

Many neighborly spheres

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Abstract

The result of Padrol [13] asserts that for every $d \geq 4$, there exist $2^{\Omega(n \log n)}$ distinct combinatorial types of $\lfloor d/2 \rfloor$ -neighborly simplicial $(d-1)$ -spheres with n vertices. We present a construction showing that for every $d \geq 5$, there are at least $2^{\Omega(n^{\lfloor (d-1)/2 \rfloor})}$ such types.

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MSC

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1 Introduction

A simplicial complex on n vertices is s -neighborly if it has the same $(s-1)$ -skeleton as the $(n-1)$ -simplex on the same vertex set. Of special interest are $\lfloor d/2 \rfloor$ -neighborly $(d-1)$ -spheres. They arise, for instance, in the context of Stanley's upper bound theorem [16]. In this paper we address the question of how many $\lfloor d/2 \rfloor$ -neighborly $(d-1)$ -spheres with n vertices there are.

This question is ultimately related to the questions of how many combinatorial types of (convex) simplicial d -polytopes with n labeled vertices there are and how many combinatorial types of simplicial $(d-1)$ -spheres with n labeled vertices there are. Denote these numbers by $c(d, n)$ and $s(d, n)$, respectively. The asymptotic answer to the first question was given by Goodman and Pollack [5] followed by the work of Alon [1]. They showed that there are very few polytopes: $c(d, n) = 2^{\Theta(n \log n)}$ for $d \geq 4$. In contrast to these results, Kalai [7] proved that there is a very large number of simplicial spheres: for $d \geq 5$, $s(d, n) \geq 2^{\Omega(n^{\lfloor (d-1)/2 \rfloor})}$. Furthermore, Pfeifle and Ziegler [14] showed that $s(4, n) \geq 2^{\Omega(n^{5/4})}$. The current record on the number of odd-dimensional simplicial spheres is due to Nevo, Santos, and Wilson [11] who established the following bound: $s(2k, n) \geq 2^{\Omega(n^k)}$ for all $k \geq 2$. In short, the best to-date lower bound for any $d \geq 4$ is $s(d, n) \geq$

$2^{\Omega(n^{\lfloor d/2 \rfloor})}$. On the other hand, Stanley’s upper bound theorem implies that $s(d, n) \leq 2^{O(n^{\lfloor d/2 \rfloor} \log n)}$ (see [7, Section 4.2]). This is the current best upper bound.

Despite the fact that most of spheres constructed in [7, 11, 14] are not neighborly, Kalai [7, Section 6.3] speculated that the number $\text{sn}(d, n)$ of $\lfloor d/2 \rfloor$ -neighborly simplicial $(d-1)$ -spheres with n labeled vertices is very large and posited the following conjecture.

Conjecture 1.1. *For all $d \geq 4$,*

$$\lim_{n \rightarrow \infty} (\log \text{sn}(d, n) / \log s(d, n)) = 1.$$

Indeed, the currently best known lower bound on the number of $\lfloor d/2 \rfloor$ -neighborly d -polytopes with n labeled vertices (due to Padrol [13]) is also the best known lower bound on the total number of combinatorial types of d -polytopes with n labeled vertices. Padrol’s paper built on and generalized Shemer’s sewing construction [15], which was used to produce the previous record number of neighborly polytopes. In addition to neighborly polytopes, Padrol was also able to construct a record number of non-realizable neighborly oriented matroids. Yet, Padrol’s bounds only imply that $\text{sn}(d, n) \geq 2^{\Omega(n \log n)}$.

While we are still very far from being able to shed light on Kalai’s conjecture, we improve Padrol’s bound and prove the following result.

Theorem 1.2. *For all $d \geq 5$, $\text{sn}(d, n) \geq 2^{\Omega(n^{\lfloor (d-1)/2 \rfloor})}$.*

Our construction utilizes Kalai’s squeezed balls [7]. In fact, the key to our proof is an observation that for certain choices of parameters, the difference of two squeezed $(2k-1)$ -balls on n vertices forms a $(k-1)$ -neighborly and $(k-1)$ -stacked $(2k-1)$ -ball on the same vertex set, see Theorem 3.1. These “difference” balls are contained in the boundary complex of the cyclic $2k$ -polytope on n vertices, denoted as $\partial C_{2k}(n)$. They are extremely useful for our constructions in both even- and odd-dimensional cases. Indeed, on one hand, the boundary of such a ball B is a $(k-1)$ -neighborly $(2k-2)$ -sphere on n vertices. On the other hand, removing B from $\partial C_{2k}(n)$ and patching the resulting hole with the cone over the boundary of B produces a k -neighborly $(2k-1)$ -sphere on $n+1$ vertices.

A few historical remarks are in order. The first construction of polytopal d -balls with n vertices that are both r -neighborly and r -stacked (for all parameters r, d, n with $2 \leq 2r \leq d$ and $n \geq d+1$) is due to McMullen and Walkup [10]. The idea of finding inside a triangulated manifold M a full-dimensional ball B that is both 1-neighborly (i.e., B contains all vertices of M) and 1-stacked, and using such balls to construct 2-neighborly triangulations of manifolds was pioneered by Walkup [19]; for a much more recent use of the same idea see [18, Section 5].

The structure of the rest of the paper is as follows. In Section 2 we review basic definitions related to neighborliness, stackedness, and Kalai’s squeezed balls. In Section 3 we describe our main construction, the relative squeezed balls. Finally, in Section 4 we prove Theorem 1.2.

2 Preliminaries

2.1 Simplicial complexes

We start by providing a quick overview of the main objects of this paper — simplicial complexes. A *simplicial complex* Δ with vertex set $V(\Delta)$ is a collection of subsets of $V(\Delta)$ that is closed under

inclusion and contains all singletons: $\{v\} \in \Delta$ for all $v \in V(\Delta)$. The elements of Δ are called *faces*. The *dimension of a face* $\tau \in \Delta$ is $\dim \tau := |\tau| - 1$. The *dimension of Δ* , $\dim \Delta$, is the maximum dimension of its faces. A face of a simplicial complex Δ is a *facet* if it is maximal w.r.t. inclusion. We say that Δ is *pure* if all facets of Δ have the same dimension. We distinguish between the *empty complex* $\Delta = \{\emptyset\}$ whose only face is the empty set and the *void complex* $\Delta = \emptyset$ that has no faces (not even the empty set).

Let Δ be a simplicial complex. The *k-skeleton* of Δ , $\text{Skel}_k(\Delta)$, is the subcomplex of Δ consisting of all faces of dimension $\leq k$. If τ is a face of Δ , then the *antistar of τ* and the *link of τ in Δ* are the following subcomplexes of Δ :

$$\text{ast}(\tau, \Delta) = \{\sigma \in \Delta : \sigma \not\supseteq \tau\}, \quad \text{lk}(\tau, \Delta) := \{\sigma \in \Delta : \sigma \cap \tau = \emptyset \text{ and } \sigma \cup \tau \in \Delta\}.$$

If Δ is a pure simplicial complex and Γ is a full-dimensional pure subcomplex of Δ , then $\Delta \setminus \Gamma$ is the subcomplex of Δ generated by those facets of Δ that are not in Γ . Finally, if Δ and Γ are simplicial complexes on disjoint vertex sets, then the *join* of Δ and Γ is the simplicial complex $\Delta * \Gamma = \{\sigma \cup \tau : \sigma \in \Delta \text{ and } \tau \in \Gamma\}$. In particular, the join of Δ with the empty complex is Δ while the join of Δ with the void complex is the void complex.

