The merging operation and \((d - i)\)-simplicial \(i\)-simple \(d\)-polytopes

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Abstract

We define a certain merging operation that given two \(d\)-polytopes \(P\) and \(Q\) such that \(P\) has a simplex facet and \(Q\) has a simple vertex produces a new \(d\)-polytope \(P \triangleleft Q\) with \(f_0(P) + f_0(Q) - (d+1)\) vertices. We show that if for some \(1 \leq i \leq d-1\), \(P\) and \(Q\) are \((d-i)\)-simplicial \(i\)-simple \(d\)-polytopes, then so is \(P \triangleleft Q\). We then use this operation to construct new families of \((d-i)\)-simplicial \(i\)-simple \(d\)-polytopes. Specifically, we prove that for all \(2 \leq i \leq d-2 \leq 6\) with the exception of \((3,8)\) and \((5,8)\), there is an infinite family of \((d-i)\)-simplicial \(i\)-simple \(d\)-polytopes; furthermore, for all \(2 \leq i \leq 4\), there is an infinite family of self-dual \(i\)-simplicial \(i\)-simple \(2i\)-polytopes. Finally, we show that for every \(d \geq 4\), there are \(2^{\Omega(N)}\) combinatorial types of \((d-2)\)-simplicial \(2\)-simple \(d\)-polytopes with at most \(N\) vertices.

1 Introduction

A polytope is the convex hull of finitely many points in \(\mathbb{R}^d\). For brevity, we refer to \(d\)-dimensional polytopes as \(d\)-polytopes. While polytopes have been studied since antiquity, many central questions about them remain wide open. In this paper we present progress on one of these questions.

A \(d\)-polytope \(P\) is called simplicial if every facet of \(P\) contains exactly \(d\) vertices. Similarly, a \(d\)-polytope \(P\) is simple, if every vertex of \(P\) is in exactly \(d\) facets. (Equivalently, \(P\) is simple if its dual \(P^*\) is simplicial.) Much progress has been made on the study of

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simplicial and simple polytopes, but much less is known about general $d$-polytopes that are neither simplicial nor simple already when $d = 4$. We refer the reader to [8, 16] as excellent books on the theory of polytopes, to [3, 14] for one of the most celebrated results on the face numbers of simplicial polytopes, and to [2, 5, 12, 17, 18] for results on general $4$-polytopes.

Let $1 \leq i \leq d - 1$. A $d$-polytope $P$ is called $i$-simplicial if all of its $i$-faces are simplices, and it is $i$-simple if its dual $P^*$ is $i$-simplicial (equivalently, if every $(d - i - 1)$-face of $P$ is contained in exactly $i + 1$ facets). In particular, the class of $(d - 1)$-simplicial $d$-polytopes coincides with the class of simplicial $d$-polytopes, while the class of $(d - 1)$-simple $d$-polytopes is the class of simple $d$-polytopes. The $d$-simplex is both simple and simplicial, and it is known that a $j$-simplicial $i$-simple $d$-polytope must be a simplex if $i + j > d$. The question of whether $j$-simplicial $i$-simple $d$-polytopes exist when $i, j > 1$, and especially when $i + j = d$, was raised in the mid-1960s. Such polytopes can be compared to rare combinatorial objects like designs, and the constructions presented in this paper substantially advance our state of knowledge.

Let $2 \leq i \leq d - 2$. While various conjectures (see, for instance [8, Exercise 9.7.7(iii)]) suggest that there should be a large number of $(d - i)$-simplicial $i$-simple $d$-polytopes, not many examples are known. The first infinite family of $2$-simplicial $2$-simple $4$-polytopes was constructed by Eppstein, Kuperberg, and Ziegler [7]. Their approach was generalized by Paffenholz and Ziegler [13] who established the existence of infinite families of $(d - 2)$-simplicial $2$-simple $d$-polytopes for all $d \geq 4$. Notably, the minimum number of vertices in their $d$-dimensional construction is $2(d + 1)$, realized by $\text{conv}(\Sigma \cup \Sigma^*)$, where $\Sigma$ is a $d$-simplex whose $(d - 3)$-faces are tangent to the unit sphere $S^{d-1}$. Additional infinite families of $2$-simplicial $2$-simple $4$-polytopes were constructed by Paffenholz and Werner [12]: all their polytopes are elementary (i.e., have $g_2^{\text{toric}} = 0$) and have at least one simplex facet.

As for larger values of $i$, the $d$-dimensional demicube with $d \geq 4$ (also known as the half-cube) is $3$-simplicial $(d - 3)$-simple while its dual is $(d - 3)$-simplicial $3$-simple (see [8, Exercise 4.8.18]). Furthermore, the Gosset–Elte polytopes that arise from Wythoff’s construction provide finitely many examples of $(d - i)$-simplicial $i$-simple $d$-polytopes for $d \leq 8$ and $2 \leq i \leq d - 2$ [6]. These are essentially all known to-date examples of $(d - i)$-simplicial $i$-simple $d$-polytopes with $2 \leq i \leq d - 2$. In particular, it is not known whether a $5$-simplicial $5$-simple $10$-polytope exists. In light of this, we further pose the following questions.

**Question 1.1.**

1. Let $d \geq 4$. What is the minimum number of vertices that a non-simplex $(d - 2)$-simplicial $2$-simple $d$-polytope can have?

2. Let $d \geq 6$ and let $3 \leq i \leq d/2$. Are there infinite families of $(d - i)$-simplicial $i$-simple $d$-polytopes? What is the minimum number of vertices that such a non-simplex polytope can have?
The goal of this paper is to provide new infinite families of \((d-i)\)-simplicial \(i\)-simple \(d\)-polytopes for some values of \(i\) and \(d\). To achieve this, we define a certain merging operation that given two \(d\)-polytopes \(P\) and \(Q\), where \(P\) has a simplex facet and \(Q\) has a simple vertex, outputs a new \(d\)-polytope. This operation is modeled on a familiar notion of connected sums of simplicial polytopes, but designed in a way that preserves the property of being \((d-i)\)-simplicial \(i\)-simple. Using this operation, we establish the following results:

1. There exist infinite families of \((d-i)\)-simplicial \(i\)-simple \(d\)-polytopes for all pairs \((i, d)\) such that \(2 \leq i \leq d - 2 \leq 6\) and \((i, d)\) is not \((3, 8)\) or \((5, 8)\); see Theorem 5.1. This partially answers Question 1.1(2) and [10, Problem 19.5.23].

2. There exist infinite families of self-dual \(i\)-simplicial \(i\)-simple \(2i\)-polytopes for \(2 \leq i \leq 4\); see Theorem 5.4. This partially answers [10, Problem 19.5.24].

3. For all \(d \geq 4\), there are \(2^{\Omega(N)}\) combinatorial types of \((d - 2)\)-simplicial \(2\)-simple \(d\)-polytopes with at most \(N\) vertices; see Theorem 6.13.

To prove the last result, we construct a higher-dimensional analog of the unique \(2\)-simplicial \(2\)-simple 4-polytope with nine vertices. (This 4-polytope is called \(P_9\) in [12]; it has the minimum number of vertices among all non-simplex \(2\)-simplicial \(2\)-simple 4-polytopes.) We then apply the merging operation to produce new infinite families of \((d - 2)\)-simplicial \(2\)-simple \(d\)-polytopes.

As for the second result, several examples of (non-simplex) self-dual \(2\)-simplicial \(2\)-simple 4-polytopes were known before, among them polytopes \(P_9\) and \(P_{10}\) from [12]. In fact, [11] provides a (different) infinite family of self-dual \(2\)-simplicial \(2\)-simple 4-polytopes, that, for instance, includes the 24-cell. An interesting infinite family of self-dual \(d\)-polytopes that are neither \(j\)-simplicial nor \(i\)-simple (for any \(d \geq 3\) and \(j, i > 1\)) is the family of multiplexes constructed by Bisztriczky [4].

The outline of the paper is as follows. We review several definitions related to polytopes and face lattices in Section 2. Section 3 serves as a warm-up section where we discuss the minimum number of vertices that a non-simplex \(3\)-simplicial \(2\)-simple 5-polytope can have. In Section 4, we introduce and study the merging operation that applies to pairs of polytopes one of which has a simplex facet and another a simple vertex. This operation has several interesting properties; see, for instance, Theorem 4.6 and Theorem 4.12. Sections 5 and 6 form the most crucial part of this paper: there, we utilize the merging operation and its properties to provide our promised constructions of new \((d - i)\)-simplicial \(i\)-simple \(d\)-polytopes. Specifically, in Section 5.1, we construct infinite families of \((d - i)\)-simplicial \(i\)-simple \(d\)-polytopes for \(d \leq 8\). In Section 5.2, we construct infinite families of self-dual \(i\)-simplicial \(i\)-simple \(2i\)-polytopes for \(i \leq 4\). In Section 6.1, we revisit the \(2\)-simplicial \(2\)-simple 4-polytopes providing several new constructions. Finally, in Section 6.2, we produce a higher-dimensional analog of \(P_9\) and use it to construct exponentially many (in \(N\)) combinatorial types of \((d - 2)\)-simplicial \(2\)-simple \(d\)-polytopes with at most \(N\) vertices.
2 Preliminaries

A polytope $P \subseteq \mathbb{R}^d$ is the convex hull of a finite set of points in $\mathbb{R}^d$. The dimension of $P$ is the dimension of the affine span of $P$. For brevity, we say that $P$ is a $d$-polytope if $P$ is $d$-dimensional. In what follows, we always assume that $P \subseteq \mathbb{R}^d$ is a $d$-polytope.

A hyperplane $H \subseteq \mathbb{R}^d$ is a supporting hyperplane of $P$ if $P$ is contained in one of the two closed half-spaces determined by $H$. A (proper) face of $P$ is the intersection of $P$ with any supporting hyperplane of $P$. A face of a polytope is by itself a polytope. We refer to $(d-1)$-faces of $P$ as facets of $P$, to $(d-2)$-faces as ridges, to 1-faces as edges, and to 0-faces as vertices. We denote by $V(P)$ the vertex set of $P$. If $V(P)$ consists of $d+1$ affinely independent points, then $P$ is a $d$-simplex; we denote it by $\sigma_d$.

The face poset of $P$, $\mathcal{L}(P)$, is the set of faces of $P$ (including $P$ and $\emptyset$) ordered by inclusion, and two polytopes $P$ and $Q$ have the same combinatorial type if $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ are isomorphic. The face poset of $P$ is a lattice. We usually write the maximum element of $\mathcal{L}(P)$ (namely, $P$) as $\hat{1}$ and the minimum element (namely, $\emptyset$) as $\hat{0}$. For a subset $S$ of $\mathcal{L}(P)$, we let $\vee S$ and $\wedge S$ denote the join and the meet of elements of $S$, respectively.

By using translation, if necessary, we can always assume that the origin, $0$, lies in the interior of $P$. The set
\[ P^* = \{ y \in \mathbb{R}^d : y^t x \leq 1, \forall x \in P \} \]
is then a polytope called the dual polytope of $P$; see [16, Chapter 2]. The dual construction has the following properties: for every $d$-polytope $P \subseteq \mathbb{R}^d$ (with $0$ in the interior of $P$), $P^{**} = P$ and there are order-reversing bijective maps $\phi : \mathcal{L}(P) \rightarrow \mathcal{L}(P^*)$ and $\phi : \mathcal{L}(P^*) \rightarrow \mathcal{L}(P^{**}) = \mathcal{L}(P)$, which by slight abuse of notation we denote by the same symbol, such that $\phi(\phi(G)) = G$ for all $G \in \mathcal{L}(P) \cup \mathcal{L}(P^*)$. If $\mathcal{L}(P)$ is self-dual, that is, if there is an order reversing bijection from $\mathcal{L}(P)$ to itself, then we say that $P$ is a self-dual polytope.

Let $1 \leq i \leq d-1$. A $d$-polytope $P$ is $i$-simplicial if all of its $i$-faces are simplices; equivalently, if all of its $i$-faces have $i+1$ vertices. Similarly, $P$ is $i$-simple if every $(d-i-1)$-face is contained in exactly $i+1$ facets. The class of $(d-1)$-simplicial $d$-polytopes is known as the class of simplicial $d$-polytopes, while the class of $(d-1)$-simple $d$-polytopes is known as the class of simple $d$-polytopes. In particular, if $P$ is $i$-simplicial, then the interval $[0, \tau]$ is a Boolean lattice for any face $\tau$ with dim $\tau \leq i$. Likewise, if $P$ is $i$-simple, then $[\tau, 1]$ is Boolean for any face $\tau$ with dim $\tau \geq d-i-1$. Hence $P$ is $i$-simplicial if and only if $P^*$ is $(d-i)$-simple.

