From acute sets to centrally symmetric 2-neighborly polytopes

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Abstract

What is the maximum number of vertices that a centrally symmetric 2-neighborly polytope of dimension d can have? It is known that the answer does not exceed 2^d. Here we provide an explicit construction showing that it is at least $2^{d-1}+2$.

1 Introduction

The goal of this note is to construct centrally symmetric 2-neighborly polytopes with many vertices. Recall that a polytope is the convex hull of a set of finitely many points in $\mathbb{R}^d$. The dimension of a polytope P is the dimension of its affine hull. We say that P is a d-polytope if the dimension of P is equal to d. A polytope P is *centrally symmetric* (cs, for short) if for every $x \in P$, $-x$ belongs to P as well.

A cs polytope P is called *k-neighborly* if every set of k of its vertices, no two of which are antipodes, is the vertex set of a face of P. In addition to being of intrinsic interest, the study of cs k-neighborly polytopes is motivated by the recently discovered tantalizing connections (initiated by Donoho and his collaborators [6, 7]) between such polytopes and the seemingly distant areas of error-correcting codes and sparse signal reconstruction. It is also worth mentioning that in contrast with the situation for polytopes without a symmetry assumption, a cs d-polytope with sufficiently many vertices cannot even be 2-neighborly [4, 10].

A few more definitions are to follow. A set $S \subset \mathbb{R}^d$ is *acute* if every three points from S determine an acute triangle. A set $S \subset \mathbb{R}^d$ is *antipodal* if for every two points $x, y \in S$, there exist two (distinct) parallel hyperplanes H_x and H_y such that $x \in H_x$, $y \in H_y$, and all elements of S lie in the closed strip defined by H_x and H_y.

It is well-known and easy to check that any acute set is antipodal. Similarly, as was observed in [10, Lemma 2.1], the vertex set of any cs 2-neighborly polytope $P \subset \mathbb{R}^d$ is antipodal. Furthermore, according to a celebrated theorem of Danzer and Grünbaum [5] (see also [1, Ch. 17]), an antipodal subset of $\mathbb{R}^d$ has at most 2^d elements, and it has exactly 2^d elements if and only if it is the vertex set of a parallelotope. Since the vertex set of a parallelotope is not acute and since a

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d-parallelotope (for $d > 2$) is not cs 2-neighborly, it follows that any acute set $S \subset \mathbb{R}^d$ has at most $2^d - 1$ elements, while any cs 2-neighborly d-polytope P with $d \geq 3$ has at most $2^d - 2$ vertices.

Although the size of the largest acute set in $\mathbb{R}^d$ remains a mystery, in a very recent breakthrough paper [8], Gerencsér and Harangi constructed an acute set in $\mathbb{R}^d$ of size $2^{d-1} + 1$. The previous record size was $F_{d+2} = \Theta\big((\frac{1+\sqrt{5}}{2})^d\big)$, where F_n is the n-th Fibonacci number, see [11].

Similarly, the current record size of the vertex set of a cs 2-neighborly d-polytope is about $\sqrt{3}^d$, see [2]. Here we establish the following new bound.

Theorem 1. *There exists a cs 2-neighborly d-polytope with $2^{d-1} + 2$ vertices.*

We say that a (finite) set $S \subset \mathbb{R}^d \setminus \{0\}$ is *cs* if for every $x \in S$, the point $-x$ is also in S. Observe that a cs set of size larger than two can never be acute: indeed, for any $x, y \in \mathbb{R}^d$, the parallelogram determined by $x, y, -x, -y$ has a non-acute angle. The main insight of this note is the notion of an almost acute set: a set S is *almost acute* if it is cs and for every ordered triple (x, y, z) of distinct points in S, the angle $\angle xyz$ is acute as long as x and z are not antipodes.

With this definition in hand, the following two results yield Theorem 1.

Lemma 2. *Let $S \subset \mathbb{R}^d$ be a cs set that spans $\mathbb{R}^d$, and let $P = \mathrm{conv}(S)$. If S is almost acute, then P is a cs 2-neighborly d-polytope whose vertex set is S.*

Lemma 3. *There exists an almost acute subset of $\mathbb{R}^d$ of size $2^{d-1} + 2$.*

To prove Lemma 3 we modify the Gerencsér–Harangi construction: as in [8], we start with the vertex set of the $(d-1)$-cube $[-1, 1]^{d-1}$ embedded in the coordinate hyperplane $\mathbb{R}^{d-1} \times \{0\}$ of $\mathbb{R}^d$. We then use the extra dimension to perturb the vertices in such a way that the resulting set in $\mathbb{R}^d$ is almost acute. (In particular, any pair of antipodes is perturbed to a pair of antipodes.) Adding to this set a pair of antipodes of the form $(0, \dots, 0, c)$ and $(0, \dots, 0, -c)$, where $c \in \mathbb{R}$ is sufficiently large, completes the construction.

The proofs of Lemmas 2 and 3 are given in Sections 2 and 3, respectively. We close in Section 4 with some remarks and open problems.