Let V be a set of size $d+1$. Denote by \bar{V} the d -dimensional simplex on V . Its boundary complex is $\partial\bar{V} := \{\tau : \tau \subsetneq V\}$. Most of complexes considered in this paper are PL balls or PL spheres. (Here PL stands for piecewise linear.) A *PL d -ball* is a simplicial complex PL homeomorphic to \bar{V} . Similarly, a *PL $(d-1)$ -sphere* is a simplicial complex PL homeomorphic to $\partial\bar{V}$. If Δ is a PL d -sphere and $\Gamma \subset \Delta$ is a PL d -ball, then so is $\Delta \setminus \Gamma$, see [6]. Furthermore, the link of any face in a PL sphere is a PL sphere. On the other hand, the link of a face τ in a PL d -ball B is either a PL ball or a PL sphere; in the former case we say that τ is a *boundary face* of B , and in the latter case that τ is an *interior face* of B . The *boundary complex* of B , ∂B , is the subcomplex of B that consists of all boundary faces of B ; in particular, ∂B is a PL $(d-1)$ -sphere.

For a $(d-1)$ -dimensional complex Δ and $-1 \leq i \leq d-1$, we let $f_i = f_i(\Delta)$ be the number of i -dimensional faces of Δ . The vector $f(\Delta) = (f_{-1}, f_0, \dots, f_{d-1})$ is called the *f-vector* of Δ . We also define the *h-vector* of Δ , $h(\Delta) = (h_0, \dots, h_d)$, by the following relation: $\sum_{j=0}^d h_j \lambda^{d-j} = \sum_{i=0}^d f_{i-1} (\lambda - 1)^{d-i}$. In particular, $f_{-1} = h_0 = 1$ and $f_{d-1} = \sum_{j=0}^d h_j$.

2.2 Cyclic polytopes, neighborliness, and stackedness

Let $i \geq 1$. We say that a simplicial complex Δ is *i-neighborly* w.r.t. V (or simply *i-neighborly*) if $\text{Skel}_{i-1}(\Delta) = \text{Skel}_{i-1}(\bar{V})$.

Let $m : \mathbb{R}^d \rightarrow \mathbb{R}$, $t \mapsto (t, t^2, \dots, t^d)$, be the *moment curve* in \mathbb{R}^d , and let $t_1 < t_2 < \dots < t_n$ be distinct real numbers, where $n > d$. The *cyclic d -polytope* $C_d(n)$ is defined as the convex hull $\text{conv}(m(t_1), \dots, m(t_n))$. It is known that $C_d(n)$ is a simplicial d -polytope with n vertices, that it is $\lfloor d/2 \rfloor$ -neighborly and that its combinatorial type is independent of the choice of t_1, \dots, t_n . In the rest of the paper we treat the boundary complex of $C_d(n)$, $\partial C_d(n)$, as an abstract simplicial complex. In particular, we identify a vertex $m(t_i)$ with $i \in [n] := \{1, 2, \dots, n\}$ and the vertex set of $\partial C_d(n)$ with $[n]$. The facets of $C_d(n)$ have a particularly nice description known as Gale's evenness condition [4]:

Lemma 2.1. *Let $n > d \geq 2$, and let $C_d(n)$ be the cyclic d -polytope. A d -subset $F \subset [n]$ forms a facet of $\partial C_d(n)$ if and only if for every $i < j$ not in F , the number of elements $\ell \in F$ between i and*

j is even. In particular, for $d = 2k$, $F = \{i_1, i_1 + 1, i_2, i_2 + 1, \dots, i_k, i_k + 1\}$ is a facet of $\partial C_{2k}(n)$ if $1 \leq i_1$, $i_k \leq n - 1$, and $i_j \leq i_{j+1} - 2$ for all $1 \leq j \leq k - 1$.

A property that seems to be inseparable from neighborliness is that of stackedness. A PL d -ball B is called i -stacked (for some $0 \leq i \leq d$), if all interior faces of B are of dimension $\geq d - i$, that is, $\text{Skel}_{d-i-1}(B) = \text{Skel}_{d-i-1}(\partial B)$. In particular, 0-stacked balls are simplices; 1-stacked balls are also known in the literature as stacked balls. We will rely on the following basic properties, see for instance [12, Lemma 2.2]. Part 2 of the statement below is stronger than the one provided in [12, Lemma 2.2(2)], but the proof is identical, so we omit it.

Lemma 2.2. *Let B_1 and B_2 be PL balls of dimension d_1 and d_2 , respectively. If B_1 is i_1 -stacked and B_2 is i_2 -stacked, then*

1. *The complex $B_1 * B_2$ is an $(i_1 + i_2)$ -stacked PL $(d_1 + d_2 + 1)$ -ball.*
2. *Furthermore, if $d_1 = d_2 = d$ and $B_1 \cap B_2 \subseteq \partial B_1 \cap \partial B_2$ is a PL $(d - 1)$ -ball that is i_3 -stacked, then $B_1 \cup B_2$ is an i -stacked PL d -ball, where $i = \max\{i_1, i_2, i_3 + 1\}$.*

We close this subsection with the following theorem that summarizes a few properties of the h -vectors of PL balls.

Theorem 2.3. *Let Δ be a PL $(d - 1)$ -ball with n vertices. Then*

1. *The h -numbers of Δ satisfy $0 \leq h_i \leq \binom{n-d+i-1}{i}$ for all $1 \leq i \leq d$.*
2. *Δ is i_0 -neighborly if and only if $h_i(\Delta) = \binom{n-d+i-1}{i}$ for all $i \leq i_0$.*
3. *Δ is $(r - 1)$ -stacked if and only if $h_i(\Delta) = 0$ for all $i \geq r$.*

The first two statements are due to Stanley [16], see also [17, Chapter II.3]; they hold not only for balls but for all Cohen–Macaulay complexes. For the last statement, see [9, Proposition 2.4].

2.3 Kalai’s squeezed balls and spheres

To review the definition of squeezed balls and spheres, we will use some terminology from partially ordered sets. Specifically, recall that an *antichain* \mathcal{A} in a poset (\mathcal{Q}, \leq) is a subset of \mathcal{Q} no two of which elements are comparable to each other. If $\mathcal{A} \subseteq \mathcal{Q}$ is an antichain, we denote by $\mathcal{Q}(\mathcal{A})$ the *order ideal* (also known as the *initial set*) generated by \mathcal{A} : $\mathcal{Q}(\mathcal{A}) = \{x \in \mathcal{Q} : x \leq a \text{ for some } a \in \mathcal{A}\}$. When \mathcal{Q} is finite, there is a natural bijection ϕ from the set of antichains of \mathcal{Q} to the set of order ideals of \mathcal{Q} defined by $\phi(\mathcal{A}) = \mathcal{Q}(\mathcal{A})$; the inverse map ϕ^{-1} takes an order ideal \mathcal{I} to the set of maximal elements of \mathcal{I} .

In what follows, every d -subset of $[n]$ is written in an increasing order and is identified with an element of \mathbb{N}^d . In particular, we compare two d -subsets using the standard *partial order* \leq_p on \mathbb{N}^d : for $F = \{i_1, i_2, \dots, i_d\}$ and $G = \{j_1, j_2, \dots, j_d\}$, we say that $G \leq_p F$ if $j_\ell \leq i_\ell$ for all $1 \leq \ell \leq d$. Denote by $\mathbf{1}_d$ the all-ones vector of length d . We also say that $G \prec_p F$ if $G \leq_p F - \mathbf{1}_d$.

