Face numbers: the upper bound side of the story

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Abstract

We survey the theory of face numbers of simplicial complexes through the lens of upper bound type results and neighborliness. We focus on the classes of polytopes, simplicial spheres, and simplicial manifolds, along with the classes of centrally symmetric polytopes and centrally symmetric simplicial spheres. Along the way, we sketch some of the ideas and methods used in the proofs. We also highlight some of the many open problems in the field.

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1. Introduction

The focus of this survey is simplicial polytopes and more general simplicial complexes with nice topological properties such as triangulations of spheres and manifolds. Simplicial complexes have played an important role in topology since its early days, as topological spaces were often studied through their triangulations. Due to their discrete nature, simplicial complexes have also been studied by combinatorialists and discrete geometers. The theory of simplicial complexes has always been inseparable from the theory of polytopes. Although polytopes were studied since antiquity, the field has become extremely active since the last half of the 20th century, partly due to rapid developments in optimization and statistics.

There are many excellent surveys on combinatorics of polytopes and simplicial complexes, see for instance [12, 44]. Here we concentrate on polytopes and simplicial complexes with and without symmetry, with a particular emphasis on the upper bound type results on their face numbers and the related notion of neighborliness. Among the questions we discuss are: what is the largest number of i-faces that a simplicial sphere of dimension $d-1$ with n vertices can have? How many combinatorially distinct neighborly spheres of dimension $d-1$ with n vertices are there? How neighborly can a centrally symmetric d-polytope be? How different is the answer for centrally symmetric $(d-1)$-spheres? Along the way, we mention some of the algebraic, combinatorial, analytical, and topological tools that have been developed and used over the last fifty years and have brought the field to its current state. The interplay between these various methods is a really fascinating part of the story. We also discuss some of the many open problems in the field. We are only able to touch on a limited number of topics and the choice of these topics is rather subjective. Yet we hope this paper provides the reader with a view into the beautiful theory of face numbers.

2. Some basics

A convex *d-dimensional polytope* (a *d-polytope*, for short) is the convex hull of a finite set of points in $\mathbb{R}^d$ that affinely span $\mathbb{R}^d$. A *supporting hyperplane* of a polytope P is a hyperplane H in $\mathbb{R}^d$ that intersects P non-trivially and such that all points of P lie on the same (closed) side of H. A *proper face* of P is the intersection of P with a supporting hyperplane. The empty set and P itself are the *improper faces* of P. A polytope P is *simplicial* if all of its proper faces are simplices. Faces of dimension 0, 1, and $d-1$ are called *vertices, edges*, and *facets*, respectively; faces of dimension i are called *i-faces*.

A *simplicial complex* Δ on a (finite) vertex set $V = V(\Delta)$ is a collection of subsets of V that is closed under inclusion. The elements $F \in \Delta$ are called *faces*. The *dimension of a face* $F \in \Delta$ is $\dim(F) = |F| - 1$ and the *dimension of* Δ is $\dim(\Delta) = \max\{\dim(F) \mid F \in \Delta\}$. As in the case of polytopes, an i-dimensional face is abbreviated as an *i-face*. A $(d-1)$-dimensional simplicial complex is *pure* if all of its maximal faces (w.r.t. inclusion) are $(d-1)$-faces; in this case the $(d-1)$-faces are called *facets* and the $(d-2)$-faces are called *ridges*. Unless Δ is the void complex $\emptyset$, the empty set is a face of Δ.

Although simplicial complexes are defined as purely combinatorial objects, each simplicial complex Δ admits a geometric realization $\|\Delta\|$ that contains a geometric i-simplex for each i-face of Δ. We typically do not distinguish between the combinatorial object Δ and the geometric object $\|\Delta\|$ and often say that Δ has certain geometric or topological properties in addition to certain combinatorial properties. For instance, we say that Δ is a *simplicial sphere* (respectively a *simplicial manifold*) if $\|\Delta\|$ is homeomorphic to a sphere (a compact topological manifold without boundary, respectively). Each simplicial d-polytope P gives rise to a simplicial $(d-1)$-sphere, namely the *boundary complex* of P, denoted ∂P.

For a $(d-1)$-dimensional simplicial complex Δ, we denote by $f_i(\Delta)$ the number of i-faces of Δ, and by $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta))$ the *f-vector* of Δ. For reasons that will become apparent below, it is often more natural to study a certain invertible integer transformation of the f-vector called the *h-vector* of Δ, $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$; it is defined by the following polynomial relation:

$$\sum_{j=0}^{d} h_j(\Delta) \cdot t^{d-j} = \sum_{j=0}^{d} f_{j-1}(\Delta) \cdot (t-1)^{d-j}.$$

For instance, $h_d = f_{d-1} - f_{d-2} + \cdots + (-1)^{d-1} f_0 + (-1)^d = (-1)^{d-1}\tilde{\chi}(\Delta)$ is, up to a sign, the *reduced Euler characteristic* of Δ. Abusing notation, for a simplicial polytope P, we write $f(P)$ and $h(P)$ instead of $f(\partial P)$ and $h(\partial P)$, respectively.

3. The cyclic polytope and McMullen's Upper Bound Theorem

Our story begins with an amazing object — the *cyclic polytope*. This polytope, $C_d(n)$, is the convex hull of $n \geq d+1$ distinct points on the d-th *moment curve* $M(t) = M_d(t) = (t, t^2, t^3, \ldots, t^d)$, that is, $C_d(n) = \operatorname{conv}\big(M_d(t_1), \ldots, M_d(t_n)\big)$, where $t_1 < t_2 < \cdots < t_n$ are real numbers. It is a d-dimensional simplicial polytope with n vertices whose combinatorial type is independent of the choice of $t_1, t_2, \ldots, t_n$. The most remarkable property of $C_d(n)$ is that it is $\lfloor d/2 \rfloor$*-neighborly*, meaning that every $k \leq d/2$ vertices of $C_d(n)$ form the vertex set of a face. (No d-polytope except a d-simplex can be $(\lfloor d/2 \rfloor + 1)$-neighborly.)