If $v$ is a vertex of $P$, then the vertex figure of $P$ at $v$, denoted $P/v$, is the polytope obtained by intersecting $P$ with a hyperplane $H$ that has $v$ on one side and all other vertices of $P$ on the other side. The combinatorial type of $P/v$ does not depend on the choice of $H$. In fact, $\mathcal{L}(P/v)$ is exactly the interval $[v, 1]$ in $\mathcal{L}(P)$. We say that a vertex $v$ of a $d$-polytope $P$ is simple if $P/v$ is a simplex, or equivalently, if $v$ belongs to exactly $d$ facets of $P$.

If $P$ is a simplicial polytope, then the collection of vertex sets of faces of $P$, including $\emptyset$ but not including $P$ itself, forms an abstract simplicial complex $\partial P$ called the boundary.
complex of $P$. When $V$ is a finite set, we let $\partial V := \{ \tau \subset V : \tau \neq V \}$ denote the boundary complex of an abstract simplex with vertex set $V$.

Consider a $d$-polytope $P \subset \mathbb{R}^d \times \{0\} \subset \mathbb{R}^d \times \mathbb{R}^d$ and a $d'$-polytope $Q \subset \{0\} \times \mathbb{R}^{d'} \subset \mathbb{R}^d \times \mathbb{R}^{d'}$ such that the origin is in the relative interior of both $P$ and $Q$. The polytope $P \oplus Q := \text{conv}(P \cup Q)$ is called the free sum of $P$ and $Q$. All faces of $P \oplus Q$ are of the form $\text{conv}(F \cup G)$, where $F \neq P$ is a face of $P$ and $G \neq Q$ is a face of $Q$. Consequently, if $P$ and $Q$ are simplicial polytopes then the boundary complex of $P \oplus Q$ coincides with the join of $\partial P$ and $\partial Q$:

$$\partial(P \oplus Q) = \partial P \ast \partial Q := \{ \sigma \cup \tau : \sigma \in \partial P, \tau \in \partial Q \}.$$ 

For a $d$-polytope $P$, we let $f(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P))$ be the $f$-vector of $P$; here $f_i(P)$ denotes the number of $i$-faces of $P$. Also, for $0 \leq i < j \leq d-1$, we let $f_{i,j}(P)$ denote the number of pairs of faces $F_i \subset F_j$ of $P$ such that $\dim F_i = i$ and $\dim F_j = j$.

To conclude this section, we note that for all $0 \leq i \leq d-1$, $f_i(P) = f_{d-1-i}(P^*)$. This is immediate from the existence of an order-reversing bijection $\phi : \mathcal{L}(P) \rightarrow \mathcal{L}(P^*)$.

3 A warm-up: the minimum number of vertices

As mentioned in the introduction, for every $d \geq 4$, there exists a $(d-2)$-simplicial $2$-simple $d$-polytope with $2(d+1)$ vertices. Furthermore, for $d = 4$, there is a $2$-simplicial $2$-simple $4$-polytope with only $9$ vertices. Are there non-simplex $(d-2)$-simplicial $2$-simple $d$-polytopes with fewer than $2d+2$ vertices for $d > 4$? (Cf. Question 1.1(1).) The goal of this warm-up section is to answer this question for $d = 5$; see Proposition 3.3. To do this, we first establish a criterion that the $f$-vectors of $(d-i)$-simplicial $i$-simple $d$-polytopes (if they exist) must satisfy; cf. [8, Exercise 9.7.7(ii)]. We include the proof for completeness.

**Lemma 3.1.** Let $d \geq 2$ and $1 \leq i \leq d-1$. Let $P$ be a $(d-i)$-simplicial $d$-polytope. Then $P$ is $i$-simple if and only if $(d-i+1)f_{d-i}(P) = (i+1)f_{d-i-1}(P)$.

**Proof:** If $P$ is $(d-i)$-simplicial, then every $(d-i)$-face of $P$ is a simplex; hence, every $(d-i)$-face contains $d-i+1$ faces of dimension $d-i-1$. This means that $f_{d-i-1,d-i}(P) = (d-i+1)f_{d-i}(P)$. On the other hand, a $(d-i-1)$-face of any $d$-polytope is contained in at least $i+1$ faces of dimension $d-i$. Thus, $f_{d-i-1,d-i}(P) \geq (i+1)f_{d-i-1}(P)$, and we conclude that $(d-i+1)f_{d-i}(P) = f_{d-i-1,d-i}(P) \geq (i+1)f_{d-i-1}(P)$. Furthermore, equality holds if and only if every $(d-i-1)$-face is in exactly $i+1$ faces of dimension $d-i$ which happens if and only if $P$ is $i$-simple. \qed

**Corollary 3.2.** For all $i \geq 1$, an $i$-simplicial $2i$-polytope $P$ is $i$-simple if and only if $f_{i-1}(P) = f_i(P)$.

**Proposition 3.3.** The minimum number of vertices that a non-simplex $3$-simplicial $2$-simple $5$-polytope can have is $12$. 


Proof: There exists a 3-simplicial 2-simple 5-polytope with 2(5 + 1) = 12 vertices. Thus, we only need to show that there is no non-simplex 3-simplicial 2-simple 5-polytope with fewer than 12 vertices.

It is known (see [12]) that every non-simplex 2-simplicial 2-simple 4-polytope has at least 9 vertices, and the only such polytope with 9 vertices is the polytope denoted by $P_9$ in [12]. Since vertex figures of 3-simplicial 2-simple 5-polytopes are 2-simplicial 2-simple, it follows that a non-simplex 3-simplicial 2-simple polytope $Q$ must have at least 10 vertices.

Assume that $f_0(Q) = 10$. Then each vertex figure is either the 4-simplex $σ_4$ or $P_9$, and so each vertex of $Q$ has degree 5 or 9. Since $Q$ is not simple, at least one of the vertex figures of $Q$ is $P_9$. Consider $Q^*$; it has 10 facets each of which is either $σ_4$ or $P_9$. (This is because both $σ_4$ and $P_9$ are self-dual.) Now consider a facet $F$ of $Q^*$ that is isomorphic to $P_9$. It has 7 non-simplex facets (one cross-polytope, also known as an octahedron, and six bipyramids); see Construction 6.1. Each of these seven 3-faces must lie in $F$ and one additional facet of $Q^*$, which cannot be a simplex. This shows that $Q^*$ has at least eight facets isomorphic to $P_9$. Then in $Q$, at least 8 out of 10 vertices are of degree 9. This implies that all vertices of $Q$ have degree $≥ 8$. Consequently, all vertices of $Q$ have degree 9, and so $f_1(Q) = \binom{10}{2} = 45$.

Since $Q$ is 3-simplicial 2-simple, $4f_3(Q) = 3f_2(Q)$ by Lemma 3.1. Furthermore, since $Q$ is 3-simplicial and since the toric $h$-vector of a 5-polytope is symmetric [15],

$$0 = g_3^{\text{toric}}(Q) = f_2(Q) - 4f_1(Q) + 10f_0(Q) - 20.$$

Finally, by the Euler relation, $f_0(Q) - f_1(Q) + f_2(Q) - f_3(Q) + f_4(Q) = 2$. This uniquely determines the $f$-vector of $Q$: $f(Q) = (10, 45, 100, 75, 12)$. But then we must have 75 = $f_3(Q) ≤ \binom{14}{2} = 66$, which is a contradiction.

Similarly, if $f_0(Q) = 11$, then $f_2(Q) = 4f_1(Q) - 10f_0(Q) + 20 = 4f_1(Q) - 90$, which is not a multiple of 4. On the other hand, $4f_3(Q) = 3f_2(Q)$ still holds, so $f_3(Q)$ is not an integer, which is again a contradiction. □

While a 2-simplicial 2-simple 4-polytope with 9 vertices is unique, this is not the case with 3-simplicial 2-simple 5-polytopes with 12 vertices. (For instance, in Section 6 we will see that there is such a polytope with a simplex facet.) For $d ≥ 6$, Question 1.1(1) remains unsolved. It would be very interesting to shed any light on whether the answer is $2d + 2$ or smaller than $2d + 2$.

4 The merging operation

Throughout, let $d ≥ 2$. Recall that a connected sum of two simplicial $d$-polytopes\(^1\) is a simplicial $d$-polytope. In other words, taking connected sums preserves the property

\(^1\)The connected sum of two simplicial polytopes $P$ and $Q$ is defined by gluing them along a common facet whose hyperplane separates $P$ and $Q$. To guarantee that the result is a polytope we first apply an appropriate projective transformation to $P$ (or $Q$).
of being \((d-1)\)-simplicial 1-simple. Is there an analogous operation that preserves the property of being \((d-i)\)-simplicial \(i\)-simple for an arbitrary \(2 \leq i \leq d-1\)? The goal of this section is to discuss one such operation that can be applied to two \(d\)-polytopes as long as one of them has a simplex facet and another one has a simple vertex. The order in which we list the vertices will be important for our construction. Specifically, we write \([a_1, \ldots, a_m]\) to denote the polytope \(\text{conv}(a_1, \ldots, a_m)\) whose vertices are ordered as \(a_1, \ldots, a_m\). We will mainly use this notation to describe faces of a given polytope. For brevity, we also write the edge \([u, v]\) as \(uv\).

### 4.1 The definition and basic properties

Let \(P_1\) and \(P_2\) be two \(d\)-polytopes such that \(P_1\) has a simplex facet \(F := [u_1, \ldots, u_d]\) and \(P_2\) has a simple vertex \(v\) whose neighbors are ordered as \(u'_1, \ldots, u'_d\). We adopt the following notation: for \(1 \leq j \leq d\), let \(H_j\) be the facet of \(P_1\) that is adjacent to \(F\) along the ridge \(G_j := [u_1, \ldots, \bar{u}_j, \ldots, u_d]\). Similarly, for \(1 \leq j \leq d\), let \(H'_j\) be the facet of \(P_2\) that contains all the edges of \(P_2\) incident with \(v\) but \(vu'_j\).

By applying a projective transformation to \(P_1\), we may assume that the hyperplanes \(\text{aff}(F), \text{aff}(H_1), \ldots, \text{aff}(H_d)\) define a \(d\)-simplex \(\Sigma\) that contains \(P_1\). Denote the vertex of \(\Sigma\) that does not lie in \(F\) by \(u\). By applying the unique affine transformation that maps \(v\) to \(u\), and \(u'_k\) to \(u_k\) for \(1 \leq k \leq d\), we may further assume that the \(d\)-simplices \(\Sigma' = [v, u'_1, \ldots, u'_d]\) and \(\Sigma\) coincide, and in particular that \(P_1 \subseteq \Sigma = \Sigma'\) is a convex subset of \(P_2\).

Finally, let \(P'_2 := \text{conv}(V(P_2) \setminus v)\) and \(F' := [u'_1, \ldots, u'_d]\) be two subpolytopes of \(P_2\). Note that if \(P_2\) is a \(d\)-simplex, then \(P'_2\) is \(F'\), and otherwise, \(F'\) is a facet of \(P'_2\).

**Definition 4.1.** Under the above assumptions on \(P_1\) and \(P_2\), define a new \(d\)-polytope \(P_1 \triangleright P_2\) obtained from \(P_2\) by replacing \(\Sigma' = \Sigma\) with \(P_1\). Alternatively, \(P_1 \triangleright P_2\) is the union of \(P_1\) and \(P'_2\) where we identify \(u_k\) with \(u'_k\) for \(1 \leq k \leq d\). (Observe that \(P_1\) and \(P'_2\) share the facet \(F = F'\), lie on the opposite sides of \(F\) and that their union is a polytope.) The new polytope is called the merge of \(P_1\) and \(P_2\) along \(F\) and \(v\).

**Example 4.2.** Consider two polygons \(P_1\) and \(P_2\) whose boundary complexes are cycles \((u_1, \ldots, u_n, u_1)\) and \((v_0, v_1, \ldots, v_k, v_0)\). Then the merge of \(P_1\) and \(P_2\) along the edge \(F = u_1u_n\) and the vertex \(v_0\) is the polygon whose boundary complex is the cycle \((v_1 = u_1, u_2, \ldots, u_{n-1}, u_n = v_k, v_{k-1}, \ldots, v_2, v_1 = u_1)\). In other words, in dimension 2, \(P_1 \triangleright P_2\) is exactly the connected sum of \(P_1\) and \(P'_2 = \text{conv}(V(P_2) \setminus v_0)\).

Figure 1 illustrates how to merge two 3-polytopes.

**Remark 4.3.** For \(d \geq 3\), the set of facets of \(P_1 \triangleright P_2\) consists of

- old facets: all facets of \(P_1\) with the exception of \(F, H_1, \ldots, H_d\), and all facets of \(P_2\) with the exception of \(H'_1, \ldots, H'_d\).
Figure 1: $P_1 \subseteq \Sigma$, $P_2 \supseteq \Sigma'$, and $P_1 \triangleright P_2$, where the merge is along $[u_1, u_2, u_3] \cong [u'_1, u'_2, u'_3]$ and $v$.