Let $[m, n]$ denote the set $\{m, m+1, \dots, n\}$. (It is the empty set if $m > n$.) Our main construction relies on the poset $\mathcal{F}_{2k}^{[m, n]}$ defined as follows. When $k = 0$, this poset consists only of the empty set. When $k \geq 1$, as a set $\mathcal{F}_{2k}^{[m, n]}$ consists of the following facets of the cyclic polytope $C_{2k}(n)$:

$$\{\{i_1, i_1 + 1, i_2, i_2 + 1, \dots, i_k, i_k + 1\} : m \leq i_1, i_k \leq n - 1, i_j \leq i_{j+1} - 2, \forall 1 \leq j \leq k - 1\};$$

these facets are ordered by the partial order \leq_p . Note that if $m > n - 2k + 1$, then $\mathcal{F}_{2k}^{[m,n]}$ is a void poset. Otherwise, $[n - 2k + 1, n] = \{n - 2k + 1, n - 2k + 2, \dots, n\}$ is the unique maximal element of $\mathcal{F}_{2k}^{[m,n]}$, that is, $\mathcal{F}_{2k}^{[m,n]}$ is the order ideal generated by the antichain $\{[n - 2k + 1, n]\}$.

When k and n are fixed or understood from context, we abbreviate $\mathcal{F}_{2k}^{[1,n]}$ as \mathcal{F}_{2k} or as \mathcal{F} . We say that two antichains \mathcal{S} and \mathcal{T} in \mathcal{F} satisfy $\mathcal{T} \leq_p \mathcal{S}$ if $\mathcal{F}(\mathcal{T}) \subseteq \mathcal{F}(\mathcal{S})$; equivalently, if for every $G \in \mathcal{T}$ there is an element $F \in \mathcal{S}$ such that $G \leq_p F$. Similarly, we say that $\mathcal{T} \prec_p \mathcal{S}$ if for every element $G \in \mathcal{T}$, there exists an element $F \in \mathcal{S}$ such that $G \prec_p F$. For instance, if $k = 2$, $n = 8$, $\mathcal{S} = \{\{1, 2, 7, 8\}, \{3, 4, 6, 7\}\}$, $\mathcal{T} = \{\{2, 3, 5, 6\}\}$, and $\mathcal{T}' = \{\{1, 2, 6, 7\}\}$, then $\mathcal{T} \prec_p \mathcal{S}$, $\mathcal{T}' \leq_p \mathcal{S}$, but $\mathcal{T}' \not\prec_p \mathcal{S}$.

For an antichain \mathcal{S} in \mathcal{F} , let $B(\mathcal{S})$ be the pure simplicial complex whose facets are the sets in the order ideal $\mathcal{F}(\mathcal{S})$. (In particular, $B(\mathcal{S}) \neq B(\mathcal{T})$ if $\mathcal{S} \neq \mathcal{T}$.) For example, if $k = 2$ and $\mathcal{S} = \{\{1, 2, 5, 6\}, \{2, 3, 4, 5\}\}$, then the complex $B(\mathcal{S})$ is a 3-ball with facets $\{1, 2, 3, 4\}$, $\{1, 2, 4, 5\}$, $\{1, 2, 5, 6\}$ and $\{2, 3, 4, 5\}$.

Kalai [7] proved that the complexes $B(\mathcal{S})$ are PL balls and called them *squeezed balls*. The boundary complex $\partial B(\mathcal{S})$ of $B(\mathcal{S})$ is a *squeezed sphere*. Some of the properties of these objects are summarized in the following theorem. We refer to [20] for the definition of shellability and only mention a known fact that shellable balls and spheres are always PL.

Theorem 2.4. *Fix k and n , and let $\mathcal{S}, \mathcal{S}'$ be non-empty antichains in $\mathcal{F} = \mathcal{F}_{2k}^{[1,n]}$. Then*

1. *$B(\mathcal{S})$ is a k -stacked shellable $(2k - 1)$ -ball. Furthermore, if $\mathcal{S} = \{[n - 2k + 1, n]\}$, then $B(\mathcal{S})$ is k -neighborly w.r.t. $[n]$.*
2. *If $\partial B(\mathcal{S}) = \partial B(\mathcal{S}')$, then $B(\mathcal{S}) = B(\mathcal{S}')$.*

Part 2 is [7, Proposition 3.3], and a large portion of Part 1 is proved in [7, Corollary 3.2 and Proposition 5.3(i)]. For completeness, we discuss some of the details of the proof of Part 1 below.

Proof: By [7, Corollary 3.2], $B(\mathcal{S})$ is a shellable $(2k - 1)$ -ball. By Gale's evenness condition, the elements of $\mathcal{F} = \mathcal{F}_{2k}(\{[n - 2k + 1, n]\})$ are precisely the facets of $C_{2k}(n + 1)$ that do not contain $n + 1$. Hence $B(\{[n - 2k + 1, n]\})$ is k -neighborly w.r.t. $[n]$ (because $C_{2k}(n + 1)$ is k -neighborly w.r.t. $[n + 1]$) and

$$B(\mathcal{S}) \subseteq B(\{[n - 2k + 1, n]\}) = \text{ast}(n + 1, \partial C_{2k}(n + 1)).$$

Gale's evenness condition also implies that $\text{lk}(n + 1, \partial C_{2k}(n + 1)) = \partial C_{2k-1}(n)$. Consequently, $\text{lk}(n + 1, \partial C_{2k}(n + 1))$ is a $(k - 1)$ -neighborly (w.r.t. $[n]$) $(2k - 2)$ -sphere. Since this sphere is the boundary complex of $\text{ast}(n + 1, \partial C_{2k}(n + 1))$, all faces of $\text{ast}(n + 1, \partial C_{2k}(n + 1))$ of dimension $\leq k - 2$ are boundary faces. We conclude that $\text{ast}(n + 1, \partial C_{2k}(n + 1))$ is a k -neighborly w.r.t. $[n]$ and k -stacked $(2k - 1)$ -ball. Finally, since $B(\mathcal{S})$ is a full-dimensional subcomplex of this ball, an interior face of $B(\mathcal{S})$ is necessarily an interior face of the antistar. Thus $B(\mathcal{S})$ is also k -stacked. \square

To count the number of distinct squeezed $(2k - 2)$ -spheres, we define another poset

$$\mathcal{P} = \mathcal{P}_k^n = \{(x_1, x_2, \dots, x_k) : 1 \leq x_1 < x_2 < \dots < x_k \leq n - k\} \subseteq \mathbb{N}^k$$

also ordered by the partial order \leq_p . There is a natural bijection R between \mathcal{P} and \mathcal{F} given by

$$R : (x_1, x_2, \dots, x_k) \mapsto \{x_1, x_1 + 1, x_2 + 1, x_2 + 2, x_3 + 2, x_3 + 3, \dots, x_k + k - 1, x_k + k\}.$$

This map is an isomorphism of posets. Counting the number of distinct antichains in \mathcal{P} leads to

Theorem 2.5. [7, Theorem 4.2] *Let $k \geq 2$. The number of combinatorial types of squeezed $(2k-1)$ -balls with n labeled vertices (or equivalently, those of squeezed $(2k-2)$ -spheres with n labeled vertices) is $2^{\Omega(n^{k-1})}$.*

3 The relative squeezed balls and spheres

In this section, we introduce and study the main objects of the paper — relative squeezed balls. For an antichain \mathcal{S} in $\mathcal{F} = \mathcal{F}_{2k}^{[1,n]}$, let

$$\mathcal{S} - \mathbf{1}_{2k} := \{x - \mathbf{1}_{2k} : x = \{x_1, x_1 + 1, x_2, x_2 + 1, \dots, x_k, x_k + 1\} \in \mathcal{S}, x_1 > 1\},$$

and define $B_{\mathcal{S}} := B(\mathcal{S}) \setminus B(\mathcal{S} - \mathbf{1}_{2k})$ to be the difference of two squeezed balls. The goal of this section is to prove the following result that parallels Theorem 2.4:

Theorem 3.1. *Let $\mathcal{S}, \mathcal{S}'$ be non-empty antichains in $\mathcal{F} = \mathcal{F}_{2k}^{[1,n]}$. Then*

1. *The complex $B_{\mathcal{S}}$ is a $(k-1)$ -stacked PL $(2k-1)$ -ball. Furthermore, if \mathcal{S} contains $[1, 2] \cup [n-2k+3, n]$ as an element, then $B_{\mathcal{S}}$ has n vertices and is $(k-1)$ -neighborly.*
2. *If $\partial B_{\mathcal{S}} = \partial B_{\mathcal{S}'}$, then $B_{\mathcal{S}} = B_{\mathcal{S}'}$.*

In view of Theorem 3.1, we introduce the following terminology:

Definition 3.2. Let \mathcal{S} be a non-empty antichain in $\mathcal{F} = \mathcal{F}_{2k}^{[1,n]}$. The complex $B_{\mathcal{S}}$ is called a *relative squeezed $(2k-1)$ -ball* defined by \mathcal{S} . The boundary complex $\partial B_{\mathcal{S}}$ is the *relative squeezed $(2k-2)$ -sphere* defined by \mathcal{S} .