To see that $C_d(n)$ is $\lfloor d/2 \rfloor$-neighborly, consider any integer $0 < k \leq \lfloor d/2 \rfloor$ and a k-subset $I = \{i_1, \ldots, i_k\}$ of $[n] = \{1, 2, \ldots, n\}$. Let $P(t) = (t - t_1 + 1)^{d-2k}(t - t_{i_1})^2(t - t_{i_2})^2 \cdots (t - t_{i_k})^2$. Observe that $P(t)$ is a polynomial of degree d, and so it can be written as $P(t) = \gamma_d t^d + \gamma_{d-1} t^{d-1} + \cdots + \gamma_1 t + \gamma_0$. Observe also that $P(t_i) = 0$ for all $i \in I$ while $P(t_i) > 0$ for all $i \in [n] \setminus I$. It follows that the hyperplane

$$H = \{\vec{x} = (x_1, \ldots, x_d) \mid (\gamma_1, \ldots, \gamma_d) \cdot \vec{x} = -\gamma_0\}$$

is a supporting hyperplane of $C_d(n)$ (here $\vec{a} \cdot \vec{x}$ denotes the dot product) and that $H \cap C_d(n) = \operatorname{conv}\big(M_d(t_i) \mid i \in I\big)$. Thus $\{M(t_i) \mid i \in I\}$ is the vertex set of a face.

That the combinatorial type of $C_d(n)$ is independent of the choice of $t_1 < \cdots < t_n$ is a consequence of *Gale's evenness condition* [26]. This result asserts that for a subset $I = \{i_1, \ldots, i_d\} \subset [n]$, $\{M(t_{i_1}), M(t_{i_2}), \ldots, M(t_{i_d})\}$ is the vertex set of a facet of $C_d(n)$ if and only if for all $i, j \in [n] \setminus I$, the number of elements $\ell \in I$ that lie between i and j is even.

The cyclic polytope was discovered and rediscovered by many people, including Carathéodory [19,20], Gale [26], Motzkin [65], and others; see [34, Chapter 7] for historic remarks and additional references. The importance of the cyclic polytope is that by virtue of its neighborliness, the face numbers $f_{i-1}(C_d(n))$ for $i \leq \lfloor d/2 \rfloor$, are equal to $\binom{n}{i}$, which is the maximum possible number of $(i-1)$-faces that any $(d-1)$-dimensional simplicial complex with n vertices can have. This led Motzkin [65] to propose the following *Upper Bound Conjecture* (UBC, for short): *in the class of all d-polytopes with n vertices, the cyclic polytope simultaneously maximizes all the face numbers.*

The motivation for the UBC partly comes from optimization: stated in a dual form, it posits that among all d-polytopes defined by n linear constraints, the polar of the cyclic polytope has the largest number of vertices.

By a standard trick of "pulling vertices", to prove the UBC for all polytopes, it suffices to prove it for simplicial polytopes. One advantage of working with a simplicial polytope P, is that the first half of the f-vector of P determines the entire f-vector. This important fact is known as the Dehn–Sommerville relations [47]. More specifically, when $i \geq \lfloor d/2 \rfloor$, $f_i(P)$ can be written as a linear combination of $f_j(P)$ for $j < i$. This result and the fact that the cyclic polytope is $\lfloor d/2 \rfloor$-neighborly make the UBC even more plausible. Unfortunately, the main difficulty in trying to derive the UBC for the upper half of the face numbers from the lower half is that the linear combinations $f_i = \sum_{j=-1}^{i-1} a_{ij} f_j$ contain positive and negative coefficients, which makes it very hard to control the magnitude of the sums.

After many partial results and premature announcements, Motzkin's conjecture was finally proved by McMullen [58]. McMullen's insight was to work with the h-numbers instead of the f-numbers. At this point we should note that stated in terms of the h-numbers, the Dehn–Sommerville relations for simplicial polytopes take on the following elegant form: $h_i = h_{d-i}$ for all $0 \leq i \leq d$. (The $h_0 = h_d$ instance of this result is the Euler relation.)

McMullen used shellability of polytopes (established by Brugesser and Mani [17]) and the Dehn–Sommerville relations to prove that in the class of simplicial polytopes, the cyclic polytope simultaneously maximizes not only the f-numbers, but also the h-numbers. In other words, McMullen's Upper Bound Theorem (UBT for short) asserts that for every simplicial d-polytope P with n vertices,

$$h_i(P) \leq h_i(C_d(n)) \quad \text{for all } 0 \leq i \leq d.$$

The f-version of the UBC follows right away since the f-numbers are non-negative linear combinations of the h-numbers. Furthermore, the f- and the h-versions of the UBT can be stated as explicit bounds on the f-numbers (h-numbers, resp.) by using that

$$h_i(C_d(n)) = h_{d-i}(C_d(n)) = \binom{n-d+i-1}{i} \quad \text{for all } 0 \leq i \leq d/2.$$

Some additional remarks are in order. A simplicial $(d-1)$-sphere (or a simplicial $(d-1)$-ball) Δ is called *shellable* if its facets can be ordered $F_1, F_2, \ldots, F_m$ in such a way that for every $i < m$, the simplicial complex generated by the facets $F_1, \ldots, F_i$ is a simplicial $(d-1)$-ball; such an ordering of facets is called a *shelling*. While many simplicial spheres are not shellable [35,52], all spheres that arise as the boundary complexes of polytopes are.