- new facets: for each $1 \leq j \leq d$, $H_j$ and $H'_j$ merge into a single facet $H_j \triangleright H'_j$ where the merge is along $G_j = [u_1, \ldots, \hat{u}_j, \ldots, u_d]$ and $v$ (with the neighbors of $v$ in $H'_j$ ordered as $u'_1, \ldots, \hat{u}'_j, \ldots, u'_d$).

**Remark 4.4.** The description of facets of $P_1 \triangleright P_2$ leads to the following observation: the combinatorial type of $P_1 \triangleright P_2$ may depend on the ordering of vertices of $F$ and neighbors of $v$. That is, letting $F = [u_{\sigma(1)}, \ldots, u_{\sigma(d)}]$ and relabeling the neighbors of $v$ as $v_{\sigma'(1)}, \ldots, v_{\sigma'(d)}$, for some permutations $\sigma, \sigma'$ of $[d] := \{1, 2, \ldots, d\}$, may result in a polytope with a different combinatorial type; see Section 6 for examples. This is analogous to the situation with the connected sum of two simplicial polytopes.

It follows from Definition 4.1 that if $P_1$ is a simplex, then $P_1 \triangleright P_2 = P_2$, and similarly if $P_2$ is a simplex, then $P_1 \triangleright P_2 = P_1$. In all other cases, $F$ is not a facet of $P_1 \triangleright P_2$ and $v$ is not a vertex of $P_1 \triangleright P_2$. Furthermore, if both $P_1$ and $P_2$ are simplicial and $P_2$ has a simple vertex $v$, then the merge of $P_1$ and $P_2$ along any facet $F$ of $P_1$ and $v$ is the connected sum of $P_1$ and $P'_2 = \text{conv}(V(P_2) \setminus v)$.

We summarize this discussion in the following lemma.

**Lemma 4.5.** Let $d \geq 2$. Let $P_1$ be a $d$-polytope with a simplex facet and let $P_2$ be a $d$-polytope with a simple vertex. Then $f_0(P_1 \triangleright P_2) = f_0(P_1) + f_0(P_2) - (d + 1)$. In particular, $f_0(P_1 \triangleright P_2) \geq \max\{f_0(P_1), f_0(P_2)\}$ and equality holds if and only if at least one of $P_1$ and $P_2$ is a simplex. In the case that one of $P_1$ and $P_2$ is a simplex, $P_1 \triangleright P_2$ is equal to the other polytope.

The following theorem and corollary explain the significance of the merging operation.

**Theorem 4.6.** Let $d \geq 2$ and $1 \leq i, j \leq d - 1$, and let $P_1$ and $P_2$ be $d$-polytopes with a simplex facet and a simple vertex, respectively. If $P_1$ and $P_2$ are $j$-simplicial, then so is $P_1 \triangleright P_2$. If $P_1$ and $P_2$ are $i$-simple, then so is $P_1 \triangleright P_2$. 
Proof: We first discuss \( j \)-simplicial polytopes. The proof is by induction on \( d \). The statement holds for \( j = 1 \) for any \( d \) (since all polytopes are 1-simplicial). Hence the statement holds for \( d = 2 \).

Now, assume the statement holds for \( d - 1 \) and any \( 1 \leq j \leq d - 2 \). We prove that the statement holds for \( d \) and any \( 1 \leq j \leq d - 1 \). Let \( P_1 \) and \( P_2 \) be two \( j \)-simplicial \( d \)-polytopes. If one of them is a simplex, there is nothing to prove. Also, if \( j = d - 1 \), then \( P_1 \bowtie P_2 \) is the connected sum of two simplicial polytopes \( P_1 \) and \( P_2' \), which is \((d-1)\)-simplicial.

Thus assume that \( 2 \leq j \leq d - 2 \) and that neither \( P_1 \) nor \( P_2 \) is a simplex. Let \( \tau \) be a \( j \)-face of \( P_1 \bowtie P_2 \). Then either \( \tau \) is a \( j \)-face of \( P_1 \) or it is a \( j \)-face of \( P_2 \) or it is a \( j \)-face of \( H_k \bowtie H_k' \) for some \( k \). In the first two cases, \( \tau \) is a simplex because \( P_1 \) and \( P_2 \) are \( j \)-simplicial. In the last case, it is a simplex because both \( H_k \) and \( H_k' \) are \( j \)-simplicial, and so \( \tau \) is a simplex by the induction hypothesis.

We now discuss \( i \)-simple polytopes. The proof is again by induction on \( d \). The statement holds for \( i = 1 \) and any \( d \) (since all polytopes are 1-simple). Hence the statement holds for \( d = 2 \). Now assume the statement holds for \( d - 1 \) and any \( 2 \leq i \leq d - 2 \). Let \( 2 \leq i \leq d - 1 \) and let \( P_1 \) and \( P_2 \) be two \( i \)-simple \( d \)-polytopes. To see that \( P_1 \bowtie P_2 \) is \( i \)-simple, let \( \tau \) be a \((d - i - 1)\)-face of \( P_1 \bowtie P_2 \). There are two possible cases.

Case 1: \( \tau \) is a face of one of \( H_k \bowtie H_k' \). Since \( P_1 \) and \( P_2 \) are \( i \)-simple, \( H_k \) and \( H_k' \) are \((i - 1)\)-simple \((d - 1)\)-polytopes. Thus, by the induction hypothesis, \( H_k \bowtie H_k' \) is an \((i - 1)\)-simple \((d - 1)\)-polytope. Since \( \tau \) is a face of \( H_k \bowtie H_k' \), there are \( i \) facets of \( H_k \bowtie H_k' \) (and hence ridges of \( P_1 \bowtie P_2 \)) that contain \( \tau \). Each of these \( i \) ridges is contained in \( i + 1 \) facets of \( P_1 \bowtie P_2 \), namely, \( H_k \bowtie H_k' \) and the \( i \) additional facets just described.

Case 2: \( \tau \) is not contained in any \( H_k \bowtie H_k' \) (for \( k = 1, \ldots, d \)). Then either \( \tau \) is a face of \( P_1 \) not contained in any of \( F, H_1, \ldots, H_d \), or \( \tau \) is a face of \( P_2 \) that does not contain \( v \) and is not contained in any of \( H_1', \ldots, H_d' \). In the former case, the facets of \( P_1 \bowtie P_2 \) that contain \( \tau \) are the facets of \( P_1 \) that contain \( \tau \) and there are \( i + 1 \) of them since \( P_1 \) is \( i \)-simple. Similarly, in the latter case, the facets of \( P_1 \bowtie P_2 \) that contain \( \tau \) are the facets of \( P_2 \) that contain \( \tau \) and there are \( i + 1 \) of them.

Corollary 4.7. Let \( d \geq 2 \) and \( 1 \leq i \leq d - 1 \). Let \( P \) be a \((d - i)\)-simplicial \( i \)-simple \( d \)-polytope such that (1) \( P \) is not a simplex, (2) \( P \) has a simplex facet \( F \), and (3) \( P \) has a simple vertex \( v \) not contained in \( F \). Finally, let \( P \bowtie P \) be the merge of \( P \) with itself along \( F \) and \( v \). Then \( P \bowtie P \) is a \((d - i)\)-simplicial \( i \)-simple \( d \)-polytope that has a simplex facet and a simple vertex not contained in that facet; furthermore, \( f_0(P \bowtie P) > f_0(P) \). Consequently, there exists an infinite family of \((d - i)\)-simplicial \( i \)-simple \( d \)-polytopes obtained by iterative merging with \( P \).

Proof: Consider two copies of \( P \): \( P_1 \) and \( P_2 \). Denote the copy of \( F \) in \( P_j \) by \( F_j \), and the copy of \( v \) in \( P_j \) by \( v_j \). Merge \( P_1 \) and \( P_2 \) along \( F_1 \) and \( v_2 \). By Theorem 4.6, \( P_1 \bowtie P_2 \) is \((d - i)\)-simplicial and \( i \)-simple; it has a simplex facet \( F_2 \) and a simple vertex \( v_1 \notin F_2 \). \( \square \)
In this subsection, we assume that \( P \) that will be merged along a simplex facet \( i \).

Question 4.8. Let \( d \geq 4 \) and \( 2 \leq i \leq d - 2 \). Are there infinite families of \((d - i)\)-simplicial \( i \)-simple \( d \)-polytopes, each of which has a simplex facet and a simple vertex?

4.2 The face lattice

In this subsection, we assume that \( P_1 \) and \( P_2 \) are two \((d - i)\)-simplicial \( i \)-simple \( d \)-polytopes that will be merged along a simplex facet \( F = \{u_1, \ldots, u_d\} \) of \( P_1 \) and a simple vertex \( v \) of \( P_2 \).

Our goal is to describe the face lattice of \( P \), and let \( L \) be their \((d - i)\)-face of \( P \) and \( \tau \) a \((d - i)\)-face of \( P_1 \). Consider the following two subposets of \( L \):

\[
\mathcal{L}(P_1) := \{ \sigma : \sigma \subseteq F, \dim \sigma \geq d - i \},
\]

\[
\mathcal{L}(P_2) := \{ \sigma : v \in \sigma, \dim \sigma < d - i \},
\]

and let \( \mathcal{L}(P_1) \sqcup \mathcal{L}(P_2) \) be their disjoint sum, i.e., the disjoint union of \( \mathcal{L}(P_1) \) and \( \mathcal{L}(P_2) \) with the original partial orders on \( \mathcal{L}(P_1) \) and \( \mathcal{L}(P_2) \), and no other comparable pairs.

Definition 4.9. Consider the following two subposets of \( \mathcal{L}(P_1) \) and \( \mathcal{L}(P_2) \):

\[
\mathcal{L}(P_1)^{-} := \mathcal{L}(P_1) \setminus \{ \sigma : \sigma \subseteq F, \dim \sigma \geq d - i \},
\]

\[
\mathcal{L}(P_2)^{-} := \mathcal{L}(P_2) \setminus \{ \sigma : v \in \sigma, \dim \sigma < d - i \},
\]

and let \( \mathcal{L}(P_1)^{-} \sqcup \mathcal{L}(P_2)^{-} \) be their disjoint sum, i.e., the disjoint union of \( \mathcal{L}(P_1)^{-} \) and \( \mathcal{L}(P_2)^{-} \), with no other comparable pairs.

Definition 4.10. Let \( \mathcal{L} \) be the following quotient poset of \( \mathcal{L}(P_1)^{-} \sqcup \mathcal{L}(P_2)^{-} \). As a set, it is \( (\mathcal{L}(P_1)^{-} \sqcup \mathcal{L}(P_2)^{-}) / \sim \), where

\[
[u_k : k \in S] \sim [u'_k : k \in S] \quad \text{for all } S \subseteq [d], \ |S| \leq d - i,
\]

and \( \cap_{k \in S} H_k \sim \cap_{k \in S} H'_k \) for all \( S \subseteq [d], \ |S| \leq i \).

The partial order on \( \mathcal{L} \) is inherited from \( \mathcal{L}(P_1)^{-} \sqcup \mathcal{L}(P_2)^{-} : [\tau] < [\sigma] \) if there are representatives \( \tau' \) and \( \sigma' \) of the equivalence classes \([\tau]\) and \([\sigma]\) such that \( \tau' < \sigma' \) in \( \mathcal{L}(P_1)^{-} \sqcup \mathcal{L}(P_2)^{-} \).

The main result of this subsection —Theorem 4.12— asserts that \( \mathcal{L} \) is the face lattice of \( P_1 \sqcup P_2 \). The proof relies on the following lemma.

Lemma 4.11. Let \( S \subseteq [d] \).

1. If \( |S| \leq i \), then \( \cap_{k \in S} H_k \) is a \((d - |S|)\)-face of \( P_1 \) not contained in \( F \), while \( \cap_{k \in S} H'_k \) is a \((d - |S|)\)-face of \( P_2 \) containing \( v \).

2. If \( |S| \leq d - i \), then \([u_k : k \in S]\) is an \(|S| - 1\)-face of \( P_1 \) and \([u'_k : k \in S]\) is an \(|S| - 1\)-face of \( P_2 \).

3. If \( H \) is a facet of \( P_1 \) that is not one of \( F, H_1, \ldots, H_d \), then \( H \) shares with \( F \) at most \( d - i - 1 \) vertices, and \( H \) does not contain any intersection of the form \( \cap_{k \in S} H_k \), for \( S \subseteq [d], \ |S| \leq i \). Hence, \( \mathcal{L}(H) \) is equal to \([0, H]\) computed in both \( \mathcal{L}(P_1)^{-} \) and \( \mathcal{L} \).
4. If $H$ is a facet of $P_2$ that does not contain $v$, then $H$ does not contain any intersection of the form $\cap_{k \in S} H'_k$. Thus $\mathcal{L}(H)$ is equal to $[\hat{0}, H]$ computed in both $\mathcal{L}(P_2^-)$ and $\mathcal{L}$.