To motivate this definition and Theorem 3.1, consider the following example: let $k = 2$, $n = 8$, and $\mathcal{S} = \{\{1, 2, 7, 8\}, \{3, 4, 6, 7\}\}$. Then $B(\mathcal{S})$ is not 1-stacked as it has too many facets (ten facets instead of five). On the other hand, $\mathcal{S} - \mathbf{1}_4 = \{\{2, 3, 5, 6\}\}$ and the facets of $B_{\mathcal{S}}$ consist of

$$\{1, 2, 7, 8\}, \{1, 2, 6, 7\}, \{2, 3, 6, 7\}, \{3, 4, 6, 7\}, \text{ and } \{3, 4, 5, 6\}.$$

The above order of facets of $B_{\mathcal{S}}$ shows that $B_{\mathcal{S}}$ is indeed a 1-stacked 1-neighborly (w.r.t. [8]) 3-ball.

The proof of Theorem 3.1 requires quite a bit of preparation. To verify that $B_{\mathcal{S}}$ is a $(k-1)$ -stacked PL ball, we will utilize Lemma 2.2 along with inductive arguments on dimension. To this end, the elements of the form $[i, i'] \cup [j, j']$ in an antichain of $\mathcal{F}^{[i, j']}$ will play a special role (e.g., notice the element $[1, 2] \cup [n-2k+3, n]$ in the statement of Theorem 3.1) and the following definition will be indispensable.

Definition 3.3. Let \mathcal{S} be an antichain in $\mathcal{F} = \mathcal{F}_{2k}^{[1,n]}$, let $1 \leq \ell \leq k$, let $J = [j, j+2\ell-1]$ be a subset of $[n]$ of size 2ℓ , and let $m \geq 1$.

- Consider the following subcollection of $\mathcal{F}_{2(k-\ell)}^{[1,n]}$:

$$\{H \subseteq [j+2\ell, n] : J \cup H \in \mathcal{F}(\mathcal{S})\}.$$

Define $\mathcal{S}(J) \subseteq \mathcal{F}_{2(k-\ell)}^{[1,n]}$ to be the set of maximal elements of this collection.

- Define $\mathcal{F}(\mathcal{S}, m) = \mathcal{F}(\mathcal{S}) \cap \mathcal{F}^{[m, n]}$; that is, $\mathcal{F}(\mathcal{S}, m)$ is the collection of all sets G in $\mathcal{F}(\mathcal{S})$ such that the minimum of G is at least m . Similarly, define $\mathcal{F}_{2(k-\ell)}(\mathcal{S}(J), m)$ as $\mathcal{F}_{2(k-\ell)}(\mathcal{S}(J)) \cap \mathcal{F}_{2(k-\ell)}^{[m, n]}$, where as before $\mathcal{F}_{2(k-\ell)}(\mathcal{S}(J))$ denotes the order ideal of $\mathcal{F}_{2(k-\ell)}^{[1, n]}$ generated by $\mathcal{S}(J)$.
- Let $B(\mathcal{S}, m)$ be the pure simplicial complex whose collection of facets is $\mathcal{F}(\mathcal{S}, m)$. Similarly, let $B(\mathcal{S}(J), m)$ be the pure simplicial complex whose collection of facets is $\mathcal{F}_{2(k-\ell)}(\mathcal{S}(J), m)$. In particular, $B(\mathcal{S}, 1) = B(\mathcal{S})$ and $B(\mathcal{S}(J), 1) = B(\mathcal{S}(J))$.

Example 3.4. Consider the following antichain in $\mathcal{F} = \mathcal{F}_6^{[1, 14]}$:

$$\mathcal{S} := \{\{1, 2, 3, 4, 13, 14\}, \{1, 2, 6, 7, 11, 12\}, \{2, 3, 4, 5, 12, 13\}, \{2, 3, 5, 6, 10, 11\}, \{2, 3, 7, 8, 9, 10\}\}.$$

By definition,

$$\mathcal{S}([1, 2]) = \{\{3, 4, 13, 14\}, \{6, 7, 11, 12\}, \{4, 5, 12, 13\}, \{7, 8, 9, 10\}\},$$

$$\mathcal{S}([2, 3]) = \{\{4, 5, 12, 13\}, \{5, 6, 10, 11\}, \{7, 8, 9, 10\}\}, \quad \text{and} \quad \mathcal{S}([3, 4]) = \emptyset.$$

Furthermore, $B(\mathcal{S}([2, 3]), 6)$ is a 2-stacked 3-ball generated by the facets

$$\{6, 7, 8, 9\}, \{6, 7, 9, 10\}, \{7, 8, 9, 10\}.$$

Note also that $\mathcal{S}([2, 7]) = \{\emptyset\}$ since $[2, 7] \in \mathcal{F}(\mathcal{S})$, but $\mathcal{S}([3, 8]) = \emptyset$ since $[3, 8] \notin \mathcal{F}(\mathcal{S})$.

To study complexes of the form $B(\mathcal{S}) \setminus B(\mathcal{S} - \mathbf{1}_{2k})$, it will be helpful to look at complexes of the form $B(\mathcal{S}) \setminus B(\mathcal{T})$ for *all* pairs of antichains $\mathcal{T} \prec_p \mathcal{S}$. The following lemma, which is an easy consequence of Definition 3.3, is the first step in this direction. For the rest of the section, we fix $k \geq 1$, $n \geq 2k$, and we always assume that \mathcal{S}, \mathcal{T} are antichains in $\mathcal{F} = \mathcal{F}_{2k}^{[1, n]}$.

Lemma 3.5. *For $i, \ell \geq 1$,*

1. $\mathcal{S}([i+1, i+2]) \leq_p \mathcal{S}([i, i+1])$.
2. $(\mathcal{S} - \mathbf{1}_{2k})([i, i+1]) = \mathcal{S}([i+1, i+2]) - \mathbf{1}_{2k-2}$.
3. *If $\mathcal{T} \prec_p \mathcal{S}$, then $\mathcal{T}([i, i+2\ell-1]) \prec_p \mathcal{S}([i+1, i+2\ell])$.*

Proof: Parts 1 and 2 follow from Definition 3.3. Part 3 is a consequence of Part 2. Indeed,

$$\mathcal{T}([i, i+1]) \leq_p (\mathcal{S} - \mathbf{1}_{2k})([i, i+1]) = \mathcal{S}([i+1, i+2]) - \mathbf{1}_{2k-2} \prec_p \mathcal{S}([i+1, i+2]).$$

Hence, for $\ell = 2$,

$$\mathcal{T}([i, i+3]) = (\mathcal{T}([i, i+1]))([i+2, i+3]) \prec_p (\mathcal{S}([i+1, i+2]))([i+3, i+4]) = \mathcal{S}([i+1, i+4]).$$

The result now follows by induction on ℓ . □

As the proof of Theorem 3.1 is rather long and technical, it is worth to pause and outline the plan for the proof. Fix $\mathcal{T} \prec_p \mathcal{S}$. The first step is to decompose each complex $B(\mathcal{S}, i) \setminus B(\mathcal{T}, i)$ into analogous lower-dimensional objects joined with simplices, see Lemma 3.6 and Corollary 3.7. The minimum of each facet F determines which component of this decomposition F is placed in. The second step is to study the intersections of components appearing in this decomposition, see Lemma 3.10. With these results at our disposal, the last step is to use induction on the dimension to show that each complex $B(\mathcal{S}, i) \setminus B(\mathcal{T}, i)$ is a ball with the desired properties, see Lemmas 3.11 and 3.14.

