaum simplicial poset P with $\beta_j(|P|) = b_j$ and $h'_j(P) = h'_j$ for all $1 \le j \le d-1$ if and only if $h'_j \ge \binom{d}{j} b_{j-1}$ for all $1 \le j \le d-1$.*

If $b_i = 0$ for all $i \neq d-1$, then one can even find a shellable poset satisfying the conditions of the proposition, see [36, Theorem 3.10]. For the other combinations of b_i satisfying the hypotheses, the proposition is an immediate consequence of the next four lemmas.

Lemma 7.6 *There exists a $(d-1)$-dimensional Buchsbaum simplicial poset $X = X(1,d)$ such that $\beta_1(X) = 1, \beta_i(X) = 0$ for $i \neq 1$ and*

$$h'_i(X) = \begin{cases} \binom{d}{i}, & \text{if } i = 0 \text{ or } i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Proof: One such X is given by taking a stacked ball whose facets are

$$\{1, 2, \ldots, d\}, \{2, 3, \ldots, d+1\}, \ldots, \{d, d+1, \ldots, 2d-1\},$$

and identifying the codimension one face spanned by $1, 2, \ldots, d-1$ with the codimension one face spanned by $d+1, d+2, \ldots, 2d-1$ (where vertex i is identified with vertex $d+i$). The realization of X, $|X|$, is a $(d-2)$-disk bundle over $\mathbb{S}^1$, orientable or not depending on the parity of d. Hence X is a Buchsbaum simplicial poset satisfying $\beta_1(X) = 1$ and $b_i = 0$ for $i \neq 1$. A straightforward computation shows that $f_{i-1}(X) = d\binom{d-1}{i-1}$ for all $i \geq 1$. Hence $h_0 = 1$, $h_1 = 0$, and $h_i = (-1)^i\binom{d}{i}$ for $i \geq 2$, which together with the above count of Betti numbers implies that all h'_i numbers of X vanish except for h'_0 and h'_2, and those two are equal to 1 and $\binom{d}{2}$, respectively. $\square$

Lemma 7.7 *There exists a $(d-1)$-dimensional Buchsbaum simplicial poset $X = X(d-2, d)$ such that $\beta_{d-2}(X) = 1, \beta(X)_i = 0$ for $i \neq d-2$ and*

$$h'_i(X) = \begin{cases} \binom{d}{i}, & \text{if } i = 0 \text{ or } i = d-1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof: One possible construction for X is as follows. The vertices of X are $1, 2, \ldots, d$. The $(d-3)$-skeleton of X is the $(d-3)$-skeleton of the $(d-1)$-simplex. For every subset of vertices of cardinality $d-1$ give X two distinct codimension one faces. Label these faces $A_1, A_2, \ldots, A_d, B_1, B_2, \ldots, B_d$, where A_i and B_i are the two faces whose vertices do not contain i. Any potential facet of X is described by choosing one of A_i or B_i for each i as the boundary faces of the facet. The facets of X are the d possible ways of choosing exactly one boundary face of type B and the rest of type A.

Since X has the $(d-3)$-skeleton of the simplex and also contains the $(d-2)$-skeleton of the simplex, $\beta_i(X) = 0$ for $i < d-2$. It is easy to see that the kernel of the boundary map from the $(d-1)$-chains to the $(d-2)$-chains is zero, hence $\beta_{d-1}(X) = 0$. A check of the Euler characteristic of X shows that $\beta_{d-2}(X) = 1$. Now that the Betti numbers of X are known, direct computation shows that X has the required h' numbers. To see that $\tilde{H}_i(\mathrm{lk}\,\sigma) = 0$ for a face σ and $i < d - |\sigma| - 1$, use the same argument, except that the kernel of the boundary map in dimension $(d-1-|\sigma|)$ is of dimension $|\sigma|-1$. $\square$