The h-numbers of a shellable complex Δ have a simple and well-known interpretation in terms of any shelling of Δ, see [101, Section 8.3]. The Dehn–Sommerville relations for all shellable spheres are a consequence of this interpretation and the following easy fact: if $F_1, F_2, \ldots, F_m$ is a shelling order of facets of a simplicial sphere, then so is the reverse order $F_m, F_{m-1}, \ldots, F_1$.

A far-reaching generalization of the UBT for full-dimensional subcomplexes of the boundary complex of a simplicial d-polytope was proved by Kalai [42]; another generalization is due to Björner [14].

4. Spheres, manifolds, and Eulerian complexes

For a simplicial complex Δ and its face F, the *link of F in Δ* is the following subcomplex of Δ that captures the local behavior of Δ near F:

$$\mathrm{lk}(F, \Delta) := \{G \in \Delta \mid G \cap F = \emptyset,\ G \cup F \in \Delta\}.$$

In particular, $\mathrm{lk}(\emptyset, \Delta) = \Delta$. Understanding how various algebraic, topological, and combinatorial properties of links of non-empty faces affect the properties of the entire complex is a common theme in this part of combinatorics.

Klee [47] introduced *Eulerian complexes* as a combinatorial analog and a vast generalization of simplicial spheres: a pure $(d-1)$-dimensional simplicial complex Δ is Eulerian if for every face F of Δ, including the empty face, $\mathrm{lk}(F, \Delta)$ has the same Euler characteristic as a $(d - |F| - 1)$-dimensional sphere, $S^{d-|F|-1}$. (In particular, every ridge is in exactly two facets.) In addition to simplicial spheres, the class of Eulerian complexes includes among others all *Eulerian manifolds*, that is, all odd-dimensional simplicial manifolds, and all even-dimensional simplicial manifolds whose Euler characteristic is two.

Klee [47] proved that the Dehn–Sommerville relations $h_i = h_{d-i}$ for $0 \le i \le d$, hold for *all* Eulerian complexes of dimension $d-1$.* His proof relied on the Euler relation for the links of faces as well as on the observation that for a simplicial complex Δ and $j > i$, every $(j-1)$-face F of Δ contains exactly $\binom{j}{i}$ faces of dimension $i-1$ while every $(i-1)$-face G of Δ is contained in $f_{j-i-1}(\mathrm{lk}(G, \Delta))$ faces of dimension $j-1$. Klee then applied the Dehn–Sommerville relations along with some results from extremal combinatorics to prove the following astonishing fact: the assertion of the Upper Bound Theorem continues to hold for *all Eulerian simplicial complexes* provided they have sufficiently many vertices ($d^2/2$ is enough), see [48]. In view of this result, Klee [48] proposed the following far reaching extension of Motzkin's UBC:

Conjecture 4.1. *Let Δ be an Eulerian simplicial complex of dimension $d-1$ with n vertices. Then $f_i(\Delta) \le f_i(C_d(n))$ for all $1 \le i \le d-1$.*

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* Klee also established a version of the Dehn–Sommerville relations for *semi-Eulerian complexes*, i.e., pure complexes all of whose vertex links are Eulerian. Very recently Sawaske and Xue [86] extended this result to arbitrary pure simplicial complexes by expressing $h_{d-i}(\Delta) - h_i(\Delta)$ in terms of the Euler characteristics of links of faces of Δ.

While in this generality the conjecture remains wide open, at present it is known to hold for all simplicial spheres* (Stanley [94]), all Eulerian manifolds (Novik [68, 69]), and even some pseudomanifolds with very mild singularities (Hersh and Novik [37], and Novik and Swartz [73]).

Stanley's proof of the UBT for simplicial spheres relied on the theory of Cohen–Macaulay rings. In fact, it was one of the first applications of commutative algebra to combinatorics. Here are some highlights of Stanley's proof. Let Δ be a simplicial complex on the vertex set $[n]$. Consider the polynomial ring $\Bbbk[X] = \Bbbk[x_1, \dots, x_n]$ over an infinite field $\Bbbk$. Let $I_\Delta = (x_{i_1} \dots x_{i_s} \mid \{i_1 < i_2 < \dots < i_s\} \notin \Delta) \subset \Bbbk[X]$ be the squarefree ideal generated by *non-faces* of Δ. The *face ring* of Δ (also known as the *Stanley–Reisner ring* of Δ) is the quotient ring $\Bbbk[\Delta] := \Bbbk[X]/I_\Delta$.

The face ring of Δ is a finitely-generated standard graded $\Bbbk$-algebra. We denote by $\Bbbk[\Delta]_j$ its j-th graded component. Stanley's and Hochster's insight (independently from each other) in defining this ring [38, 94] was that algebraic properties of $\Bbbk[\Delta]$ reflect many combinatorial and topological properties of Δ. For instance, if Δ is $(d-1)$-dimensional, then the Hilbert series of $\Bbbk[\Delta]$, $\mathrm{Hilb}_{\Bbbk}(\Bbbk[\Delta], t) := \sum_{j=0}^{\infty} \dim_{\Bbbk} \Bbbk[\Delta]_j \cdot t^j$, is equal to

$$\sum_{i=0}^{d} \frac{f_{i-1}(\Delta) \cdot t^i}{(1-t)^i} = \frac{\sum_{i=0}^{d} h_i(\Delta) \cdot t^i}{(1-t)^d}.$$