2 Polytopes with an almost acute vertex set

The goal of this section is to prove Lemma 2. For all undefined terminology pertaining to polytopes, we refer our readers to Ziegler's book [12]. Assume that $S \subset \mathbb{R}^d$ is an *almost acute* set that spans $\mathbb{R}^d$. Then $P := \mathrm{conv}(S)$ is a cs d-polytope whose vertex set is contained in S. To prove the lemma, we have to show that (i) every $x \in S$ is a vertex of P, and (ii) for every $x, y \in S$ with $y \notin \{x, -x\}$, the line segment $[x, y]$ is an edge of P.

Let x be any element of S. Let H be the hyperplane that contains x and is perpendicular to the line segment $[-x, x]$. Since S is an almost acute set, for every $y \in S \setminus \{-x, x\}$, the angle $\angle(-x)xy$ is acute, and so y lies in the same open half-space of $\mathbb{R}^d$ defined by H as $-x$. It follows that H is a supporting hyperplane of P and that $H \cap P = \{x\}$. Hence x is a vertex of P.

Now, let x, y be any elements of S with $y \notin \{x, -x\}$. Consider the parallelogram Q with vertices $x, y, -x, -y$. There are two possible cases:

Case 1: Q is a **rectangle**. Let H be the hyperplane perpendicular to the line segment $[-y, x]$ and passing through x, and hence also through y. Since S is an almost acute set, for every

$z \in S \setminus \{x, -y, y\}$, the angle $\angle(-y)xz$ is acute, and so z lies in the same open half-space of $\mathbb{R}^d$ defined by H as $-y$. We conclude that H is a supporting hyperplane of P and that $H \cap P = \mathrm{conv}(x, y) = [x, y]$. Thus $[x, y]$ is an edge of P.

Case 2: Q is **not a rectangle**. In this case exactly one of the angles $\angle(-y)xy$, $\angle(-x)yx$ is obtuse. Without loss of generality (by switching the roles of x and y if necessary), we may assume that $\angle(-y)xy$ is obtuse. As in Case 1, let H be the hyperplane perpendicular to the line segment $[-y, x]$ and passing through x. Since $\angle(-y)xy$ is obtuse, the points y and $-y$ lie on the opposite (open) sides of H. Furthermore, our assumption that S is almost acute means that all elements of $S \setminus \{x, y\}$ lie on the same side of H as $-y$; we denote the closure of this side of H by H^-.

Our goal is to prove that $[x, y]$ is an edge of P. Indeed, if it is not an edge, then all neighbors of x in P belong to the half-space H^-. Since x itself is in H, we obtain that the cone based at x and spanned by the rays from x to the neighbors of x is a subset of H^-; in particular, it does not contain y. This contradicts the well-known fact (see [12, Lemma 3.6]) that such a cone must contain P, and hence also y. The lemma follows.

3 Construction of an almost acute set

In this section we prove Lemma 3. To do so, we construct an almost acute set in $\mathbb{R}^d$ of size $2^{d-1} + 2$. We start with the set S^0 described below; we then perturb the points of S^0 to obtain an almost acute set. As our construction/proof is a simple modification of that in [8], we only sketch the main ideas and leave out some of the details.

Pick a real number $c > \sqrt{d-1}$ and consider the following subset of $\mathbb{R}^d$ of size $2^{d-1} + 2$:

$$S^0 := \big\{(\delta_1, \dots, \delta_{d-1}, 0) \mid \delta_1, \dots, \delta_{d-1} \in \{\pm 1\}\big\} \cup \big\{(0, \dots, 0, \pm c)\big\}.$$

Thus, S^0 consists of the vertex set V^0 of the $(d-1)$-cube $[-1, 1]^{d-1} \times \{0\} \subset \mathbb{R}^d$ and two additional points, x_0 and $-x_0$, positioned high above and far below the center of the cube, respectively. An easy computation shows that for all distinct $y, z \in V^0$, the angles $\angle(\pm x_0)yz$, $\angle y(\pm x_0)z$, and $\angle(\pm x_0)(\mp x_0)y$ are acute, and, assuming also that $z \neq -y$, so is $\angle y(-y)z$. Hence there exists an $\epsilon_0 > 0$ such that if all vertices of the cube are perturbed by no more than ϵ_0, then all of the above angles remain acute. Therefore, to complete the proof, it suffices to perturb the vertices of V^0 in such a way that (i) antipodes are perturbed to antipodes, and (ii) for all $x, y, z \in V^0$ no two of which are antipodes, the perturbed triangle is acute.