Lemma 7.8 *Let P_1 and P_2 be two disjoint $(d-1)$-dimensional Buchsbaum simplicial posets. If Q is obtained from P_1 and P_2 by identifying a facet of P_1 with that of P_2, then Q is also a Buchsbaum poset. Moreover,*

$$\begin{aligned} \beta_i(Q) &= \beta_i(P_1) + \beta_i(P_2), \quad i = 0, 1, \ldots, d-1, \quad \textit{and} \quad (21)\\ h'_i(Q) &= h'_i(P_1) + h'_i(P_2) \quad i = 1, 2, \ldots, d. \quad (22)\end{aligned}$$

Proof: That the Betti numbers add when P_1 and P_2 are glued along a facet is an easy application of the Mayer-Vietoris sequence and the fact that the intersection of P_1 and P_2 is contractible. The same Mayer-Vietoris sequence also shows that Q is Buchsbaum. Since $f_{i-1}(Q) = f_{i-1}(P_1) + f_{i-1}(P_2) - \binom{d}{i}$, the defining relation for the h-numbers implies that $h_i(Q) = h_i(P_1) + h_i(P_2)$ for $i \geq 1$, which together with Eq. (21) yields (22). □

Lemma 7.9 *Let P be a $(d-1)$-dimensional Buchsbaum simplicial poset and let $g'_1, \ldots, g'_d$ be nonnegative integers satisfying $g'_i \geq h'_i(P)$ for all $i = 1, \ldots, d$. Then there exists a $(d-1)$-dimensional Buchsbaum simplicial poset Q whose Betti numbers, except possibly for β_{d-1}, coincide with those of P and such that $h'_i(Q) = g'_i$ for all $1 \leq i \leq d$.*

Proof: By [36, Theorem 3.10] there exists a shellable simplicial poset R such that $h_i(R) = g'_i - h'_i(P)$ for all $1 \leq i \leq d$. Attaching R to P along a facet (as in the proof of Lemma 7.8) produces a required poset Q. □

In view of the last two lemmas, to answer Question 7.4 in the affirmative, it is enough to construct for every d and $i \leq d-1$ a $(d-1)$-dimensional Buchsbaum simplicial poset $X = X(i, d)$ such that

$$\beta_j(X) = \begin{cases} 0, & \text{if } j \neq i \\ 1, & \text{if } j = i \end{cases} \quad \text{and } h'_j(X) = \begin{cases} 0, & \text{if } j \neq 0, i+1 \\ \binom{d}{j}, & \text{otherwise.} \end{cases}$$

Lemmas 7.6 and 7.7 provide such a construction for $i = 1$ and $i = d-2$ (and any d), $X(0, d)$ is the disjoint union of two $(d-1)$-simplices, while $X(d-1, d)$ can be obtained by gluing two $(d-1)$-simplices along their boundaries. A construction for $X(2, 5)$ is also known. A simplicial poset homeomorphic to $\mathbb{C}P^2$ with h-vector $(1, 0, 0, 10, -5, 2)$ is described in [9]. Removing any facet (or more precisely, the open cell of a facet) is an example satisfying the requirements of $X(2, 5)$.

The problem of determining all possible h-vectors of Buchsbaum complexes (as opposed to posets) was previously considered by Terai [41] and in dimension 2 ($d = 3$) by Hanano [10]. The linear inequalities established in [41, Theorem 2.4] also hold for Buchsbaum posets. In fact, the stronger inequalities, $i\, h_i + (d-i+1)h_{i-1} \geq 0,\ 1 \leq i \leq d$, hold for arbitrary simplicial pure posets whose vertex links have nonnegative h-vectors [39, Proposition 2.3]. At this time we do not have enough examples to make a firm conjecture which determines all possible (h, β) pairs for Buchsbaum complexes. Hence we finish with

Question 7.10 *Are there other restrictions on pairs* (h, β) *for* $(d-1)$*-dimensional Buchsbaum complexes other than those coming from Theorem 3.4,*

$$h'_i \geq \binom{d}{i}\beta_{i-1},$$

and Theorem 4.3

$$h'_{i+1} \leq \left(h_i - \binom{d}{i}\beta_{i-1}\right)^{<i>} ?$$

Very recently, Murai [26] has shown that in Question 7.10 *some* additional restrictions are necessary.

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