Using techniques from homological algebra, Reisner [82] proved that if Δ is a simplicial $(d-1)$-sphere, then $\Bbbk[\Delta]$ is Cohen–Macaulay. This means that a sequence $\theta_1, \dots, \theta_d$ of *generic* linear forms in $\Bbbk[\Delta]$ is a *regular sequence*, i.e., θ_{s+1} is a non-zero divisor on $\Bbbk[\Delta]/(\theta_1, \dots, \theta_s)$ for all $0 \le s \le d-1$. Put differently, for every $0 \le s \le d-1$ and $j > 0$, the following sequence of $\Bbbk$-vector spaces is exact:

$$0 \to \Bbbk[\Delta]/(\theta_1, \dots, \theta_s)_{j-1} \xrightarrow{\cdot \theta_{s+1}} \Bbbk[\Delta]/(\theta_1, \dots, \theta_s)_j \to \Bbbk[\Delta]/(\theta_1, \dots, \theta_s, \theta_{s+1})_j \to 0. \tag{4.1}$$

The Cohen–Macaulayness of $\Bbbk[\Delta]$ is a key to Stanley's proof of the UBT. Indeed, standard manipulations with Hilbert series using (4.1), show that for a simplicial $(d-1)$-sphere Δ,

$$\mathrm{Hilb}_{\Bbbk}\big(\Bbbk[\Delta]/(\theta_1, \dots, \theta_d), t\big) = (1-t)^d \,\mathrm{Hilb}_{\Bbbk}\big(\Bbbk[\Delta], t\big) = \sum_{i=0}^{d} h_i(\Delta) \cdot t^i. \tag{4.2}$$

The fact that $\Bbbk[\Delta]/(\theta_1, \dots, \theta_d)$ is generated as a $\Bbbk$-algebra by $n-d$ elements of degree one then implies that $h_i(\Delta)$ cannot exceed the number of monomials of degree i in $n-d$ variables. The number of such monomials is $\binom{n-d+i-1}{i}$, which coincides with $h_i(C_d(n))$ when $i \le d/2$. This observation and the Dehn–Sommerville relations yield the UBT for simplicial spheres.

How does one extend Stanley's result to Eulerian manifolds? One immediate obstacle is that the face rings of simplicial manifolds are in general not Cohen–Macaulay.

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* As we will see in Section 7, for $d \ge 4$, there are many more simplicial $(d-1)$-spheres than simplicial d-polytopes.

However, they satisfy a weaker property, known as *Buchsbaumness*, and Buchsbaum face rings are reasonably well understood [87, 97], see also [71, 72] for more recent results.* Specifically, for a simplicial $(d-1)$-manifold Δ and a sequence $\theta_1, \ldots, \theta_d$ of generic linear forms in $\Bbbk[\Delta]$, Schenzel [87] computed the dimensions of kernels of maps $\cdot\theta_{s+1}$: $\Bbbk[\Delta]/(\theta_1, \ldots, \theta_s)_{j-1} \to \Bbbk[\Delta]/(\theta_1, \ldots, \theta_s)_j$. He then used this result to derive the following formula for the Hilbert function of the quotient $\Bbbk(\Delta) := \Bbbk[\Delta]/(\theta_1, \ldots, \theta_d)$ (such quotient is called an *Artinian reduction* of $\Bbbk[\Delta]$):

$$\dim_{\Bbbk} \Bbbk(\Delta)_i = h_i(\Delta) + \binom{d}{i}\big(\beta_{i-2}(\Delta) - \beta_{i-3}(\Delta) + \cdots + (-1)^i\beta_0(\Delta)\big) \quad \text{for all } 0 \le i \le d.$$

We denote the right-hand side of this equation by $h_i'(\Delta)$. Here $\beta_j(\Delta)$ is the dimension of the j-th reduced simplicial homology of Δ computed with coefficients in $\Bbbk$. In particular, if Δ is a simplicial sphere, then $h_i'(\Delta) = h_i(\Delta)$ for all i (which recovers Stanley's formula (4.2)).

Now consider the socle $\mathcal{S}$ of $\Bbbk(\Delta)$, i.e., the collection of elements μ of $\Bbbk(\Delta)$ such that $x_\ell \cdot \mu = 0$ for all variables $x_\ell \in X$. In the spirit of Schenzel's results, Novik and Swartz [72] proved that the dimension of $\mathcal{S}_i$ is at least $\binom{d}{i}\beta_{i-1}(\Delta)$. (For a weaker bound that suffices to prove the UBT, see [68]). As $\Bbbk(\Delta)/(\mathcal{S}_i)$ is a standard graded $\Bbbk$-algebra**, the dimensions of its homogeneous components satisfy Macaulay's inequalities [56], [96, page 56]. In particular, one obtains an upper bound on $h_{i+1}'(\Delta)$ in terms of $h_i' - \binom{d}{i}\beta_{i-1}$. These bounds (together with the Dehn–Sommerville relations) in turn lead to the desired upper bounds on the h-numbers of an Eulerian manifold Δ.

Related to the UBT is a conjecture by Kühnel [49, Conjecture B]. It asserts that the *reduced Euler characteristic of a simplicial $2k$-manifold Δ on n vertices satisfies*

$$(-1)^k\binom{2k+1}{k}(\tilde{\chi}(\Delta) - 1) \le \binom{n-k-2}{k+1}.$$

Moreover, equality holds if and only if Δ is $(k+1)$-neighborly. Kühnel's conjecture was proved by Novik and Swartz [72] using machinery similar to the one outlined above. What is important to note is that while an Eulerian $2k$-manifold cannot be $(k+1)$-neighborly unless it is the boundary of a simplex, there do exist non-Eulerian simplicial $2k$-manifolds that are $(k+1)$-neighborly. For surfaces, there are the 2-neighborly triangulations in [40, 84] (such as the six-vertex triangulation of the real projective plane). For $k > 1$, examples include the nine-vertex triangulation of the complex projective plane [50] and several 13-vertex triangulations of $S^3 \times S^3$ [55]; for additional examples see [15, 21].