Lemma 3.6. *The following decomposition holds:*

$$\begin{aligned} B(\mathcal{S}, i) &= \bigcup_{j \geq i} \left(B(\mathcal{S}([j, j+1]), j+2) * \overline{[j, j+1]} \right) \\ &= \left(B(\mathcal{S}([i, i+1]), i+2) * \overline{[i, i+1]} \right) \cup B(\mathcal{S}, i+1). \end{aligned} \quad (3.1)$$

Proof: By definition of $\mathcal{F}_{2(k-1)}(\mathcal{S}([j, j+1]), j+2)$, the facets of the complex on the right-hand side of (3.1) are also the facets of the complex on the left-hand side of (3.1). Conversely, let G be a facet of $B(\mathcal{S}, i)$ and let j be the minimal element of F . Then $[j, j+1] \subseteq G$ and by definition of $\mathcal{F}(\mathcal{S}([j, j+1]))$, $G \setminus \{j, j+1\} \in \mathcal{F}_{2(k-1)}(\mathcal{S}([j, j+1]), j+2)$. Thus G is also a facet of the complex on the right-hand side of (3.1). \square

Corollary 3.7. *If $\mathcal{T} \prec_p \mathcal{S}$, then*

$$B(\mathcal{S}, i) \setminus B(\mathcal{T}, i) = \bigcup_{j \geq i} \left((B(\mathcal{S}([j, j+1]), j+2) \setminus B(\mathcal{T}([j, j+1]), j+2)) * \overline{[j, j+1]} \right).$$

Our next step is to understand the intersections of components of the decomposition provided by Corollary 3.7. With this goal in mind, we fix \mathcal{S} and \mathcal{T} such that $\mathcal{T} \prec_p \mathcal{S}$ and introduce the following definition:

Definition 3.8. Define D_j as

$$D_j := \left(B(\mathcal{S}([j, j+1]), j+2) \setminus B(\mathcal{T}([j, j+1]), j+2) \right) * \overline{[j, j+1]}.$$

In plain English, the complex D_j is generated by all facets of $\mathcal{F}(\mathcal{S})$ that are of the form $[j, j+1] \cup H$ with $H \subseteq [j+2, n]$, and are not facets of $\mathcal{F}(\mathcal{T})$. Define also $\Gamma_{j,\ell}$ as

$$\Gamma_{j,\ell} := B(\mathcal{S}([j+1, j+2\ell]), j+2\ell+1) \setminus B(\mathcal{T}([j, j+2\ell-1]), j+2\ell+1).$$

That is, the complex $\Gamma_{j,\ell}$ is generated by the facets $H \subseteq [j+2\ell+1, n]$ such that $[j+1, j+2\ell] \cup H \in \mathcal{F}(\mathcal{S})$ but $[j, j+2\ell-1] \cup H \notin \mathcal{F}(\mathcal{T})$.

Lemma 3.9. *If D_{j+1} is not the void complex, then $D_j \cap D_{j+1}$ has the following decomposition according to initial segments of facets:*

$$D_j \cap D_{j+1} = \bigcup_{\ell=1}^k \left(\Gamma_{j,\ell} * \overline{[j+1, j+2\ell-1]} \right).$$

Proof: By definition, D_j and D_{j+1} are pure $(2k-1)$ -dimensional simplicial complexes that do not share common facets. We first show that $D_j \cap D_{j+1}$ is pure $(2k-2)$ -dimensional. Let F be a maximal (w.r.t. inclusion) face of $D_j \cap D_{j+1}$. Let G be a minimal (w.r.t. \leq_p) facet of D_j containing F . Note that G must contain $[j, j+1]$. In addition, since $G \in \mathcal{F}_{2k}$, G is a disjoint union of k pairs of the form $[q, q+1]$. This implies that $G = [j, j+2\ell-1] \cup M$, where $1 \leq \ell \leq k$ and $M \in \mathcal{F}_{2(k-\ell)}^{[j+2\ell+1, n]}$. It suffices to show that $\bar{G} := G \setminus \{j\} \cup \{j+2\ell\} \in D_{j+1}$ and hence $F = G \setminus \{j\} = \bar{G} \setminus \{j+2\ell\}$.

Suppose, to the contrary, that $\bar{G} \notin D_{j+1}$. Since $G \leq_p \bar{G}$ and $G \notin B(\mathcal{T})$, it follows that $\bar{G} \notin B(\mathcal{T})$, and so $\bar{G} \notin B(\mathcal{S})$ (or else, \bar{G} would be in D_{j+1}). The fact that $G \in B(\mathcal{S})$ then forces G

to be in \mathcal{S} . (Indeed, if G is in $B(\mathcal{S})$ but not in \mathcal{S} , then there must exist $G' \in \mathcal{S}$ such that $G <_p G'$. By definition of \bar{G} , such $G' \in B(\mathcal{S})$ satisfies $\bar{G} \leq_p G'$, which is impossible because $\bar{G} \notin B(\mathcal{S})$.)

Let $s := \max(G \setminus F)$. If $s \in [j, j + 2\ell - 1]$, then $M \subseteq F$, which together with $F \in D_{j+1}$ implies that $\bar{G} = [j + 1, j + 2\ell] \cup M \in D_{j+1}$, contradicting our assumption. If $s \notin [j, j + 2\ell - 1]$, we let H be the maximum (w.r.t. \leq_p) facet of \mathcal{F}_{2k} such that $H <_p G$ and $G \setminus \{s\} \subseteq H$. Our assumption that $j + 2\ell \notin G$ and the definition of \mathcal{F}_{2k} imply that H exists and that it can be expressed as $H = [j, j + 2\ell - 1] \cup (M \setminus \{s\}) \cup \{t\}$ for some t between $j + 2\ell$ and $s - 1$. Hence, $\min(H) = j$ and $H <_p G <_p H + \mathbf{1}_{2k}$. The fact that $G \in \mathcal{S}$, then implies that $H \in B(\mathcal{S}) \setminus B(\mathcal{S} - \mathbf{1}_{2k}) \subseteq B(\mathcal{S}) \setminus B(\mathcal{T})$. Thus, $H \in D_j$. This, however, contradicts our choice of G as a minimal facet of D_j containing F .

The above discussion shows that any maximal face $F \in D_j \cap D_{j+1}$ is a $(2k - 2)$ -face with the property that for some $1 \leq \ell \leq k$,

$$[j + 1, j + 2\ell - 1] \subseteq F, \quad j \notin F, \quad j + 2\ell \notin F,$$

and furthermore $F \cup \{j\}$ is a facet of D_j while $F \cup \{j + 2\ell\}$ is a facet of D_{j+1} . We conclude that $F \setminus [j + 1, j + 2\ell - 1]$ is a common facet of complexes

$$\begin{aligned} & B(\mathcal{S}([j, j + 2\ell - 1]), j + 2\ell + 1) \setminus B(\mathcal{T}([j, j + 2\ell - 1]), j + 2\ell + 1) \quad \text{and} \\ & B(\mathcal{S}([j + 1, j + 2\ell]), j + 2\ell + 1) \setminus B(\mathcal{T}([j + 1, j + 2\ell]), j + 2\ell + 1). \end{aligned}$$

Since by Part 3 of Lemma 3.5, $B(\mathcal{T}([j, j + 2\ell - 1]), j + 2\ell + 1) <_p B(\mathcal{S}([j + 1, j + 2\ell]), j + 2\ell + 1)$, all common facets of the above two complexes, including $F \setminus [j + 1, j + 2\ell - 1]$, are facets of

$$B(\mathcal{S}([j + 1, j + 2\ell]), j + 2\ell + 1) \setminus B(\mathcal{T}([j, j + 2\ell - 1]), j + 2\ell + 1) = \Gamma_{j,\ell}.$$

We infer that $D_j \cap D_{j+1} \subseteq \bigcup_{\ell=1}^k \left(\Gamma_{j,\ell} * \overline{[j + 1, j + 2\ell - 1]} \right)$.