Proof: For part (1), we only need to show that $\cap_{k \in S} H_k$ is $(d - |S|)$-dimensional and that it is not contained in $F$. Consider $\tau := (\cap_{k \in S} H_k) \cap F = \cap_{k \in S} (H_k \cap F)$. Since $F$ is a $(d - 1)$-simplex, $\tau$ is a face of $P_1$ of dimension $d - |S| - 1$. Now, since $|S| \leq i$, and so $d - |S| - 1 \geq d - i - 1$, the assumption that $P_1$ is $i$-simple implies that the interval $[\tau, \hat{1}]$ is a Boolean lattice whose coatoms are $H_k$, for $k \in S$, and $F$. This, in turn, implies the desired properties of $\cap_{k \in S} H_k$.

For part (2), since $F$ is a simplex facet of $P_1$, $[u_k : k \in S]$ must be a simplex $(|S| - 1)$-face of $P_1$. Also, since $v$ is simple, the edges $vu'_k$ for $k \in S$ determine an $|S|$-face of $P_2$, and this face must be a simplex since $P_2$ is $(d - i)$-simplicial. Thus $[u'_k : k \in S]$ is an $(|S| - 1)$-face of $P_2$.

For part (3), note that if $H$ contained $d - i$ vertices of $F$, say, $u_1, \ldots, u_{d-i}$, then $[u_1, \ldots, u_{d-i}]$ would be a $(d - i - 1)$-face of $P_1$ contained in at least $i + 2$ facets, namely, $F$, $H_{d-i+1}, \ldots, H_d$, and $H$; this is impossible since $P$ is $i$-simple. Similarly, if $H$ contained, say, the face $H_1 \cap \cdots \cap H_i$, then this $(d - i)$-face would be in at least $i + 1$ facets, namely, $H_1, \ldots, H_i$, and $H$, which is again a contradiction.

Part (4) follows from the fact that $v \in \cap_{k \in S} H'_k$ but $v \notin H$, and from the definition of $\mathcal{L}(P_2^-)$ and $\mathcal{L}$.

Let $S$ be a subset of $[d]$. Note that $\hat{0}_{P_1} = \vee_{k \in S} u_k \sim \vee_{k \in \emptyset} u'_k = \hat{0}_{P_2}$ is the minimum element of $\mathcal{L}$, while $1_{P_2} = \wedge_{k \in S} H_k \sim \wedge_{k \in \emptyset} H'_k = 1_{P_2}$ is the maximum element. Furthermore, Lemma 4.11 implies that if $|S| \leq d - i$, then $\vee_{k \in S} u_k \in \mathcal{L}(P_1)$ and $\vee_{k \in S} u'_k \in \mathcal{L}(P_2)$ are both elements of $\mathcal{L}(P_1)^- \sqcup \mathcal{L}(P_2)^-$, and that they have the same rank. Similarly, if $|S| \leq i$, then $\wedge_{k \in S} H_k$ and $\wedge_{k \in S} H'_k$ both belong to $\mathcal{L}(P_1)^- \sqcup \mathcal{L}(P_2)^-$ and have the same rank there. We are now ready to prove that $\mathcal{L}$ is the face lattice of $P_1 \triangleright P_2$. Specifically, for $S \subseteq [d], |S| \leq i$, the class $\wedge_{k \in S} H_k \sim \wedge_{k \in S} H'_k$ in $\mathcal{L}$ represents the face $\cap_{k \in S} (H_k \triangleright H'_k)$ of $P_1 \triangleright P_2$.

Theorem 4.12. Let $d \geq 2$ and $1 \leq i \leq d - 1$. Let $P_1$ and $P_2$ be $(d - i)$-simplicial $i$-simple polytopes such that $P_1$ has a simplex facet $F = [u_1, \ldots, u_d]$ and $P_2$ has a simple vertex $v$ whose neighbors are $u'_1, \ldots, u'_d$. Then $\mathcal{L} = \mathcal{L}(P_1 \triangleright P_2)$.

Proof: The proof is by induction on $d$ and $i$. First we consider the case where $P_1$ and $P_2$ are both $(d - 1)$-simplicial $1$-simple $d$-polytopes. This case splits into two subcases:

1. If $P_2$ is not a simplex, then $P_1 \triangleright P_2 = P_1 \# P_2'$. The lattice $\mathcal{L}(P_1 \triangleright P_2)$ is obtained from $\mathcal{L}(P_1)$ and $\mathcal{L}(P_2')$ by removing facets $[u_1, \ldots, u_d]$ and $[u'_1, \ldots, u'_d]$ and identifying their boundary complexes; this agrees with our definition of $\mathcal{L}(P_1)^- \sqcup \mathcal{L}(P_2)^- / \sim = \mathcal{L}$.

2. If $P_2$ is a simplex, then $P_1 \triangleright P_2$ is $P_1$. That $\mathcal{L}$ is equal to $\mathcal{L}(P_1)$ in this case, again follows easily from the definition of $\mathcal{L}$. 

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This discussion completes the proof of the base case \( i = 1 \) and arbitrary \( d \geq 2 \).

Now assume that the statement holds in dimension \( \leq d - 1 \) and consider two \((d - i)\)-simplicial \( i \)-simple \( d \)-polytopes \( P_1 \) and \( P_2 \), where \( i \geq 2 \). By definition, \( \mathcal{L} \) and \( \mathcal{L}(P_1 \triangleright P_2) \) have the same coatoms. So it suffices to show that for every facet \( H \) of \( P_1 \triangleright P_2 \), the interval \([0, H]\) in \( \mathcal{L} \) is equal to \( \mathcal{L}(H) \).

First, if \( H \) is a facet of \( P_1 \) not equal to \( F, H_1, \ldots, H_d \), or \( H \) is a facet of \( P_2 \) that does not contain \( v \), then by Lemma 4.11, the interval \([0, H]\) in \( \mathcal{L} \) is equal to \( \mathcal{L}(H) \). For \( 1 \leq k \leq d \), both \( H_k \) and \( H'_k \) are \((d - i)\)-simplicial \((i - 1)\)-simple \((d - 1)\)-polytopes. In particular,

\[
\mathcal{L}(H_k)^- = \mathcal{L}(H_k)\backslash\{\sigma : \sigma \subseteq F \backslash u_k, \ \dim \sigma \geq (d - 1) - (i - 1) = d - i\},
\]

\[
\mathcal{L}(H'_k)^- = \mathcal{L}(H'_k)\backslash\{\sigma : \sigma \subseteq v, \ u'_k \notin \sigma, \ \dim \sigma < (d - 1) - (i - 1) = d - i\}.
\]

Hence \([0, H_k]\) computed in \( \mathcal{L}(P_1) \) is \( \mathcal{L}(H_k)^- \) and \([0, H'_k]\) computed in \( \mathcal{L}(P_2) \) is \( \mathcal{L}(H'_k)^- \). Then the inductive hypothesis implies that \([0, H_k \triangleright H'_k]\) in \( \mathcal{L} \) is equal to \( \mathcal{L}(H_k \triangleright H'_k) \). This proves that \( \mathcal{L} = \mathcal{L}(P_1 \triangleright P_2) \).

One application of Theorem 4.12 is the following result on the \( f \)-numbers of \( P_1 \triangleright P_2 \).

**Corollary 4.13.** Let \( d \geq 2 \) and \( 1 \leq i \leq d - 1 \). Let \( P_1 \) and \( P_2 \) be \((d - i)\)-simplicial \( i \)-simple \( d \)-polytopes that can be merged along a simplex facet \( F \) of \( P_1 \) and a simple vertex \( v \) of \( P_2 \). Then for all \( 0 \leq j \leq d - 1 \), \( f_j(P_1 \triangleright P_2) = f_j(P_1) + f_j(P_2) - \binom{d+1}{j+1} \).

**Proof:** First assume that \( 0 \leq j \leq d - i - 1 \). By definition of \( \mathcal{L}(P_1 \triangleright P_2) \), each \( j \)-face of \( F \) (i.e., each \((j + 1)\)-subset of \( \{u_1, \ldots, u_d\} \)), is identified with the corresponding \( j \)-face of \( F' \) (i.e., the corresponding \((j + 1)\)-subset of \( \{u'_1, \ldots, u'_d\} \)). In addition, all \((j + 1)\)-subsets of \( v, u'_1, \ldots, u'_d \) that contain \( v \) are removed from \( \mathcal{L}(P_1 \triangleright P_2) \). Hence

\[
f_j(P_1 \triangleright P_2) = f_j(P_1) + f_j(P_2) - \binom{d}{j+1} = f_j(P_1) + f_j(P_2) - \binom{d+1}{j+1}.
\]

Similarly, for \( d - i \leq j \leq d - 1 \), by definition of \( \mathcal{L}(P_1 \triangleright P_2) \), all \( j \)-faces of \( P_1 \) contained in \( F \) (i.e., \((j + 1)\)-subsets of \( \{u_1, \ldots, u_d\} \)) are removed from \( \mathcal{L}(P_1 \triangleright P_2) \), while for each \((d - j)\)-subset \( S \) of \( \{d\} \), the \( j \)-face \( \cap_{k \in S} H_k \) is identified with the \( j \)-face \( \cap_{k \in S} H'_k \). Hence

\[
f_j(P_1 \triangleright P_2) = f_j(P_1) + f_j(P_2) - \binom{d}{j+1} - \binom{d}{d-j} = f_j(P_1) + f_j(P_2) - \binom{d+1}{j+1}.
\]

\( \square \)

## 5 Applications: part I

### 5.1 Infinite families of \((d - i)\)-simplicial \( i \)-simple polytopes for small \( d \)

The goal of this section is to answer Question 4.8 in the affirmative for small values of \( d \). Our starting point is the uniform \( 8 \)-polytope \( 2_{41} \) constructed within the symmetry of
the $E_8$ group. (It was first discovered by Gosset and Elte; see also [6, Section 11]). This polytope has 17280 simplex facets and it is 4-simplicial and 4-simple. The polytope $2_{41}$ gives rise to the following 7-polytopes:

- Each nonsimplex facet of $2_{41}$ is the 7-polytope $2_{31}$. It is 4-simplicial 3-simple and it has 576 simplex facets.
- Each vertex figure of $2_{41}$ is the 7-demicube.

Recall that the $d$-demicube is defined as follows (see [8, Exercise 4.8.18]). Consider the $d$-cube $C_d = [0,1]^d$. For each vertex $v$ in $C_d$ whose coordinates have an even number of ones, truncate $C_d$ along the hyperplane that contains all $d$ vertices adjacent to $v$. The resulting polytope is called the $d$-demicube; we denote it by $Q_d$. This polytope has the following properties:

- When $d > 4$, $Q_d$ has exactly $2^{d−1}$ simplex facets (these are the facets defined by truncating hyperplanes), and $2d$ non-simplex facets (these are the facets obtained by truncating the facets of $C_d$). Moreover, no two simplex facets are adjacent in $Q_d$.
- When $d ≥ 4$, $Q_d$ is 3-simplicial and $(d−3)$-simple.

We are now in a position to prove the main result of this subsection:

**Theorem 5.1.** For every element of $\{(i,d) : 2 ≤ i ≤ d−2 ≤ 6\}\{(3,8), (5,8)\}$, there exists an infinite family of $(d−i)$-simplicial $i$-simple $d$-polytopes, each of which has a simplex facet and a simple vertex not in that facet.

**Proof:** By considering dual polytopes, it suffices to prove the statement for $i ≤ d/2 ≤ 4$. The case of $i = 2$ and an arbitrary $d ≥ 4$ will be discussed in Section 6. For now, we mention that for $i = 2$ and $d = 4$, the result follows by applying Corollary 4.7 to $P_9$. (For the description of facets of $P_9$, see Construction 6.1.) Consider the case of $i = 3$ and $d = 6$. Since both $Q_6$ and $Q_6^*$ are 3-simplicial 3-simple, and since $Q_6$ has a simplex facet (in fact, 32 of them) and $Q_6^*$ has a simple vertex (in fact, 32 of them), the merge of $Q_6$ and $Q_6^*$, $P = Q_6 \triangleright Q_6^*$, is well-defined; furthermore, $P$ has a simplex facet $F$ and a simple vertex $v$ not contained in $F$. Hence, Corollary 4.7 applies to $P$ and results in a desired infinite family of 3-simplicial 3-simple 6-polytopes. Similarly, in the case of $i = 3$ and $d = 7$, apply Corollary 4.7 to $P = 2_{31} \triangleright Q_7^*$. Finally, in the case of $i = 4$ and $d = 8$, apply Corollary 4.7 to $P = 2_{41} \triangleright 2_{41}^*$.

The proof of Theorem 5.1 provides the following partial answer to Question 4.8.