5. Witt spaces

Computations similar to the ones sketched near the end of the previous section can be used to show that the bounds of the UBC continue to hold for some non-Eulerian

* For an alternative very recent treatment of Cohen-Macaulay and Buchsbaum face rings, where the use of local cohomology and other homological algebra techniques is replaced with a more topological approach, see [5].

** A much stronger result [5, 71] asserts that if Δ is a connected simplicial $(d-1)$-manifold orientable over $\Bbbk$, then $\Bbbk(\Delta)/\oplus_{i=0}^{d-1}\mathcal{S}_i$ is a Poincaré duality algebra.

manifolds, most notably, they hold for all even-dimensional manifolds with vanishing middle homology (Novik [68]).* In light of this result, Kalai [43] proposed to extend the UBC to complexes triangulating Witt spaces. The notion of Witt spaces was introduced by Siegel [90]; it relies on Goresky and MacPherson's intersection homology theory [30, 31] — the theory that was developed as a generalization of the Poincaré–Lefschetz theory to stratified singular spaces such as piecewise-linear pseudomanifolds. Here we only consider simplicial strata. A simplicial *pseudomanifold* is a pure simplicial complex Δ such that every ridge of Δ is contained in exactly two facets. An oriented simplicial pseudomanifold Δ is a *Witt space* if its intersection homology groups w.r.t. lower-middle perversity $\bar{m} = (0, 0, 1, 1, 2, 2, \dots)$ have the following property: $IH_j^{\bar{m}}(\Delta'; \mathbb{Q}) = 0$ for every j and every $2j$-dimensional subcomplex $\Delta' \subset \Delta$ that is the link of a non-empty face of Δ. Kalai's conjecture posits the following:

Conjecture 5.1. *Let Δ be a simplicial complex that is a Witt space. Assume that Δ is $(d-1)$-dimensional and has n vertices. If $d-1$ is even, assume further that $IH^{\bar{m}}_{(d-1)/2}(\Delta) = 0$. Then $f_i(\Delta) \le f_i(C_d(n))$ for all $1 \le i \le d-1$.*

An even stronger conjecture (in the spirit of [14,42]) asserts that if Δ is a $(d-1)$-dimensional Witt space and $f_i(\Delta) \le f_i(C_d(n'))$ for some i and n', then $f_j(\Delta) \le f_j(C_d(n'))$ for all $j > i$.

At present, Conjecture 5.1 is known to hold for even-dimensional simplicial manifolds with vanishing middle homology (even on the level of the h-numbers) [68] and also for odd-dimensional pure complexes all of whose vertex links are manifolds with vanishing middle homology (but only on the level of the f-numbers) [37]. It is open in all other cases. The main difficulty seems to be our lack of understanding how various topological characteristics other than simplicial homology can be traced in the face rings.

6. The g-conjecture

This part of the story would be incomplete if we do not mention some very recent spectacular developments on the g-conjecture. The g-conjecture posits a characterization of the set of f-vectors of simplicial spheres; as such, it contains the UBC for spheres as a special case. (For the class of simplicial polytopes the g-conjecture was proposed by McMullen [59].) The conjecture states that an integer vector $(h_0, h_1, h_2, \dots, h_d)$ is the h-vector of a simplicial $(d-1)$-sphere if and only if (i) $h_j = h_{d-j}$ for all $0 \le j \le d$, (ii) $1 = h_0 \le h_1 \le \cdots \le h_{\lfloor d/2 \rfloor}$, and (iii) the numbers $(g_j := h_j - h_{j-1})_{j=0}^{\lfloor d/2 \rfloor}$ form an M-sequence, i.e., they satisfy Macaulay's inequalities. (Here $g_0 := 1$.) The sufficiency of these conditions was established by Billera and Lee [13]. The necessity of conditions for the case of simplicial polytopes was proved by Stanley [95]; a more elementary proof was found by McMullen [60, 61], see also [25]. The necessity of conditions for simplicial spheres remained open until very recently and was considered one of the most outstanding problems in the field.

* On the other hand, the existence of $(k+1)$-neighborly $2k$-manifolds that are not boundaries of simplices demonstrates that not all simplicial manifolds satisfy the bounds of the UBC.

The recent striking news is that Adiprasito [3], Papadakis and Petrotou [79], and Adiprasito, Papadakis, and Petrotou [4] proved the g-conjecture for simplicial spheres. Along the way, they established surprising algebraic properties of Artinian reductions of face rings of simplicial spheres. For instance, [79] asserts that if Δ is a simplicial $(d-1)$-sphere and $\Bbbk$ is a field of characteristic two, then there exists a purely transcendental field extension $\mathbb{K}$ of $\Bbbk$ and linear forms $\theta_1, \ldots, \theta_d \in \mathbb{K}[\Delta]$ such that for every nonzero homogeneous element $u \in \mathbb{K}[\Delta]/(\theta_1, \ldots, \theta_d)$ of degree at most $d/2$, its square u^2 is also *nonzero*. The paper [4] provides far reaching generalizations of these algebraic results to face rings of much more general simplicial complexes such as normal pseudomanifolds. It is now more pressing than ever to compute the Hilbert functions of generic Artinian reductions of face rings and certain further quotients of these rings for the purpose of understanding complexes with singularities more complicated than those studied in [73,74].