For the other inclusion, assume that H is a facet of $\Gamma_{j,\ell}$. We need to show that

$$[j, j + 2\ell - 1] \cup H \in D_j \quad \text{and} \quad [j + 1, j + 2\ell] \cup H \in D_{j+1},$$

or, equivalently, that

$$\begin{aligned} & [j + 2, j + 2\ell - 1] \cup H \in B(\mathcal{S}([j, j + 1]), j + 2) \setminus B(\mathcal{T}([j, j + 1]), j + 2) \quad \text{and} \\ & [j + 3, j + 2\ell] \cup H \in B(\mathcal{S}([j + 1, j + 2]), j + 3) \setminus B(\mathcal{T}([j + 1, j + 2]), j + 3). \end{aligned}$$

This follows easily from our assumption that $H \in \Gamma_{j,\ell}$ using the definition of $\mathcal{S}([r, r + 2\ell - 1])$. \square

We are now ready to present a much more elegant description of $D_j \cap D_{j+1}$.

Lemma 3.10. *If D_{j+1} is not the void complex, then*

$$D_j \cap D_{j+1} = \left(B(\mathcal{S}([j + 1, j + 2]), j + 2) \setminus B(\mathcal{T}([j, j + 1]), j + 2) \right) * \overline{\{j + 1\}}.$$

Proof: It suffices to prove that

$$\begin{aligned} & \bigcup_{\ell=m}^k \left(\Gamma_{j,\ell} * \overline{[j + 1, j + 2\ell - 1]} \right) \\ & = \left(B(\mathcal{S}([j + 1, j + 2m]), j + 2m) \setminus B(\mathcal{T}([j, j + 2m - 1]), j + 2m) \right) * \overline{[j + 1, j + 2m - 1]}. \end{aligned}$$

Indeed, the case $m = 1$ of this equation together with Lemma 3.9 immediately yield the statement.

The proof is by reverse induction on m . The base case $m = k$ follows from the fact that $\mathcal{S}([j + 1, j + 2k]) = \{\emptyset\}$, while $\mathcal{T}([j, j + 2k - 1])$ is either $\{\emptyset\}$ or \emptyset . In any case,

$$\begin{aligned} B(\mathcal{S}([j + 1, j + 2k]), j + 2k + 1) &= B(\mathcal{S}([j + 1, j + 2k]), j + 2k) \quad \text{and} \\ B(\mathcal{T}([j, j + 2k - 1]), j + 2k + 1) &= (\mathcal{T}([j, j + 2k - 1]), j + 2k). \end{aligned}$$

The inductive step is a consequence of the following two observations. (Recall that $\bar{\emptyset} = \{\emptyset\}$.)

$$\begin{aligned} &B(\mathcal{S}([j + 1, j + 2m]), j + 2m) \setminus B(\mathcal{T}([j, j + 2m - 1]), j + 2m) \\ \stackrel{(*)}{=} &\left((B(\mathcal{S}([j + 1, j + 2m + 2]), j + 2m + 2) \setminus B(\mathcal{T}([j, j + 2m + 1]), j + 2m + 2)) * \overline{[j + 2m, j + 2m + 1]} \right) \\ &\cup \left(B(\mathcal{S}([j + 1, j + 2m]), j + 2m + 1) \setminus B(\mathcal{T}([j, j + 2m - 1]), j + 2m + 1) \right) \\ \stackrel{(**)}{=} &\bigcup_{\ell=m}^k \left(\Gamma_{j,\ell} * \overline{[j + 2m, j + 2\ell - 1]} \right). \end{aligned}$$

Here $(**)$ follows from the inductive hypothesis along with the definition of $\Gamma_{j,\ell}$. For $(*)$, note that $H = [j + 2m, j + 2m + 1] \cup G$ is a facet of $B(\mathcal{S}([j + 1, j + 2m]), j + 2m)$ if and only if the minimum element of G is at least $j + 2m + 2$, and furthermore there exists a minimal (w.r.t. \leq_p) facet H' such that $H \leq_p H'$ and $[j + 1, j + 2m] \cup H' \in B(\mathcal{S})$. This H' must be of the form $[j + 2m + 1, j + 2m + 2] \cup G'$ for some $G' \geq_p G$. Hence $G' \in B(\mathcal{S}([j + 1, j + 2m + 2]), j + 2m + 3)$ and $G \in B(\mathcal{S}([j + 1, j + 2m + 2]), j + 2m + 2)$. Thus by Lemma 3.6,

$$\begin{aligned} B(\mathcal{S}([j + 1, j + 2m]), j + 2m) &= \left(B(\mathcal{S}([j + 1, j + 2m + 2]), j + 2m + 2) * \overline{[j + 2m, j + 2m + 1]} \right) \\ &\cup B(\mathcal{S}([j + 1, j + 2m]), j + 2m + 1). \end{aligned}$$

This expression, along with the expression for $B(\mathcal{T}([j, j + 2m - 1]), j + 2m)$ given by Lemma 3.6, establishes $(*)$ and completes the proof of the lemma. \square

With Corollary 3.7 and Lemma 3.10 at our disposal, we are now in a position to prove the portion of Theorem 3.1 asserting that $B_{\mathcal{S}}$ is a $(k - 1)$ -stacked PL $(2k - 1)$ -ball. In fact, we prove the following stronger statement.

Lemma 3.11. *If $\mathcal{T} \prec_p \mathcal{S}$ and $B(\mathcal{S}, i)$ is not the void complex, then $B(\mathcal{S}, i) \setminus B(\mathcal{T}, i)$ is a k -stacked PL $(2k - 1)$ -ball. Furthermore, it is $(k - 1)$ -stacked if $\mathcal{T} = \mathcal{S} - \mathbf{1}_{2k}$.*

Proof: The proof is by induction on k . For $k = 1$, there exist integers $i < j < j'$ such that $B(\mathcal{S}, i)$ is generated by edges $\{i, i + 1\}, \{i + 1, i + 2\}, \dots, \{j' - 1, j'\}$ while $B(\mathcal{T}, i)$ is generated by edges $\{i, i + 1\}, \{i + 1, i + 2\}, \dots, \{j - 1, j\}$. Hence $B(\mathcal{S}, i) \setminus B(\mathcal{T}, i)$ is a path, and so it is indeed a 1-stacked 1-ball. If $j = j' - 1$ or, equivalently, if $T = S - \mathbf{1}_2$, this path consists of a single edge, and hence it is 0-stacked.