**Corollary 5.2.** Let $2 ≤ i ≤ 4$. There exists an infinite family of $i$-simplicial $i$-simple $2i$-polytopes, each of which has a simplex facet and a simple vertex not in that facet.
5.2 Self-dual polytopes

Kalai [10, Problem 19.5.24] asked for which values of $i$ and $d$ there are self-dual $i$-simplicial $d$-polytopes other than the $d$-simplex. For the rest of this section, assume that $d = 2i$ and consider an $i$-simplicial $i$-simple $2i$-polytope $P$ with a simplex facet $F = [u_1, \ldots, u_{2i}]$. As before, assume that $H_1, \ldots, H_d$ are the facets of $P$ adjacent to $F$, where $H_k \cap F = [u_1, \ldots, \hat{u}_k, \ldots, u_d]$. Let $\phi : \mathcal{L}(P) \to \mathcal{L}(P^*)$, $\phi : \mathcal{L}(P^*) \to \mathcal{L}(P)$ be the order-reversing bijections on the face lattices. Then $P^*$ is an $i$-simplicial $i$-simple $2i$-polytope with a simple vertex $v := \phi(F)$. The neighbors of $v$ are $u'_k := \phi(H_k)$ for $1 \leq k \leq d$. Let $H'_k$ be the facet of $P^*$ determined by the edges $vu'_1, \ldots, vu'_k, \ldots, vu'_d$. In other words, $H'_k = (\vee_{j \in [d] \setminus k} u'_j) \cap v$, and hence

$$\phi(H'_k) = (\wedge_{j \in [d] \setminus k} \phi(u'_j)) \wedge \phi(v) = (\wedge_{j \in [d] \setminus k} H_j) \wedge F = u_k. \quad (5.1)$$

The next proposition is our main tool for constructing self-dual $i$-simplicial $i$-simple $2i$-polytopes. We follow assumptions and notation introduced in the previous paragraph.

**Proposition 5.3.** The merge of $P$ and $P^*$ along $F = [u_1, \ldots, u_d]$ and $v$ (whose neighbors are ordered as $u'_1, \ldots, u'_d$) is a self-dual polytope.

**Proof:** The map $\phi : \mathcal{L}(P) \to \mathcal{L}(P^*), \mathcal{L}(P^*) \to \mathcal{L}(P)$ provides us with an order-reversing involution on $\mathcal{L}(P) \cup \mathcal{L}(P^*)$. Since $\phi(H_k) = u'_k$ and $\phi(H'_k) = u_k$, it follows that for $S \subseteq [d]$,

$$\phi(\vee_{k \in S} u_k) = \wedge_{k \in S} H'_k, \quad \phi(\vee_{k \in S} u'_k) = \wedge_{k \in S} H_k. \quad (5.1)$$

In particular, $\phi$ maps $\ell$-faces of $F$ to $(d - \ell - 1)$-faces containing $v$. Since $d = 2i$, it follows that $\phi$ induces an order-reversing involution on $\mathcal{L}(P)^- \cup \mathcal{L}(P^*)^-$. Furthermore, by (5.1), this involution descends to an order-reversing involution on the quotient $\mathcal{L}$ described in Definition 4.10. Thus $\mathcal{L}$ is a self-dual lattice. The result follows since by Theorem 4.12, $\mathcal{L} = \mathcal{L}(P \triangleright P^*)$.

**Theorem 5.4.** For all $2 \leq i \leq 4$, there exists an infinite family of self-dual $i$-simplicial $2i$-polytopes.

**Proof:** Let $2 \leq i \leq 4$. By Corollary 5.2, there exists an infinite family of $i$-simplicial $i$-simple $2i$-polytopes each of which has a simplex facet. The result follows by applying Proposition 5.3 to this family.

6 Applications: part II

This section is devoted to $(d - 2)$-simplicial $2$-simple $d$-polytopes for all $d \geq 4$. We show that for such values of parameters, the answer to Question 4.8 is yes, and, in fact, that for every $d \geq 4$, there are $2^{\Omega(N)}$ combinatorial types of $(d - 2)$-simplicial $2$-simple $d$-polytopes with at most $N$ vertices, each of which has a simplex facet and a simple vertex. Section 6.1 concentrates on a few constructions for $d = 4$; Section 6.2 treats the general case.
6.1 Revisiting 2-simplicial 2-simple 4-polytopes

By a result of Paffenholz and Werner [12], there exist infinite families of 2-simplicial 2-simple 4-polytopes each of which has a simplex facet and a simple vertex. This solves Question 4.8 in the affirmative in dimension $d = 4$.

In this section, we provide alternative (and more symmetric) constructions. We start by revisiting the construction from [12] of $P_9$ — the unique 2-simplicial 2-simple 4-polytope with nine vertices — casting it in a way that will help us construct higher-dimensional analogs of $P_9$ in Section 6.2. We then provide another construction of a highly symmetric 2-simplicial 2-simple 4-polytope with 18 vertices that appears to be new. The promised infinite families are obtained by merging $k$ copies of $P_9$ (respectively, $P_{18}$) for all natural numbers $k \geq 2$. The cross-polytope is featured prominently in our constructions, and we often abbreviate it as CP. (The notion of a point beyond or beneath a facet is defined in [8, page 78].)

Construction 6.1. To construct $P_9$, start with a regular 4-simplex $\Sigma := [u'_1, u'_2, u'_3, u'_4, u'_5]$. Now add the vertices $u_1, u_2, u_3, v_2$ in the following way. (Why we label the vertices in this fashion will become clear in Section 6.2.) For $i = 1, 2, 3$, place $u_i$ in the affine hull of the facet $\Sigma \setminus u'_i$ of $\Sigma$ so that it is positioned beyond the 2-face $\Sigma \setminus u'_5$ and so that $[u_1, u_2, u_3, u'_1, u'_2, u'_3]$ is a 3-cross-polytope; cf. Definition 6.8 below. (Hence $u_i$ can be thought of as a perturbation of the barycenter of $[u'_j, u'_k, u'_\ell]$, where $\{i, j, k, \ell\} = \{4\}$.) Then position $v_2$ on the intersection of the affine hulls of $[u'_1, u'_4, u_2, u_3]$, $[u'_2, u'_4, u_1, u_3]$, and $[u'_3, u'_4, u_1, u_2]$ (this intersection is a line) and beyond the hyperplane aff$([u'_4, u_1, u_2, u_3])$; cf. Definitions 6.7 and 6.9. (Thus, $v_2$ is a special perturbation of the barycenter of $[u_1, u_2, u_3, u'_4]$).

The resulting polytope has nine vertices $\{v_2, u_1, u_2, u_3, u'_1, \ldots, u'_5\}$; it is also convenient to let $v_1 = u'_4$. Figure 2 shows part of the Schlegel diagram of $P'_9 = \text{conv}(V(P_9) \setminus u'_5)$. The complete list of facets of $P_9$ is given as follows (cf. Lemma 6.10):

1. a CP with antipodal facets $[u_1, u_2, u_3]$ and $[u'_1, u'_2, u'_3]$ (colored in blue) and a simplex $[u'_1, u'_2, u'_3, u'_5]$;
2. three bipyramids $[u_1, u'_5, u'_2, u'_3, u'_4]$, $[u_2, u'_5, u'_1, u'_3, u'_4]$, and $[u_3, u'_5, u'_1, u'_2, u'_4]$, where the pairs of suspension vertices are $(u_1, u'_5)$, $(u_2, u'_5)$, and $(u_3, u'_5)$, respectively;
3. three more bipyramids $[v_2, u'_1, u_2, u_3, v_1]$ (colored in purple), $[v_2, u'_2, u_1, u_3, v_1]$, and $[v_2, u'_3, u_1, u_2, v_1]$, where the pairs of suspension vertices are $(v_2, u'_1)$, $(v_2, u'_2)$, and $(v_2, u'_3)$, respectively;
4. another simplex $[v_2, u_1, u_2, u_3]$ (colored in orange).

The list of facets shows that $P_9$ is 2-simplicial. The $f$-vector of $P_9$ is symmetric, namely, $f(P_9) = (9, 26, 26, 9)$. Thus, by Corollary 3.2, $P_9$ is also 2-simple. Furthermore, $P_9$ has two pairs of a simplex facet and a simple vertex not in that facet: $([v_2, u_1, u_2, u_3], u'_5)$ and
Figure 2: Parts of the Schlegel diagrams of $P'_9$. 

Take two copies of $P_9$, $P'_9$ and $P''_9$, and consider the merge $P'_9 \triangledown P''_9$ along $[v_2, u_1, u_2, u_3]$ from $P'_9$ and $u'_5$ from $P''_9$. Since the facets of $P_9$ containing $u'_5$ consist of a simplex and three bipyramids, depending on the order in which we list the neighbors of $u'_5$, the cross-polytopal facet of $P'_9$ will either be merged with a 3-simplex or with a bipyramid of $P''_9$, resulting in two distinct combinatorial types of 2-simplicial 2-simple 4-polytopes, each of which has a simplex facet and a simple vertex not in that facet. This observation will allow us to construct exponentially many (in the number of vertices) 2-simplicial 2-simple 4-polytopes. We will return to this discussion (and provide many more details) in Section 6.2 after we construct a $d$-dimensional analog of $P_9$ for all $d \geq 4$; see Theorem 6.13 and Remark 6.14.

How does merging with $P_9$ affect the $f$-numbers? Let $Q$ be a 2-simplicial 2-simple 4-polytope that has a simplex facet and a simple vertex not in this facet (for instance, $Q = P_9$). Then $P_9 \triangledown Q$ and $Q \triangledown P_9$ are both defined and by Corollary 4.13,

$$f(P_9 \triangledown Q) - f(Q) = f(Q \triangledown P_9) - f(Q) = f(P_9) - \left( \binom{5}{1}, \binom{5}{2}, \binom{5}{3}, \binom{5}{4} \right) = (9, 26, 26, 9) - (5, 10, 10, 5) = (4, 16, 16, 4).$$

Recall that the toric $g_2$-number of a 2-simplicial 4-polytope is given by $g_2^{\text{toric}} = f_1 - 4f_0 + 10$ and that any polytope with $g_2^{\text{toric}} = 0$ is called an elementary polytope. It then follows that $P_9$ is an elementary polytope and that $g_2^{\text{toric}}(P_9 \triangledown Q) = g_2^{\text{toric}}(Q \triangledown P_9) = g_2^{\text{toric}}(Q)$. In other words, if $Q$ is also an elementary polytope, then so are $P_9 \triangledown Q$ and $Q \triangledown P_9$. (Elementary polytopes play an important role in the Lower Bound Theorem, see [9].)
It is worth pointing out that if one applies to $Q$ the second construction from [12, Section 3.2], the resulting polytope $T^2(Q)$ has the same $f$-vector as $f(P_9 \triangleright Q) = f(Q \triangleright P_9)$; see [12, Theorem 3.7]. At the same time, both polytopes $P_9 \triangleright Q$ and $Q \triangleright P_9$ are different from $T^2(Q)$. Indeed, merging with $P_9$, on the left or on the right, always generates a facet (contributed by the cross-polytopal facet of $P_9$) that is isomorphic to either CP or the connected sum of CP with another 3-polytope, while in the second construction of [12], all new facets are stacked 3-polytopes with either 4, 5, or 6 vertices.

Our next task is to describe another highly-neighborly 2-simplicial 2-simple 4-polytope with a simplex facet and a simple vertex. This polytope has 18 vertices and we denote it by $P_{18}$. 