7. Numbers of neighborly polytopes and neighborly spheres

As we saw in Section 3, the cyclic d-polytopes have the remarkable property of being $\lfloor d/2 \rfloor$-neighborly. At the same time, it follows easily from the Dehn–Sommerville relations that no simplicial $(d-1)$-sphere except for the boundary of a simplex can be $(\lfloor d/2 \rfloor + 1)$-neighborly. This naturally leads to the question of how rare, or how common, the property of being $\lfloor d/2 \rfloor$-neighborly is in the class of simplicial d-polytopes (or $(d-1)$-spheres). To make this discussion more precise, we first need to understand how many combinatorial types of simplicial d-polytopes (or $(d-1)$-spheres) with n *labeled* vertices there are. We denote these numbers by $c(d,n)$ and $s(d,n)$, respectively. (That is, we fix the vertex set to be $[n]$ and count the number of relevant simplicial complexes up to equality. In the case of *unlabeled* vertices, we count the number of complexes up to isomorphism.)

We say that a simplicial $(d-1)$-sphere is *polytopal* if it is isomorphic to the boundary complex of a d-polytope. It follows from Steinitz's theorem, see [34, Chapter 13], that all simplicial (and even polyhedral) 2-spheres are polytopal. Hence $c(3,n) = s(3,n)$. The asymptotic behavior of $s(3,n)$ (as well as of the number of unlabeled 2-spheres with n vertices) was worked out by Tutte [98,99], Brown [16], and Richmond and Wormland [83].

For $d \geq 4$ the results become more surprising. While by a result of Mani [57], every simplicial $(d-1)$-sphere with $n \leq d+3$ vertices is polytopal, there are examples of non-polytopal simplicial spheres already for $(d,n) = (4,8)$, see Grünbaum [34, Section 11.5] and Barnette [8]. In fact, Goodman and Pollack [29], followed by the work of Alon [7], proved that there are far fewer polytopes than what was expected at the time:

$$\big((n/d) - 1\big)^{\lfloor d/2 \rfloor n/2} \leq c(d,n) \leq n^{d(d+1)n}.$$

In other words, for a fixed d, $c(d,n) = 2^{\Theta(n \log n)}$.

The proof of the inequality $c(d,n) \leq n^{d(d+1)n}$ relies on a theorem of Milnor [64] that gives bounds on the sum of the Betti numbers of real algebraic varieties. Assume n is even. Alon's construction [7] providing lower bounds on $c(d,n)$ starts with the cyclic polytope $C_d(n/2)$ on the first $n/2$ vertices; he then adds the last $n/2$ labeled vertices in all possible ways by placing each of them close to a facet of $C_d(n/2)$. That for $n > 2d$, $f_{d-1}(C_d(n/2))$ is at least $\big((n/d)-1\big)^{\lfloor d/2\rfloor}$, implies that $c(d,n) \geq ((n/d)-1)^{\lfloor d/2\rfloor n/2}$.

In striking contrast to these results, Kalai [41] proved that there is an enormous number of simplicial spheres: for $d \geq 5$, $s(d,n) \geq 2^{\Omega(n^{\lfloor (d-1)/2\rfloor})}$. Furthermore, Pfeifle and Ziegler [81] showed that $s(4,n) \geq 2^{\Omega(n^{5/4})}$. The current record on the number of odd-dimensional simplicial spheres is due to Nevo, Santos, and Wilson [67] who verified that $s(2k,n) \geq 2^{\Omega(n^k)}$ for all $k \geq 2$. On the other hand, Stanley's UBT for spheres implies that $s(d,n) \leq 2^{O(n^{\lfloor d/2\rfloor}\log n)}$ (see [41]). To summarize, the current best bounds on $s(d,n)$ are

$$2^{\Omega(n^{\lfloor d/2\rfloor})} \leq s(d,n) \leq 2^{O(n^{\lfloor d/2\rfloor}\log n)} \quad \text{for all } d \geq 4.$$

What proportion of simplicial d-polytopes ($(d-1)$-spheres) are $\lfloor d/2\rfloor$-neighborly? While Motzkin [65] believed that $C_d(n)$ is the only $\lfloor d/2\rfloor$-neighborly d-polytope on n vertices, Shemer [89] introduced a *sewing construction* and used it to prove that even the number of distinct unlabeled $\lfloor d/2\rfloor$-neighborly d-polytopes on n vertices is at least $n^{a_d n}$, where $\lim_{d\to\infty} a_d = 1/2$. Generalizing Shemer's sewing construction, Padrol [78] greatly improved this bound: he constructed on the order of $n^{dn/2}$ labeled $\lfloor d/2\rfloor$-neighborly d-polytopes on n vertices (for even d). This lower bound on the number of $\lfloor d/2\rfloor$-neighborly d-polytopes is, of course, also a lower bound on $c(d,n)$. In fact, it is the *current best* lower bound on $c(d,n)$. This state of affairs seems to indicate that the property of being neighborly is very common among polytopes.