For the inductive step, note that by Corollary 3.7, $B(\mathcal{S}, i) \setminus B(\mathcal{T}, i) = \cup_{j \geq i} D_j$. By definition of D_j (see Definition 3.8), Lemma 3.10, the inductive hypothesis, and Part 1 of Lemma 2.2, each D_j is a $(k - 1)$ -stacked PL $(2k - 1)$ -ball, while each $D_j \cap D_{j+1}$ is a $(k - 1)$ -stacked PL $(2k - 2)$ -ball. Since $\mathcal{S}([j + 1, j + 2]) \leq_p \mathcal{S}([j, j + 1])$ and $\mathcal{T}([j + 1, j + 2]) \leq_p \mathcal{T}([j, j + 1])$ for all j , it follows from the definition of D_j that

$$D_{j+1} \cap D_i \subseteq D_{j+1} \cap D_{i+1} \subseteq \dots \subseteq D_{j+1} \cap D_j.$$

Hence $D_{j+1} \cap (D_i \cup D_{i+1} \cup \dots \cup D_j) = D_j \cap D_{j+1}$. By induction on $\ell - i$ and using Part 2 of Lemma 2.2, we infer that $\cup_{i \leq j \leq \ell} D_j$ is a k -stacked PL $(2k - 1)$ -ball. This proves the first claim.

Finally, if $\mathcal{T} = \mathcal{S} - \mathbf{1}_{2k}$, then, by inductive assumptions, every intersection $D_j \cap D_{j+1}$ is $(k - 2)$ -stacked. Hence the same argument as above using Part 2 of Lemma 2.2 yields that $B(\mathcal{S}, i) \setminus B(\mathcal{T}, i)$ is $(k - 1)$ -stacked. \square

Remark 3.12. The proof of Lemma 3.11 also implies that the ball $B(\mathcal{S}, i) \setminus B(\mathcal{T}, i)$ is constructible, see [3, Section 11] for the definition and properties of constructibility.

The part of Theorem 3.1 asserting that if $\partial B_{\mathcal{S}} = \partial B_{\mathcal{S}'}$, then $B_{\mathcal{S}} = B_{\mathcal{S}'}$ is now immediate.

Proof of Theorem 3.1(2): Consider a PL $(2k - 2)$ -sphere $K := \partial B_{\mathcal{S}} = \partial B_{\mathcal{S}'}$. By a result due to McMullen [9, Theorem 3.3] (for polytopal spheres) and due to Bagchi and Datta [2, Theorem 2.12] (for triangulated spheres), a $(k - 1)$ -stacked PL ball B that satisfies $\partial B = K$ is unique. The result follows since by Lemma 3.11 both $B_{\mathcal{S}}$ and $B_{\mathcal{S}'}$ are $(k - 1)$ -stacked PL balls. \square

To complete the proof of Theorem 3.1, it only remains to show that if \mathcal{S} contains the set $[1, 2] \cup [n - 2k + 3, n]$, then $B_{\mathcal{S}}$ is $(k - 1)$ -neighborly w.r.t. $[n]$. To do so, we first count the number of facets of such $B_{\mathcal{S}}$. For this part of the proof, it is more convenient to work with the poset $\mathcal{P} = \mathcal{P}_k^n$ (introduced at the end of Section 2.3) instead of $\mathcal{F} = \mathcal{F}_{2k}^{[1, n]}$. For an antichain \mathcal{A} of \mathcal{P} , define

$$\mathcal{A} - \mathbf{1}_k = \{x - \mathbf{1}_k : x = (x_1, x_2, \dots, x_k) \in \mathcal{A}, x_1 > 1\} \quad \text{and} \quad \mathcal{P}_{\mathcal{A}} = \mathcal{P}(\mathcal{A}) \setminus \mathcal{P}(\mathcal{A} - \mathbf{1}_k).$$

Note that the isomorphism $R : \mathcal{P} \rightarrow \mathcal{F}$ commutes with subtracting the all-ones vector: $R(\mathcal{A} - \mathbf{1}_k) = R(\mathcal{A}) - \mathbf{1}_{2k}$ and $R^{-1}(\mathcal{S} - \mathbf{1}_{2k}) = R^{-1}(\mathcal{S}) - \mathbf{1}_k$.

Lemma 3.13. *Let \mathcal{A} be an antichain of \mathcal{P} that contains $G = (1, n - 2k + 2, \dots, n - k)$. Then $|\mathcal{P}_{\{G\}}| = |\mathcal{P}_{\mathcal{A}}| = \binom{n - k - 1}{k - 1}$. Equivalently, if \mathcal{S} is an antichain of \mathcal{F} that contains $R(G) = [1, 2] \cup [n - 2k + 3, n]$, then the number of facets of $B_{\mathcal{S}}$ is $\binom{n - k - 1}{k - 1}$.*

Proof: First note that

$$\mathcal{P}_{\{G\}} = \mathcal{P}(\{G\}) = \{(1, x_2, x_3, \dots, x_k) : 1 < x_2 < x_3 < \dots < x_k \leq n - k\}.$$

Thus $|\mathcal{P}_{\{G\}}| = \binom{n - k - 1}{k - 1}$.

It remains to show that for a fixed antichain \mathcal{A} that contains G , $|\mathcal{P}_{\{G\}}| = |\mathcal{P}_{\mathcal{A}}|$. This will be done once we show that the following map is a bijection:

$$\begin{aligned} L : \mathcal{P}_{\mathcal{A}} &\rightarrow \mathcal{P}_{\{G\}} \\ x = (x_1, \dots, x_k) &\mapsto x - (x_1 - 1) \cdot \mathbf{1}_k = (1, x_2 - x_1 + 1, x_3 - x_1 + 1, \dots, x_k - x_1 + 1). \end{aligned}$$

To see that L is one-to-one, observe that for any $x, y \in \mathcal{P}$ such that $y = x + a \cdot \mathbf{1}_k$ for some $a \geq 1$, only x or y can be in $\mathcal{P}_{\mathcal{A}}$ but not both. Indeed, if $y \in \mathcal{P}_{\mathcal{A}}$, then $y \in \mathcal{P}(\mathcal{A})$. Thus $x \in \mathcal{P}(\mathcal{A} - \mathbf{1}_k)$, and hence $x \notin \mathcal{P}_{\mathcal{A}}$. To see that L is onto, note that any element of \mathcal{P} that is of the form $(1, x_2, x_3, \dots, x_k)$ is in $\mathcal{P}_{\{G\}} \subseteq \mathcal{P}(\mathcal{A})$. Consider the smallest $a \geq 1$ such that $(1, x_2, x_3, \dots, x_k) + a \cdot \mathbf{1}_k \notin \mathcal{P}(\mathcal{A})$ (it exists since for a sufficiently large, $(1, x_2, \dots, x_k) + a \cdot \mathbf{1}_k$ is not even in \mathcal{P}). Then $(1, x_2, \dots, x_k) + (a - 1) \cdot \mathbf{1}_k \in \mathcal{P}_{\mathcal{A}}$, and its image under L is $(1, x_2, \dots, x_k)$. \square

The neighborliness of $B_{\mathcal{S}}$ now follows easily:

Lemma 3.14. *Let \mathcal{S} be an antichain in \mathcal{F} that contains the set $[1, 2] \cup [n - 2k + 3, n]$. Then $B_{\mathcal{S}}$ is $(k - 1)$ -neighborly w.r.t. $[n]$.*

Proof: By Lemma 3.13, $f_{2k-1}(B_{\mathcal{S}}) = \binom{n-k-1}{k-1}$. Also, by Lemma 3.11, $B_{\mathcal{S}}$ is a $(k - 1)$ -stacked PL $(2k - 1)$ -ball, and so $h_i(B_{\mathcal{S}}) = 0$ for all $k \leq i \leq 2k$ (see Theorem 2.3(3)). Thus

$$\binom{n-k-1}{k-1} = f_{2k-1}(B_{\mathcal{S}}) = \sum_{i=0}^{2k} h_i(B_{\mathcal{S}}) = \sum_{i=0}^{k-1} h_i(B_{\mathcal{S}}).$$

Since $\binom{n-k-1}{k-1} = \sum_{i=0}^{k-1} \binom{n-2k+i-1}{i}$ and since by Theorem 2.3(1), $h_i(B_{\mathcal{S}}) \leq \binom{n-2k+i-1}{i}$ for all i , it follows that $h_i(B_{\mathcal{S}}) = \binom{n-2k+i-1}{i}$ for all $i \leq k-1$, which in turn implies that $B_{\mathcal{S}}$ is $(k-1)$ -neighborly w.r.t. $[n]$ (see Theorem 2.3(2)). \square

This concludes the proof of Theorem 3.1.