**Construction 6.2.** We start with a regular 3-simplex $F = [v_1, v_2, v_3, v_4]$ in $\mathbb{R}^3 \times \{0\}$. Specifically, let

$$v_1 = (0, 0, 0, 0), v_2 = (2, 2, 0, 0), v_3 = (2, 0, 2, 0), v_4 = (0, 2, 2, 0).$$  \hspace{1cm} (6.1)

Define $u = (1, 1, 1, h)$ for some $h > 0$. Let $0 < \epsilon \ll 1$. For all distinct $1 \leq i, j, k \leq 4$, let

$$u_{ji,k} = u_{ij,k} = \frac{1}{2}(v_i + v_j) + \epsilon(u + v_k - v_i - v_j).$$

That is,

$$u_{12,3} = (1 + \epsilon, 1 - \epsilon, 3\epsilon, h\epsilon), u_{12,4} = (1 - \epsilon, 1 + \epsilon, 3\epsilon, h\epsilon), u_{13,2} = (1 + \epsilon, 3\epsilon, 1 - \epsilon, h\epsilon),$$

$$u_{13,4} = (1 - \epsilon, 3\epsilon, 1 + \epsilon, h\epsilon), u_{14,2} = (3\epsilon, 1 + \epsilon, 1 - \epsilon, h\epsilon), u_{14,3} = (3\epsilon, 1 - \epsilon, 1 + \epsilon, h\epsilon),$$

$$u_{23,1} = (2 - 3\epsilon, 1 - \epsilon, 1 - \epsilon, h\epsilon), u_{23,4} = (2 - 3\epsilon, 1 + \epsilon, 1 + \epsilon, h\epsilon), u_{24,1} = (1 - \epsilon, 2 - 3\epsilon, 1 - \epsilon, h\epsilon),$$

$$u_{24,3} = (1 + \epsilon, 2 - 3\epsilon, 1 + \epsilon, h\epsilon), u_{34,1} = (1 - \epsilon, 1 - \epsilon, 2 - 3\epsilon, h\epsilon), u_{34,2} = (1 + \epsilon, 1 + \epsilon, 2 - 3\epsilon, h\epsilon).$$

Note that each $u_{ij,k}$ can be viewed as a certain perturbation of the barycenter of $[v_i, v_j]$ that keeps it in the hyperplane defined by $[u, v_i, v_j, v_k]$. Note also that the set of vertices $\{u_{ij} : \{i, j\} \in \{2, 3, 4\}\}$ forms a hexagon $H_1$ that lies in the plane defined by equations $x_1 + x_2 + x_3 = 2 + 3\epsilon, x_4 = h\epsilon$. Similarly, the sets of vertices $\{u_{2i,j} : \{i, j\} \in \{1, 3, 4\}\}, \{u_{3i,j} : \{i, j\} \in \{1, 2, 4\}\}$, and $\{u_{4i,j} : \{i, j\} \in \{1, 2, 3\}\}$ form hexagons $H_2, H_3, H_4$ in the planes defined by equations $x_1 + x_2 = 2 - 3\epsilon, x_4 = h\epsilon$, and $x_1 - x_2 + x_3 = 2 - 3\epsilon, x_4 = h\epsilon$, respectively. It follows that

$$\text{aff}(v_1 \cup H_1) = \{x \in \mathbb{R}^4 : -h\epsilon(x_1 + x_2 + x_3) + (2 + 3\epsilon)x_4 = 0\},$$

$$\text{aff}(v_2 \cup H_2) = \{x \in \mathbb{R}^4 : h\epsilon(x_1 + x_2 - x_3) + (2 + 3\epsilon)x_4 = 4h\epsilon\},$$

$$\text{aff}(v_3 \cup H_3) = \{x \in \mathbb{R}^4 : h\epsilon(x_1 + x_3 - x_2) + (2 + 3\epsilon)x_4 = 4h\epsilon\},$$

$$\text{aff}(v_4 \cup H_4) = \{x \in \mathbb{R}^4 : h\epsilon(x_2 + x_3 - x_1) + (2 + 3\epsilon)x_4 = 4h\epsilon\}.$$

The intersection of these four hyperplanes is the point $(1, 1, 1, \frac{3h\epsilon}{2 + 3\epsilon})$; we denote it by $w$. 

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Define $P'_{18}$ as the convex hull of all 17 vertices \{$w, v_1, \ldots, v_4, u_{ij,k} : 1 \leq i, j, k \leq 4\}$. When $\epsilon$ is very small, the polytope $P'_{18}$ has the following 19 facets (see Figure 3 for part of the Schlegel diagram). We used $\epsilon = 0.05$, $h = 2$ and verified this list with software SAGE.

1. Six simplices of the form $[v_i, v_j, u_{ij,k}, u_{ij,m}]$, where \{$i, j, k, m\} = [4]$. Parts of four of them are shown in blue in Figure 3.

2. Four simplices of the form $[u_{ij,k}, u_{ik,j}, u_{jk,i}, w]$, where $1 \leq i, j, k \leq 4$ are distinct. One such simplex is shown in purple in Figure 3.


4. Four polytopes of the form $[v_i, w, u_{ij,k}, u_{ij,m}, u_{ik,j}, u_{im,j}, u_{im,k}]$. Each is the suspension over $H_i$, with suspension vertices $v_i$ and $w$. (Here \{$i, j, k, m\} = [4]$.) One such polytope is shown in orange in Figure 3.

5. Four cross-polytopes of the form $[v_i, v_j, v_k, u_{ij,k}, u_{ik,j}, u_{jk,i}]$, where $1 \leq i, j, k \leq 4$ are distinct.

To complete the construction of $P_{18}$, we apply a projective transformation $\pi$ to $P'_{18}$ to ensure that the adjacent facets of $G = [v_1, v_2, v_3, v_4]$, i.e., the four cross-polytopes from the last item, intersect at a point $w'$ beyond $G$. We let $P_{18} = \text{conv}(\pi(P'_{18}) \cup w')$. Then $G$ is not a facet of $P_{18}$ and each facet $[v_i, v_j, v_k, u_{ij,k}, u_{ik,j}, u_{jk,i}]$ is replaced by its connected sum with $[v_i, v_j, v_k, w']$. It can be checked that $f(P_{18}) = (18, 64, 64, 18)$. Since $P_{18}$ is a 2-simplicial 4-polytope that has $f_1 = f_2$, it follows by Corollary 3.2 that $P_{18}$ is also 2-simple. A direct computation shows that $g_2^{\text{toric}}(P_{18}) = 2$. In other words, $P_{18}$ is not elementary.
Observe that $P_{15}$ has a simple vertex $w'$ and many simplex facets not containing $w'$ (see the first item in the list). Thus we can iteratively merge $P_{15}$ with itself and obtain an infinite sequence of 2-simplicial 2-simple 4-polytopes, each having at least one simplex facet and one simple vertex. By Corollary 4.13, any polytope obtained by merging $k \geq 1$ copies of $P_{15}$ will have $5 + 13k$ vertices and $g_2^{\text{toric}} = 2k$. Other families of 2-simplicial 2-simple 4-polytopes where the $k$th polytope has $g_2^{\text{toric}} = 2k$ (but $f_0 = 10 + 4k$) were constructed in [13, Corollary 4.2].

To close this section, we propose the following problem.

**Question 6.3.** Is there a sequence of 2-simplicial 2-simple 4-polytopes that approximate the unit ball?

In light of [1, Theorem 3.2], it is natural to conjecture that if such a sequence of 4-polytopes \{Q_i\} exists, then $\lim_{i \to \infty} g_2^{\text{toric}}(Q_i) = \infty$.

### 6.2 Many $(d - 2)$-simplicial 2-simple $d$-polytopes

In this section we construct a $d$-dimensional analog of $P_9$ for all $d \geq 4$. We then use this polytope along with Corollary 4.7 to show that there are $2^{\Omega(N)}$ combinatorial types of $(d - 2)$-simplicial 2-simple $d$-polytopes with at most $N$ vertices and an additional property that each of these polytopes has a simplex facet and a simple vertex.

As in Section 6.1, the $d$- and $(d - 1)$-dimensional cross-polytopes are used frequently, and we abbreviate them as CP. To start, we introduce the notion of a pseudo-regular CP and prove some of its properties. Let $0$ denote the origin of $\mathbb{R}^{d-1}$.

**Definition 6.4.** Let $G \subset \mathbb{R}^{d-1}$ be a regular $(d - 1)$-simplex centered at the origin, let $G^* \subset \mathbb{R}^{d-1}$ be the dual of $G$, and let $\alpha > 0$ be a real number. Assume also that $G$ is contained in the interior of $\alpha G^*$, denoted $\text{int}(\alpha G^*)$. A $d$-cross-polytope is called pseudo-regular if it is congruent to $\text{conv}(G \times \{1\} \cup \alpha G^* \times \{-1\})$.

Consider a regular simplex $G = [\mu_1, \ldots, \mu_d] \subset \mathbb{R}^{d-1}$ centered at the origin and let $\alpha > 0$. Then $\alpha G^* = [\mu_1', \ldots, \mu_d'] \subset \mathbb{R}^{d-1}$ is also a regular simplex centered at the origin. We label the vertices in such a way that $\mu_i'$ is an outer normal vector to the facet $[\mu_1, \ldots, \hat{\mu}_i, \ldots, \mu_d]$ of $G$. By our assumptions on $G$, this is equivalent to labeling the vertices so that for all $i \in [d]$, $\mu_i' = a \sum_{j \in [d] \setminus \{i\}} \mu_j = -a \mu_i$, where $a$ is a positive scalar independent of $i$.

For a nonempty subset $I$ of $[d]$, let $G_I = [\mu_i : i \in I]$ be a face of $G$ and $G'_I = [\mu_i' : i \in I]$ be a face of $\alpha G^*$; let $\beta_I = \frac{1}{|I|} \sum_{i \in I} \mu_i$ be the barycenter of $G_I$ and $\beta'_I = \frac{1}{|I|} \sum_{i \in I} \mu_i'$ be the barycenter of $G'_I$. Since for all $i \in [d]$, $\mu_i' = a \sum_{j \in [d] \setminus \{i\}} \mu_j = -a \mu_i$, it follows that for any proper subset $I$ of $[d]$, $\sum_{i \in I} \mu_i = -\frac{1}{a} \sum_{i \in I} \mu_i' = \frac{1}{a} \sum_{j \in [d] \setminus I} \mu_j'$. Thus, $\beta_I$ is a positive multiple of $\beta'_I$ and so the ray from $0$ and through $\beta_I$ coincides with the ray from $0$ and through $\beta'_I$. Furthermore, since $G$ is regular, the distance from $0$ to $\beta_I$ is the same for all $k$-subsets $I$ of $[d]$; we denote it by $\rho_k$ and note that $\rho_1 > \cdots > \rho_{d-1}$. Similarly, for all
of CP by $u$.

First note that indeed, consider triangles $\ell_1, \ell_2, \ldots, \ell_k$. For a subset $I$ of $[d]$, we denote the barycenter of $G_I \times \{1\}$ by $b_I$ and the barycenter of and $G_I' \times \{-1\}$ by $b_I'$. Finally, we let $H_I$ denote the hyperplane in $\mathbb{R}^d$ determined by the following set of $d$ points: \{$u_i : i \in I\} \cup \{u_j' : j \in [d] \setminus I\}$.

**Lemma 6.5.** Let $0 \leq k \leq d$. Then all hyperplanes $H_I$, where $I \subseteq [d], |I| = k$, intersect the $x_d$-axis at the same point. When $0 < k < d$, the $d$th coordinate of this point is $>1$.

**Proof:** First note that $H_{[d]}$ and $H_{\emptyset}$ intersect the $x_d$-axis at $e_d := (0, \ldots, 0, 1)$ and $-e_d$, respectively. Now let $I$ be any $k$-subset of $[d]$, where $1 \leq k \leq d-1$. Consider the points $b_I$ and $b_I'_{[d] \setminus I}$. Both of them lie in $H_I$; hence, so does the line $\ell = \text{aff}(b_I, b_I'_{[d] \setminus I})$.

We claim that $\ell$ intersects the $x_d$-axis. Consequently,

$$H_I \cap x_d = \ell \cap x_d.$$ 

To prove the claim, consider the lines $\text{aff}(e_d, b_I)$ and $\text{aff}(-e_d, b_I'_{[d] \setminus I})$. By discussion following Definition 6.4, these lines are parallel, and thus determine a 2-dimensional plane $\mathcal{L}$. For the rest of the proof, we work in this plane. It contains $\ell$ and the $x_d$-axis. Also, since $\beta_I$ is a positive multiple of $\beta_{[d] \setminus I}$, the points $b_I$ and $b_I'_{[d] \setminus I}$ lie on the same side of the $x_d$-axis in $\mathcal{L}$. Finally, since the distance from $b_I$ to the $x_d$-axis is $\rho_k$, the distance from $b_I'_{[d] \setminus I}$ to the $x_d$-axis is $\rho_{d-k}$, and $\rho_{d-k} > \rho_k$, it follows that $\ell$ and the $x_d$-axis are not parallel. Hence they intersect and the point of intersection, which we denote by $a_I = (0, \ldots, 0, c_I)$, satisfies $c_I > 1$. This proves the claim.

To complete the proof of the lemma, it remains to show that $c_I$ depends only on $|I| = k$. Indeed, consider triangles $[a_I, e_d, b_I]$ and $[a_I, -e_d, b_I'_{[d] \setminus I}]$. They are similar; hence,

$$\frac{c_I - 1}{\rho_k} = \frac{\text{dist}(a_I, e_d)}{\text{dist}(e_d, b_I)} = \frac{\text{dist}(a_I, -e_d)}{\text{dist}(-e_d, b_I'_{[d] \setminus I})} = \frac{c_I + 1}{\rho_{d-k}}.$$ 

Solving this equation yields $c_I = \frac{\rho_{d-k} + \rho_k}{\rho_{d-k} - \rho_k}$. The result follows.

Let $0 \leq k \leq d$. In view of Lemma 6.5, we denote by $a_k$ the point of intersection of $H_I$ and the $x_d$-axis, where $I$ is any subset of $[d]$ of size $k$, and by $c_k := \frac{\rho_{d-k} + \rho_k}{\rho_{d-k} - \rho_k}$ the last coordinate of $a_k$; see Figure 4 for an illustration in dimension 3.
Corollary 6.6. The heights of points $a_1, \ldots, a_d$ satisfy $c_1 > \cdots > c_{d-1} > c_d = 1$. In particular, if $q$ is a point on the $x_d$-axis that lies strictly between $a_{k-1}$ and $a_k$, then $q$ is beneath the facet $H_I = [u_i, u'_j : i \in I, j \in [d] \setminus I]$ of the CP if $|I| \leq k - 1$, and beyond the facet $H_I$ if $|I| \geq k$.