Along the same lines, Kalai [41] speculated that the number $\mathrm{sn}(d,n)$ of $\lfloor d/2\rfloor$-neighborly simplicial $(d-1)$-spheres with n labeled vertices is very large and posited the following conjecture:

Conjecture 7.1. *For all $d \geq 4$, $\lim_{n\to\infty}(\log \mathrm{sn}(d,n)/\log s(d,n)) = 1$.*

In his paper [78], Padrol also constructed a large number of $\lfloor d/2\rfloor$-neighborly $(d-1)$-spheres arising from non-realizable oriented matroids. Yet, Padrol's bound only implied that $\mathrm{sn}(d,n) \geq 2^{\Omega(n\log n)}$. While we are still very far from being able to shed light on Conjecture 7.1, very recently Novik and Zheng [77] proved that for all $d \geq 5$,

$$\mathrm{sn}(d,n) \geq 2^{\Omega(n^{\lfloor (d-1)/2\rfloor})}. \tag{7.1}$$

Note that for $d \geq 5$, the number of combinatorial types of unlabeled $\lfloor d/2\rfloor$-neighborly $(d-1)$-spheres on n vertices is also at least $2^{\Omega(n^{\lfloor (d-1)/2\rfloor})}$. Indeed, dividing the lower bound by $n! = 2^{O(n\log n)}$ does not affect its asymptotic growth if $d \geq 5$.

We now sketch some of the ideas used by Kalai [41] to show that for odd $d \geq 5$, $s(d,n) \geq 2^{\Omega(n^{(d-1)/2})}$. We then discuss some additional ideas needed to modify Kalai's construction in order to prove (7.1) for odd d. For the rest of this section, we treat the boundary complex of $C_d(n)$ as an abstract simplicial complex. In particular, a vertex $M(t_i)$ of $C_d(n)$ is identified with $i \in [n]$, and a face $\mathrm{conv}(M(t_i) \mid i \in I)$ with $I \subset [n]$. For integers $a < b$, we write $[a,b]$ to denote the set $\{a, a+1, \ldots, b\}$. If B is a simplicial d-ball, then the *boundary of B*, ∂B, is the $(d-1)$-dimensional subcomplex of B generated by ridges of B that are contained in exactly one facet of B.

Kalai's construction is a generalization of a construction used by Billera and Lee [13] to prove the sufficiency of conditions of the g-conjecture. It starts with the family $\mathcal{F} = \mathcal{F}_{2k}(n)$ of all $2k$-subsets of $[n]$ (here $k \geq 2$ is a fixed integer) of the form

$$\mathcal{F} = \big\{\{i_1, i_1+1, i_2, i_2+1, \dots, i_k, i_k+1\} \mid 1 \leq i_1 < i_1+1 < i_2 < \cdots < i_k < i_k+1 \leq n\big\}.$$

By Gale's evenness condition [26], each $F \in \mathcal{F}$ is a facet of $C_{2k}(n)$. Partially order the set $\mathcal{F}$ by $\{j_1, \dots, j_{2k}\} \leq_p \{\ell_1, \dots, \ell_{2k}\}$ if $j_1 \leq \ell_1, \dots, j_{2k} \leq \ell_{2k}$. For an antichain $\mathcal{A}$ in the poset $(\mathcal{F}, \leq_p)$, let $\mathcal{F}(\mathcal{A})$ be the order ideal of $\mathcal{F}$ generated by $\mathcal{A}$ and let $B(\mathcal{A})$ be the simplicial complex whose facets are the elements of $\mathcal{F}(\mathcal{A})$. For instance, if $k = 2$, $n = 8$, and $\mathcal{A} = \{\{1,2,7,8\}, \{2,3,6,7\}, \{3,4,5,6\}\}$, then $\mathcal{F}(\mathcal{A})$ consists of

$$\begin{aligned} &\{1,2,7,8\}, \{1,2,6,7\}, \{1,2,5,6\}, \{1,2,4,5\}, \{1,2,3,4\}, \\ &\qquad\qquad \{2,3,6,7\}, \{2,3,5,6\}, \{2,3,4,5\}, \{3,4,5,6\}, \end{aligned}$$

and $B(\mathcal{A})$ is generated by these nine facets.

Kalai [41] proved that for every (non-empty) antichain $\mathcal{A}$, $B(\mathcal{A})$ is a shellable $(2k-1)$-ball, that all vertices of $B(\mathcal{A})$ are on the boundary of $B(\mathcal{A})$, and that the boundaries $\partial(B(\mathcal{A}))$ and $\partial(B(\mathcal{A}'))$ of two such balls coincide if and only if $\mathcal{A} = \mathcal{A}'$. Estimating the number of antichains, he concluded that there are at least $2^{\Omega(n^{k-1})}$ such balls with exactly n vertices. Their boundaries provide us with the desired number of distinct $(2k-2)$-spheres with n (labeled) vertices. Kalai called these balls *squeezed balls* and their boundaries *squeezed spheres*.

To prove (7.1), the trick is to consider *differences* of appropriately chosen squeezed balls. To do so, for an antichain $\mathcal{A}$ in $\mathcal{F}$, define

$$\mathcal{A} - \mathbf{1} := \big\{\{i_1-1, i_1, i_2-1, i_2, \dots, i_{2k}-1, i_{2k}\} \mid \{i_1, i_1+1, \dots, i_{2k}, i_{2k}+1\} \in \mathcal{A} \text{ and } i_1 > 1\big\}.$$

Let $B_{\mathcal{A}}$ be the simplicial complex whose facets are the elements of $\mathcal{F}(\mathcal{A}) \setminus \mathcal{F}(\mathcal{A} - \mathbf{1})$. In the above example $\mathcal{A} = \{\{1,2,7,8\}, \{2,3,6,7\}, \{3,4,5,6\}\}$, $\mathcal{A} - \mathbf{1} = \{\{1,2,5,6\}, \{2,3,4,5\}\}$, and the facets of $B_{\mathcal{A}}$ are $\{1,2,7,8\}$, $\{1,2,6,7\}$, $\{2,3,6,7\}$, $\{2,3,5,6\}$, and $\{3,4,5,6\}$.