The key fact we will use is the following lemma from [8]:

Lemma 4. *Let V^0 be the vertex set of the $(d-1)$-cube $[-1, 1]^{d-1} \times \{0\} \subset \mathbb{R}^d$. For every $\epsilon > 0$ and $x \in V^0$, there exists $x' \in \mathbb{R}^d$ such that x' is within distance ϵ from x and the angles $\angle x'yz$ and $\angle yx'z$ are acute for all $y, z \in V^0 \setminus \{x\}$.*

Arbitrarily order the elements of V^0, so that $V^0 = \{x_1, -x_1, x_2, -x_2, \dots, x_{2^{d-2}}, -x_{2^{d-2}}\}$. We induct on $1 \le p \le 2^{d-2}$ to construct a set $V^p = \{x'_1, -x'_1, \dots, x'_p, -x'_p\} \cup \{x_j, -x_j \mid p < j \le 2^{d-2}\}$ with the property that (a) for all $1 \le i \le p$, $\|x'_i - x_i\| < \epsilon_0$, and (b) for every three points x, y, z of V^p, no two of which are antipodes and such that $x = \pm x'_i$ for some $1 \le i \le p$, the angles $\angle xyz$ and $\angle yxz$ are acute. We refer to (a) and (b) combined as the $(*_p)$-property.

Assume V^{p-1} satisfies the $(*_{p-1})$-property. In particular, for every three points x'_i, y, z of V^{p-1} no two of which are antipodes, the angles $\angle(\pm x'_i)yz$ and $\angle y(\pm x'_i)z$ are acute. Hence there exists an $0 < \epsilon_p < \epsilon_0$, such that if x_p and $-x_p$ are perturbed by no more than ϵ_p, then all of the above angles involving $\pm x_p$ (as y or z) remain acute. Furthermore, by Lemma 4, there exists x'_p within distance ϵ_p of x_p such that for all $y, z \in \{x_j, -x_j \mid p < j \le 2^{d-2}\}$, the angles $\angle x'_p yz$ and $\angle y x'_p z$ are acute. Since the set $\{x_j, -x_j \mid p < j \le 2^{d-2}\}$ is cs, it follows that the angles $\angle(-x'_p)yz$ and $\angle y(-x'_p)z$ are also acute. We conclude that V^p satisfies the $(*_p)$-property. The set $S := V^{2^{d-2}} \cup \{x_0, -x_0\}$ is then an almost acute set of size $2^{d-1}+2$. This completes the proof of Lemma 3, and hence also of Theorem 1.

Remark 5. To make this note as self-contained as possible, we sketch the strategy used in [8] to prove Lemma 4. The first step is to move x a bit towards the center of the cube (while staying in the $\mathbb{R}^{d-1} \times \{0\}$ subspace of $\mathbb{R}^d$). All angles involving x as a non-middle point then become acute. Unfortunately, all angles with x as the middle point become obtuse. This can be fixed (without spoiling acute angles) by further moving x by an appropriately chosen distance in the direction of the d-th axis, i.e., in the direction orthogonal to $\mathbb{R}^{d-1} \times \{0\}$. In fact, the computations in [8] show that for $x = (1, 1, \dots, 1, 0) \in V^0$, one can take x' to be any point of the form $(1-a, 1-a, \dots, 1-a, b)$, where $0 < a < 2$, $b^2 > 2(d-1)a$, and $(d-1)a^2+b^2 < \epsilon^2$. For instance, if $d = 5$, $\epsilon = 0.1$, and $x = (1, 1, 1, 1, 0)$, then $x' = (1 - 10^{-4}, 1 - 10^{-4}, 1 - 10^{-4}, 1 - 10^{-4}, 3 \cdot 10^{-2})$ does the job.

4 Concluding remarks and open problems

We close with a few open problems.

The main result of this note together with [10, Lemma 2.1] and the Danzer–Grünbaum theorem [5] implies that for $d \ge 3$, the maximum number of vertices that a cs 2-neighborly d-polytope can have lies in the interval $[2^{d-1} + 2, 2^d - 2]$. In dimension three, the only cs 2-neighborly polytope is the cross-polytope, which indeed has $6 = 2^2 + 2 = 2^3 - 2$ vertices. In dimension four, the maximum is $10 = 2^3 + 2$; this result is due to Grünbaum, see [9, p. 116]. However, for $d > 4$, the exact value of the maximum remains unknown.

A related question is what is the maximum number of edges, $\text{fmax}(d, N; 1)$, that a cs d-polytope with N vertices can have. At present, it is known that

$$\left(1 - 3^{-\lfloor d/2-1 \rfloor}\right)\binom{N}{2} \le \text{fmax}(d, N; 1) \le \left(1 - 2^{-d}\right)\frac{N^2}{2}; \tag{4.1}$$

see [2, Theorem 3.2(2)] and [3, Proposition 2.1] for the lower and the upper bound, respectively. However, the exact value of $\text{fmax}(d, N; 1)$ or even its asymptotics remains a mystery. The main result of this paper makes us believe that $\text{fmax}(d, N; 1)$ might be closer to the right-hand side of Eq. (4.1) than to the left one.

Finally, it would be interesting to understand the maximum number of vertices that a cs 3-neighborly d-polytope can have. It is known that there exist cs 3-neighborly d-polytopes with $\approx 2^{0.023d}$ vertices, see [2, Remark 4.3]. On the other hand, an argument similar to the proof of [10, Theorem 1.1], shows that a cs d-polytope with $\lceil 2\sqrt{2} \cdot 3^{0.5d} \rceil$ or more vertices cannot be 3-neighborly.

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