4 The number of neighborly $(d - 1)$ -spheres on n vertices

In this section we prove our main result, Theorem 1.2, asserting that $sn(d, n) \geq 2^{\Omega(n^{\lfloor (d-1)/2 \rfloor})}$. The following lemma provides an inductive method that given a neighborly sphere generates a new neighborly sphere with one additional vertex. This result is known and was used extensively to construct neighborly complexes. We refer to [13, 15] for a similar method (known as the sewing method) that was used to construct neighborly polytopes and neighborly oriented matroids; see also [12, Lemma 3.1] for an analogous statement in the centrally symmetric case.

Lemma 4.1. *Let Δ be a $\lfloor d/2 \rfloor$ -neighborly PL $(d - 1)$ -sphere on the vertex set $[n]$. Let B be a $(\lfloor d/2 \rfloor - 1)$ -neighborly (w.r.t. $[n]$) and $(\lfloor d/2 \rfloor - 1)$ -stacked PL $(d - 1)$ -ball contained in Δ . Then the complex $\Delta(B)$ obtained from Δ by replacing B with $\partial B * \overline{\{n + 1\}}$ is a $\lfloor d/2 \rfloor$ -neighborly PL $(d - 1)$ -sphere on $[n + 1]$.*

Proof: First note that B and $\partial B * \overline{\{n + 1\}}$ are PL $(d - 1)$ -balls with the same boundary. Hence $\Delta \setminus B$ is a PL $(d - 1)$ -ball and $\Delta(B) = (\Delta \setminus B) \cup (\partial B * \overline{\{n + 1\}})$ is a PL $(d - 1)$ -sphere. Moreover, since B is $(\lfloor d/2 \rfloor - 1)$ -stacked, it follows that

$$\begin{aligned} \text{Skel}_{\lfloor d/2 \rfloor - 1}(\Delta \setminus B) &= \text{Skel}_{\lfloor d/2 \rfloor - 1}(\Delta), \quad \text{and} \\ \text{Skel}_{\lfloor d/2 \rfloor - 2}(\text{lk}(n + 1, \Delta(B))) &= \text{Skel}_{\lfloor d/2 \rfloor - 2}(\partial B) = \text{Skel}_{\lfloor d/2 \rfloor - 2}(B). \end{aligned}$$

The fact that Δ is $\lfloor d/2 \rfloor$ -neighborly and B is $(\lfloor d/2 \rfloor - 1)$ -neighborly w.r.t. $[n]$ then shows that $\Delta(B)$ is $\lfloor d/2 \rfloor$ -neighborly w.r.t. $[n + 1]$. \square

Theorem 4.2. *Let $k \geq 2$. The number of distinct labeled $(k - 1)$ -neighborly (w.r.t. $[n]$) and $(k - 1)$ -stacked PL $(2k - 1)$ -balls that are contained in $\partial C_{2k}(n)$ is at least $2^{\Omega(n^{k-1})}$.*

Proof: By Theorem 3.1, $B_{\mathcal{S}}$ is a $(k - 1)$ -neighborly (w.r.t. $[n]$) and $(k - 1)$ -stacked PL $(2k - 1)$ -ball in $\partial C_{2k}(n)$ for each antichain \mathcal{S} in \mathcal{F} that contains the set $[1, 2] \cup [n - 2k + 3, n]$. All these balls are distinct labeled balls since their sets of maximal facets (w.r.t. \leq_p) are exactly the antichains \mathcal{S} . The number of such balls is the number of antichains containing $[1, 2] \cup [n - 2k + 3, n]$, which is at

least as large as the number of antichains in $\mathcal{F}_{2k}^{[3, n-2k+2]}$. As $\mathcal{F}_{2k}^{[3, n-2k+2]}$ is isomorphic to $\mathcal{F}_{2k}^{[1, n-2k]}$, the number of such antichains is at least $2^{\Omega((n-2k)^{k-1})} = 2^{\Omega(n^{k-1})}$ by Theorem 2.5. \square

We are finally ready to prove our main result, Theorem 1.2, asserting that for $d \geq 5$, the number of combinatorial types of $\lfloor d/2 \rfloor$ -neighborly $(d-1)$ -spheres on n labeled vertices is at least $2^{\Omega(n^{\lfloor (d-1)/2 \rfloor})}$.

Proof of Theorem 1.2: Consider the family \mathcal{H} of $(k-1)$ -neighborly (w.r.t. $[n]$) and $(k-1)$ -stacked PL $(2k-1)$ -balls contained in $\Delta := \partial C_{2k}(n)$. In the case of $d = 2k$, apply Lemma 4.1 to Δ and a ball B in this family to obtain the complex $\Delta(B)$ that is a k -neighborly PL $(2k-1)$ -sphere on $[n+1]$. These spheres $\Delta(B)$ are pairwise distinct because their restrictions to the vertex set $[n]$ are exactly the complexes $\Delta \setminus B$, and these are pairwise distinct. The result then follows from Theorem 4.2.

In the case of $d = 2k - 1$, consider the boundary complex of B for each $B \in \mathcal{H}$. Since B is a $(k-1)$ -stacked PL $(2k-1)$ -ball, all faces of B of dimension $\leq k-2$ are in ∂B . The fact that B is $(k-1)$ -neighborly then guarantees that ∂B is a $(k-1)$ -neighborly PL $(2k-2)$ -sphere. Furthermore, since the boundary complex of a $(k-1)$ -stacked PL $(2k-1)$ -ball uniquely determines that ball [2, Theorem 2.12], distinct elements of \mathcal{H} have distinct boundary complexes. The lower bound again follows from Theorem 4.2. \square

Remark 4.3. For $d \geq 5$, the number of combinatorial types of *unlabeled* $\lfloor d/2 \rfloor$ -neighborly $(d-1)$ -spheres on n vertices is also at least $2^{\Omega(n^{\lfloor (d-1)/2 \rfloor})}$. This is because dividing the lower bound by $n! = 2^{O(n \log n)}$ does not affect its asymptotic growth if $d \geq 5$.

We end the paper with an open problem. By the results of [7] and [8], both squeezed balls and squeezed spheres are shellable. It is natural to ask whether relative squeezed balls and spheres are also shellable. More generally, we pose the following problem.

Question 4.4. Let $k \geq 1$. Let $\mathcal{T} \prec_p \mathcal{S}$ be non-empty antichains in $\mathcal{F}_{2k}^{[1, n]}$. Is $B(\mathcal{S}) \setminus B(\mathcal{T})$ shellable? Is $\partial(B(\mathcal{S}) \setminus B(\mathcal{T}))$ shellable?

By Corollary 3.7, we write $B(\mathcal{S}) \setminus B(\mathcal{T})$ as $\cup_{j=1}^{\ell} D_j$. In the first nontrivial case $k = 2$, a shelling order for $B(\mathcal{S}) \setminus B(\mathcal{T})$ can be given as follows:

$$(F_{1,1}, \dots, F_{1,m_1}, F_{2,1}, \dots, F_{2,m_2}, \dots, F_{\ell,1}, \dots, F_{\ell,m_\ell}),$$

where $(F_{i,1}, \dots, F_{i,m_i})$ is the unique shelling order of D_i induced by the *reverse* partial order on the path $B(\mathcal{S}[i, i+1], i+2) \setminus B(\mathcal{T}([i, i+1]), i+2)$.

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