Proof: By equation (6.2), for all $1 \leq k \leq d - 1$, $\rho'_{d-k} - \rho_k > 0$. Hence $c_k = \frac{\rho'_{d-k} + \rho_k}{\rho'_{d-k} - \rho_k} > 1 = c_d$. Furthermore, for $2 \leq k \leq d - 1$,

$$c_k - c_{k-1} = \frac{\rho'_{d-k} + \rho_k}{\rho'_{d-k} - \rho_k} - \frac{\rho'_{d-k+1} + \rho_{k-1}}{\rho'_{d-k+1} - \rho_{k-1}}$$

$$= 2 \left( \frac{\rho_k}{\rho'_{d-k} - \rho_k} - \frac{\rho_{k-1}}{\rho'_{d-k+1} - \rho_{k-1}} \right)$$

$$= 2 \left( \frac{1}{\rho'_{d-k} - 1} - \frac{1}{\rho'_{d-k+1} - 1} \right) < 0,$$

where the last step follows from the fact that $\rho'_{d-k} > \rho'_{d-k+1} > \rho_{k-1} > \rho_k$; see eq. (6.2). □

Definition 6.7. Let $\text{CP} = \text{conv}(G \times \{1\} \cup \alpha G^* \times \{-1\})$ be a pseudo-regular $d$-cross-polytope. The set $\{a_k = \cap_{I \subseteq [d], |I| = k} H_I : 1 \leq k \leq d\}$ is called the sequence of points associated with CP.
Our construction of a \((d - 2)\)-simplicial 2-simple polytope starts with a certain \(d\)-polytope \(P^{d,1}\) described in Definition 6.8 and proceeds by recursively adding to \(P^{d,1}\) a total of \(d - 3\) additional vertices; see Figure 4 for an illustration of \(P^{3,1}\). As we will see below, one of the facets of \(P^{d,1}\) is a pseudo-regular CP (of dimension \(d - 1\)). By a slight abuse of notation, we continue to label the vertices of this facet by \(u_1, \ldots, u_{d-1}, u'_1, \ldots, u'_{d-1}\).

**Definition 6.8.** Let \(\Sigma = [u'_1, \ldots, u'_{d+1}]\) be a regular \(d\)-simplex. Choose an arbitrary \(0 < \epsilon \ll \text{dist}(u'_1, u'_2)\). For \(1 \leq i \leq d - 1\), let \(p_i\) be the barycenter of the \((d - 2)\)-face \(\Sigma \setminus u'_1 u'_{d+1}\), and let \(u_i := p_i + \epsilon(p_i - u'_{d+1})\). We define \(P^{d,1}\) as \(\text{conv}(u'_1, \ldots, u'_{d+1}, u_1, \ldots, u_{d-1})\).

Since \(p_i\) is the barycenter of the \((d - 2)\)-face \(\Sigma \setminus u'_1 u'_{d+1}\), it follows that \([p_1, \ldots, p_{d-1}]\) is a regular \((d - 2)\)-simplex and \([p_1, \ldots, p_{d-1}, u'_1, \ldots, u'_{d-1}]\) is a pseudo-regular \((d - 1)\)-cross-polytope. By our choice of \(u_i, [u_1, \ldots, u_{d-1}]\) is a regular \((d - 2)\)-simplex obtained from \([p_1, \ldots, p_{d-1}]\) by dilation with factor \((1 + \epsilon)\) (where \(\epsilon\) is small) followed by translation in the direction perpendicular to \(\text{aff}(p_1, \ldots, p_{d-1}, u'_1, \ldots, u'_{d-1}) = \text{aff}(\Sigma \setminus u'_{d+1})\). In particular, \(\text{aff}(u_1, \ldots, u_{d-1})\) is parallel to \(\text{aff}(u'_1, \ldots, u'_{d-1})\) and \(\text{CP} := [u_1, \ldots, u_{d-1}, u'_1, \ldots, u'_{d-1}]\) is also a pseudo-regular \((d - 1)\)-cross-polytope.

This discussion shows that the polytope \(P^{d,1}\) is the union of the simplex \(\Sigma\) and the pyramid with apex \(u'_{d}\) over the cross-polytope \(\text{CP}\) (glued along the simplex \([u'_1, \ldots, u'_{d}]\)). Furthermore, for each \(1 \leq i \leq d - 1\), the points \([u_i, u'_1, \ldots, u'_i, u'_d, u'_{d+1}]\) lie in the same hyperplane, and, in this hyperplane, the sets \(\text{conv}(u_i, u'_{d+1})\) and \(\text{conv}(u'_1, \ldots, u'_i, u'_d)\) intersect in their relative interiors. For \(1 \leq k \leq d - 1\), let \(\mathcal{H}_k\) be the set of facets \(H\) of \(\text{CP}\) with \(|H \cap \{u_1, \ldots, u_{d-1}\}| = k\). (Each such \(H\) is a \((d - 2)\)-face of \(P^{d,1}\).) Also, let \(H^+ := H \cap [u_1, \ldots, u_{d-1}]\) and \(H^- := H \cap [u'_1, \ldots, u'_{d-1}]\). Let \(v_0 := u'_{d+1}\) and \(v_1 := u'_d\). It follows that \(P^{d,1}\) has the following facets:

1. The simplex \(\Sigma \setminus u'_d\) and the pseudo-regular cross-polytope \(\text{CP}\).
2. \(d - 1\) bipyramids of the form \(\text{conv}(H \cup \{v_0, v_1\})\), where \(H \in \mathcal{H}_1\); the boundary complex of such facet is \(\partial(\text{V}(H^+) \cup v_0) * \partial(\text{V}(H^-) \cup v_1)\).
3. \(2d - 1 - d\) simplex facets of the form \(\text{conv}(H \cup v_1)\), where \(H \in \bigcup_{2 \leq k \leq d - 1} \mathcal{H}_k\).

In particular, \(\text{CP}\) is adjacent to all other facets of \(P^{d,1}\).

Since \(\text{CP}\) is pseudo-regular, by Lemma 6.5, there is a sequence of points associated with \(\text{CP}\) (lying in \(\text{aff}(\text{CP})\)): \(a_i = \bigcap_{F \in \mathcal{H}} \text{aff}(F), 1 \leq i \leq d - 1\); see Definition 6.7. The points \(\{a_i : 1 \leq i \leq d - 1\}\) all lie on the line through the barycenters \(b_{[d-1]}\) of \([u_1, \ldots, u_{d-1}]\) and \(b'_{[d-1]}\) of \([u'_1, \ldots, u'_{d-1}]\), and, according to Corollary 6.6, they appear on this line in the order \(a_1, \ldots, a_{d-2}, a_{d-1}\), with \(a_{d-2}\) closest to \(a_{d-1} = b_{[d-1]}\) and \(a_1\) farthest from \(b_{[d-1]}\).

We are now ready for the main definition of this section:

**Definition 6.9.** Consider the sequence of points \(\{a_i : 1 \leq i \leq d - 2\}\) associated with the facet \(\text{CP} = [u'_1, \ldots, u'_{d-1}, u_1, \ldots, u_{d-1}]\) of \(P^{d,1}\). Let \(v_1 = u'_d\). Inductively, for \(2 \leq
i ≤ d − 2, choose a point \(v_i\) in the relative interior of the line segment \([a_i, v_{i-1}]\) and let \(P^{d,i} = \text{conv}(P^{d,i-1} \cup v_i)\). Finally, let \(P^d = P^{d,d-2}\).

The process of adding vertices similar to the one described in Definition 6.9 is illustrated in Figure 5, where the vertices are added to the pyramid over a hexagon. (Unfortunately, Definition 6.9 itself is non-vacuous only when \(d \geq 4\), and as such is hard to illustrate.)

Our next goal is to prove that \(P^d\) is the promised high-dimensional analog of the 4-polytope \(P_9\); see Theorem 6.11. This requires describing the facets of \(P^d\). We do so by induction, showing that for \(2 ≤ k ≤ d - 2\), the set of facets of \(P^{d,k}\) is obtained from that of \(P^{d,k-1}\) as follows.

1. For each \(H \in \bigcup_{k+1 \leq i \leq d-1} \mathcal{H}_i\), the facet \(\text{conv}(H \cup v_{k-1})\) of \(P^{d,k-1}\) gets replaced with the facet \(\text{conv}(H \cup v_k)\).

2. For each \(H \in \mathcal{H}_k\), the facet \(\text{conv}(H \cup v_{k-1})\) of \(P^{d,k-1}\) gets replaced with the facet \(\text{conv}(H \cup \{v_{k-1}, v_k\})\) whose boundary complex is \(\partial(V(H^+) \cup v_{k-1}) \ast \partial(V(H^-) \cup v_k)\). There are \(\binom{d-1}{k}\) such facets.

3. The rest of the facets of \(P^{d,k-1}\) remain unchanged.

In particular, it follows by induction that \(CP\) is a facet of \(P^{d,k}\) and that it is adjacent to all other facets of \(P^{d,k}\), and, furthermore, that the collection of facets in item 3 consists of \(\Sigma \setminus u'_d\), \(CP\), and for each \(1 ≤ i ≤ k - 1\) and \(H \in \mathcal{H}_i\), a facet that contains \(H \cup v_i\).

The proof is based on:

Claim 1: For every \(H \in \mathcal{H}_k\), \(v_k \in \text{aff}(H \cup v_{k-1})\). This is because \(a_k\) lies on the hyperplane \(\text{aff}(H)\), and \(v_k \in [a_k, v_{k-1}]\).
Claim 2: For $i > k$ and $H \in \mathcal{H}_i$, $v_k$ is beneath $\text{conv}(H \cup v_{k-1})$. Indeed, by Corollary 6.6, in $\text{aff}(\text{CP})$, $a_k$ is beyond $H$. Hence in $\text{aff}(\text{CP} \cup v_{k-1}) = \mathbb{R}^d$, the point $v_k \in \text{int}[a_k, v_{k-1}]$ is beyond $\text{conv}(H \cup v_{k-1})$.

Claim 3: $v_k$ is beneath the rest of the facets of $P_{d,k-1}$. First, as easily seen from the definition of sequences $\{a_j\}$ and $\{v_j\}$, $v_k$ is beneath both $\Sigma \setminus u''_d$ and $\text{CP}$. Thus it only remains to show that if $G$ is a facet of $P_{d,k-1}$ that contains $H \cup v_i$ for some $i < k$ and $H \in \mathcal{H}_i$, then $v_k$ is beneath $G$. This follows from Corollary 6.6 along with another simple induction on $j$, where $i + 1 \leq j \leq k$. For the base case, by Corollary 6.6, in $\text{aff}(\text{CP})$, $a_{i+1}$ is beneath $H$. Hence, in $\text{aff}(\text{CP} \cup v_i) = \mathbb{R}^d$, $a_{i+1}$ is beneath $G$. Since $v_{i+1}$ is in the interior of $[v_i, a_{i+1}]$, $v_{i+1}$ is also beneath $G$. The inductive step is very similar: by the inductive hypothesis, $v_j$ is beneath $G$ and by Corollary 6.6, so is $a_{j+1}$; hence $v_{j+1} \in [v_j, a_{j+1}]$ is also beneath $G$. The claim follows.

The above three claims uniquely determine the facets of $P_{d,k}$. Claim 3 implies that the facets of $P_{d,k-1}$ from item 3 in the list are unaffected by adding $v_k$, and hence remain facets of $P_{d,k}$.

Claim 1 implies that for every $H \in \mathcal{H}_k$, the facet $\text{conv}(H \cup v_{k-1})$ of $P_{d,k-1}$ is replaced by a new facet $\text{conv}(H \cup \{v_k, v_{k-1}\})$. Note that the barycenter $b_{H^+}$ of $H^+$ lies on the line segment connecting $a_k$ and the barycenter $b_{H^-}$ of $H^-$ (see the proof of Lemma 6.5). Hence, if $v_k$ is an interior point of the line segment $[a_k, v_{k-1}]$, then $[b_{H^+}, v_{k-1}]$ and $[b_{H^-}, v_k]$ intersect at a point $p$. This implies that $\text{conv}(H^+ \cup v_{k-1}) \cap \text{conv}(H^- \cup v_k) = p$. Thus the boundary complex of $\text{conv}(H \cup \{v_k, v_{k-1}\})$ must be $\partial(V(H^+) \cup v_{k-1}) \star \partial(V(H^-) \cup v_k)$. These facets are exactly\(^2\) the facets of $P_{d,k}$ containing $v_{k-1}v_k$.

Finally, the rest of the facets of $P_{d,k}$ are those arising from $H \in \mathcal{H}_i$ for $i > k$. By Claim 2 and the previous paragraph, they must be of the form $\text{conv}(H \cup v_k)$, replacing $\text{conv}(H \cup v_{k-1})$ of $P_{d,k-1}$.

We thus obtain the following result (for convenience we let $v_{d-1} = v_d$):

**Lemma 6.10.** The polytope $P_d$ in Definition 6.9 has $3(d - 1)$ vertices and $2^{d-1} + 1$ facets. The vertex set of $P_d$ is

$$\{u_1, \ldots, u_{d-1}, u'_1, \ldots, u'_{d-1}, u_d = v_1, u'_{d+1} = v_0, v_2, \ldots, v_{d-3}, v_{d-2} = v_{d-1}\}.$$ 

The set of facets of $P_d$ naturally splits into the following $d$ subfamilies:

1. $\mathcal{F}_0$ consists of the simplex $[u'_1, \ldots, u'_{d-1}, u'_{d+1}]$ and the cross-polytope $\text{CP}$.

2. For $1 \leq k \leq d - 1$, $\mathcal{F}_k$ consists of $(d-1)$ polytopes of dimension $d - 1$ whose boundary complexes are of the form $\partial(V(H^) \cup v_{k-1}) \star \partial(V(H^-) \cup v_k)$, where $H \in \mathcal{H}_k$. In particular, $\mathcal{F}_{d-1} = \{[u_1, \ldots, u_{d-1}, v_{d-2}]\}$.

\(^2\)To see this, we invite the reader to compute the link of $v_{k-1}v_k$ in the polytopal complex generated by these facets and check that it is a $(d - 3)$-dimensional pseudomanifold (i.e., every ridge is in two facets). Thus it must coincide with the link of $v_{k-1}v_k$ in the boundary of $P_{d,k}$.
Theorem 6.11. The $d$-polytope $P^d$ is $(d - 2)$-simplicial and 2-simple. It has two pairs of a simplex facet and a simple vertex not in that facet; they are $(\{u_1, \ldots, u_{d-1}, v_{d-2}\}, u'_{d+1})$ and $(\{u_1', \ldots, u'_{d-1}, u''_{d+1}\}, v_{d-2})$.

Proof: Let $U = \{u_1, \ldots, u_{d-1}\}$ and let $U' = \{u_1', \ldots, u'_{d-1}\}$. For $M = \{u_i, \ldots, u_k\} \subseteq U$, we let $M' := \{u'_i, \ldots, u'_k\} \subseteq U'$. Also, for brevity, we write $u, u v, u w$ instead of $\{u\}$, $\{u, v\}$, and $\{u, v, w\}$.

The description of facets in Lemma 6.10 guarantees that $P^d$ is $(d - 2)$-simplicial. To show that $P^d$ is also 2-simple, it suffices to check that every $(d - 3)$-face $\tau$ of $P^d$ is contained in exactly three facets. By examining families $F_i$, $0 \leq i \leq d - 1$, of Lemma 6.10, we see that there are the following possible cases:

1. $u'_{d+1} \in V(\tau)$. In this case, $V(\tau) \subseteq U' \cup u'_{d}u'_{d+1}$. If $u'_{d}$ is also in $\tau$, then $\tau$ is contained in three bipyramids from $F_1$; otherwise, $\tau$ is contained in two bipyramids from $F_1$ and the simplex $[u'_1, \ldots, u'_{d-1}, u'_{d+1}]$ from $F_0$.

2. $V(\tau) \subseteq U'$. In this case, $\tau$ is contained in the cross-polytope and the simplex from $F_0$, and one bipyramid from $F_1$.

3. $V(\tau) = K \cup M'$, where $K \subseteq M \cup u_\ell = U$ and $|K| = i$ for some $1 \leq \ell \leq d - 1$ and $1 \leq i \leq d - 2$. Then $\tau$ is a face of CP from $F_0$, of $\partial(K \cup u_\ell v_i) * \partial(M' \cup v_{i+1})$ from $F_{i+1}$, and of $\partial(K \cup v_{i-1}) * \partial(M' \cup u'_\ell v_i)$ from $F_i$.

4. $V(\tau) = K \cup M' \cup v_i$, where $1 \leq i \leq d - 2$ and $K \subseteq M \cup u_j u_k = U$ for some $1 \leq j < k \leq d - 1$. There are two cases:

   (a) $|K| = i - 1$. Then $\tau$ is a face of $\partial(K \cup u_j u_k v_i) * \partial(M' \cup v_{i+1})$ from $F_{i+1}$ and of two facets $\partial(K \cup u_j u_{k-1} v_i) * \partial(M' \cup u'_k v_i)$, $\partial(K \cup u_k v_{i-1}) * \partial(M' \cup u'_\ell v_i)$ from $F_i$.

   (b) $|K| = i$ (and so, $i < d - 2$). Then $\tau$ is a face of $\partial(K \cup v_{i-1}) * \partial(M' \cup u'_j u'_k v_i)$ from $F_i$, and of two facets $\partial(K \cup u_j v_i) * \partial(M' \cup u'_j v_{i+1})$, $\partial(K \cup u_k v_i) * \partial(M' \cup u'_k v_{i+1})$ from $F_{i+1}$.

5. $V(\tau) = K \cup M' \cup v_{i-1} v_i$, where $2 \leq i \leq d - 2$ and $K \subseteq M \cup u_j u_k u_\ell = U$ for some $1 \leq j < k < \ell \leq d - 1$. There are two cases:

   (a) $|K| = i - 2$. Then $\tau$ is contained in three facets from $F_i$:

   $\partial(K \cup u_k u_\ell v_{i-1}) * \partial(M' \cup u'_j v_i)$, $\partial(K \cup u_j u_\ell v_{i-1}) * \partial(M' \cup u'_k v_i)$, and $\partial(K \cup u_j u_k v_{i-1}) * \partial(M' \cup u'_\ell v_i)$.

   (b) $|K| = i - 1$. Then $\tau$ is contained in three facets from $F_i$:

   $\partial(K \cup u_\ell v_{i-1}) * \partial(M' \cup u'_j u'_k v_i)$, $\partial(K \cup u_j v_{i-1}) * \partial(M' \cup u'_k v_{i+1})$, and $\partial(K \cup u_k v_{i-1}) * \partial(M' \cup u'_j u'_\ell v_i)$.
Consider all polytopes resulting from \((d - 2)\)-simple \(d\)-polytope can have in dimensions \(d = 3, 4, 5\) (cf. Proposition 3.3).

Proof: Each \(\sigma\)-copy of the CP facet denoted by \(\sigma\) in this proof we will denote by \(\sigma\) 2.

Theorem 6.13. There are \(\Omega(N)\) combinatorially distinct \((d - 2)\)-simplicial 2-simple \(d\)-polytopes with \(N = (3d - 3) + k(2d - 4)\) vertices.

Proof: Consider \(k + 1\) copies of \(P^d\), which we denote by \(T_1, \ldots, T_{k+1}\), with the corresponding copies of the CP facet denoted by \(CP_1\). Each \(T_i\) has two pairs of a simplex facet and a simple vertex not in that facet, which in this proof we will denote by \((F_i, w_i)\) and \((F'_i, w'_i)\). Consider all polytopes resulting from \((\cdots(T_1 \triangleright T_2) \triangleright T_3) \cdots) \triangleright T_{k+1}\) by the following rules:

- In the first step, we merge \(T_1\) and \(T_2\) so that the facet \(CP_1\) is merged with a bipyramid.

Remark 6.12. It is worth noting that the polytope \(P^d\) is \(d\)-dimensional and has \(3d - 3\) vertices. This is the smallest number of vertices that a non-simplex \((d - 2)\)-simplicial 2-simple \(d\)-polytope can have.

As the last theorem of the paper, we show that iteratively merging \(n\) copies of \(P^d\) from Theorem 6.11 results in exponentially many (w.r.t. the number of vertices) combinatorially distinct \((d - 2)\)-simplicial 2-simple \(d\)-polytopes. Recall from Theorem 6.11 that

- The polytope \(P^d\) has two simple vertices \(u_{d+1}'\) and \(v_{d-2}\), and two simplex facets \(F' := [u_1', \ldots, u_{d-1}', u_{d+1}']\) and \(F := [u_1, \ldots, u_{d-1}, v_{d-2}]\); \(u_{d+1}'\) is a vertex of \(F'\) but not of \(F\), and \(v_{d-2}\) is a vertex of \(F\) but not of \(F'\). All other facets containing \(u_{d+1}'\) and \(v_{d-2}\) are bipyramids.

- The CP facet \([u_1, \ldots, u_{d-1}, u_1', \ldots, u_{d-1}']\) is adjacent to all other facets of \(P^d\).

Let \(T_1\) and \(T_2\) be two copies of \(P^d\) with the copy of CP, \(F\), and \(F'\) in \(T_1\) denoted by \(CP_i\), \(F_i\), and \(F'_i\), respectively, and the copy of \(u_{d+1}'\) from \(T_2\) denoted by \(w\). We merge \(T_1\) and \(T_2\) along \(F_1\) and \(w\). Since \(CP_1\) is adjacent to \(F_1\), and since \(w\) is in one simplex facet (namely \(F_2'\)) and \(d - 1\) bipyramids, exactly as in the 4-dimensional case, there are two ways to merge leading to two distinct combinatorial types (recall that \(\sigma_{d-1}\) denotes a \((d - 1)\)-simplex):

- In \(T_1 \triangleright T_2\), the facet \(CP_1\) gets merged with the simplex \(F_2'\). The merged facet is then again a CP. Since \(CP_2\) is adjacent to all other facets of \(T_2\), including \(F_2'\), it follows that the polytope \(T_1 \triangleright T_2\) has two CP facets and that they are adjacent to each other.

- In \(T_1 \triangleright T_2\), the facet \(CP_1\) gets merged with a bipyramid, resulting in a facet of the form \(CP_{\#(d-1)}\). In this case, \(T_1 \triangleright T_2\) has two “large” facets: \(CP_{\#(d-1)}\) and \(CP_2\), and they are adjacent to each other; every other facet has at most \(d + 1\) vertices.

With these observations in hand, we are ready to prove the following.

The result follows. □
In the $i$th step, when computing the merge of $\cdots \left((T_1 \triangleright T_2) \triangleright T_3\right) \cdots \triangleright T_i$ with $T_{i+1}$, we always merge along $F_i$ and $w_{i+1}$.

Denote by $R_k$ the polytope obtained in the $k$th step. In the $i$th step ($1 \leq i < k$), $F_{i+1}$ from $T_{i+1}$ remains untouched and can be used for the $(i+1)$st step. For $1 \leq j \leq k+1$, we refer to the facet of $R_k$ resulting from CP$_j$ as the $j$th special facet. By remarks above, for each $2 \leq j \leq k$, the $j$th special facet is either a CP or a CP$\#\sigma_{d-1}$; the $(k+1)$st special facet is always a CP while the first special facet is always a CP$\#\sigma_{d-1}$. Furthermore, for all $1 \leq i, j \leq k+1$, the $i$th and $j$th special facets are adjacent if and only if $|i-j|=1$.

We show that this procedure produces at least $2^{k-1}$ pairwise non-isomorphic polytopes. First note that the boundary complexes of all non-special facets of $R_k$ are either simplices, joins of two simplices, or stackings over these, and so a non-special facet can never be isomorphic to CP or CP$\#\sigma_{d-1}$. Associate with $R_k$ its profile which is given by the following abstract graph: the nodes represent the facets of the form CP and CP$\#\sigma_{d-1}$, and two such nodes are connected by an edge if the corresponding facets are adjacent; also, label each node with a 0 or 1 depending on whether it represents a facet that is a CP or a CP$\#\sigma_{d-1}$. The resulting profile is then a path with $k+1$ nodes labeled by 0's and 1's; one of the endpoints is always labeled by 1 (the node representing the 1st special facet) and the other endpoint is always labeled by 0 (the node representing the $(k+1)$st special facet).

There are $2^{k-1}$ such 0/1-paths, and we claim that each of them is a valid profile. Indeed, given such a path, walk along it from the endpoint labeled by 1 to the endpoint labeled by 0 and read the labels of the nodes. The node at distance $i-1$ from the first endpoint corresponds to the special facet coming from $T_i$ and the label of that node simply tells us whether at the $i$th step we should merge CP$_i$ with a simplex or with a bipyramid. This claim completes the proof since isomorphic polytopes have the same profile. In other words, two polytopes with distinct profiles have different combinatorial types. \hfill \Box

**Remark 6.14.** When $d=4$, we can further merge $R_k$ with a 2-simplicial 2-simple 4-polytope with 10, 11, or 16 vertices. Such polytopes can be found in [12, Section 4.1], where they are denoted by $P_{10}, P_{11}, P_{16} = T^4(P_{11})$. This allows us to create exponentially many (in $N$) 2-simplicial 2-simple 4-polytopes with $N$ vertices for all sufficiently large integers $N$ (not just those with $N \equiv 1 \pmod{4}$). It follows from Corollary 4.13 that all resulting polytopes are elementary. Hence for $d=4$, the number of combinatorially distinct 2-simplicial 2-simple 4-polytopes that are also elementary grows exponentially with the number of vertices. This strengthens [13, Corollary 4.2].

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References