One now checks, see [77], that the following results, paralleling Kalai's theorem, hold: if $\mathcal{A}$ is an antichain in $\mathcal{F}$ that contains $[1,2] \cup [n-2k+3, n]$ as an element, then $B_{\mathcal{A}}$ is a simplicial $(2k-1)$-ball that has n vertices; this ball is $(k-1)$-*neighborly* and all faces of $B_{\mathcal{A}}$ of dimension $\leq k-1$ are on the boundary of $B_{\mathcal{A}}$; furthermore, the boundaries of two such balls $B_{\mathcal{A}}$ and $B_{\mathcal{A}'}$ coincide if and only if $\mathcal{A} = \mathcal{A}'$. We refer to $B_{\mathcal{A}}$ as a *relative squeezed ball* and to its boundary $\partial B_{\mathcal{A}}$ as a *relative squeezed sphere*. The bound (7.1) for odd d follows, since relative squeezed spheres are $(k-1)$-neighborly $(2k-2)$-spheres with n vertices and there are at least $2^{\Omega(n^{k-1})}$ of them.

A simplicial d-ball B all of whose faces of dimension $\leq d-i-1$ lie on the boundary of B is called *i-stacked*. In particular, all $(2k-1)$-balls $B_{\mathcal{A}}$ are $(k-1)$-stacked. The notion of stackedness takes its origins in the Generalized Lower Bound Theorem [63, 66, 95].

While at present Conjecture 7.1 is likely out of reach, establishing the bound $\mathrm{sn}(d,n) \geq 2^{\Omega(n^{\lfloor d/2 \rfloor})}$ might be a more feasible goal. We would also like to mention that while Kalai's squeezed balls are shellable, and so are all squeezed spheres [51], the question of whether relative squeezed balls and relative squeezed spheres are shellable is open.

8. The upper bound theorem for centrally symmetric spheres

We now shift our focus to a fascinating subclass of simplicial complexes, that of centrally symmetric complexes. Some definitions are in order. A polytope $P \subset \mathbb{R}^d$ is *centrally symmetric* (cs, for short) if $P = -P$. In the same spirit, a simplicial complex Δ is *centrally symmetric* (cs, for short) if the vertex set of Δ is endowed with a *free involution* α that induces a free involution on the set of all nonempty faces of Δ. In more detail, for every nonempty face $F \in \Delta$, the following holds:

$$\alpha(F) \in \Delta, \quad \alpha(F) \neq F, \quad \text{and } \alpha(\alpha(F)) = F.$$

A complex Δ is a cs simplicial sphere if Δ is both a simplicial sphere and a cs complex. For instance, if P is a cs simplicial polytope, then the boundary complex ∂P of P with the map $\alpha(v) = -v$ is a cs simplicial sphere.

To simplify notation, for a cs simplicial complex Δ and a face $F \in \Delta$, we write $\alpha(F) = -F$ and refer to F and $-F$ as antipodal faces of Δ. In particular, if Δ is a cs complex with $2n$ vertices, we usually assume that the vertex set of Δ is $V_n = \{\pm 1, \pm 2, \dots, \pm n\}$. Any cs complex Δ with $2n$ vertices can be naturally associated with a subcomplex of ∂C_n^*, where $C_n^* = \mathrm{conv}(\pm e_1, \pm e_2, \dots, \pm e_n)$ is the n-dimensional *cross-polytope*. (Here $e_1, \dots, e_n$ are the endpoints of the standard basis of $\mathbb{R}^n$.)

Our discussion in this and the next sections is aimed at and motivated by the following questions: what restrictions does being cs impose on the f-vectors of cs simplicial spheres and cs polytopes? What are the cs analogs of the UBT? Is there a cs version of the cyclic polytope?

To start, note that if Δ is a cs complex and v is a vertex of Δ, then v and $-v$ never form an edge. Thus the notion of neighborliness requires minor adjustments: a cs simplical complex Δ is *cs-k-neighborly* if every set of k of its vertices, no two of which are antipodes, is a face of Δ. Some examples: ∂C_d^* is cs-d-neighborly, while (the boundary complex of) $\mathrm{conv}\left(\pm e_1, \pm e_2, \dots, \pm e_d, \pm \sum_{i=1}^d e_i\right)$, which is a cs d-polytope with $2(d+1)$ vertices, is cs-$\lfloor d/2 \rfloor$-neighborly. This latter example is due to McMullen and Shephard [62].

What can be said about neighborliness of cs d-polytopes and cs $(d-1)$-spheres with more than $2(d+1)$ vertices? At this point some striking discrepancies between the cs and non-cs worlds start to emerge. On one hand, Grünbaum [34, Section 6.4] (for $d = 4$) and McMullen and Shephard [62] (for general d) proved that in contrast to the non-cs case, a cs polytope with at least $2(d+2)$ vertices cannot be cs-$(\lfloor (d+1)/3 \rfloor + 1)$-neighborly. On the other hand, Grünbaum [32, 33] constructed cs simplicial 3-spheres with 12 vertices that are cs-2-neighborly, thus leaving open the possibility that cs-$\lfloor d/2 \rfloor$-neighborly simplicial $(d-1)$-spheres with an arbitrary large number of vertices may exist.

An additional incentive for investigating if highly neighborly cs spheres exist comes from the following theorem due to Adin [2] and Stanley (unpublished): *in the class of all cs simplicial $(d-1)$-spheres with $2n$ vertices, a cs-$\lfloor d/2 \rfloor$-neighborly sphere simultaneously maximizes all the face numbers assuming such a sphere exists*. (The proof uses face rings and is similar to Stanley's proof of the UBT for spheres discussed in Section 4.)

Does such a sphere exist? In 1995, Jockusch [39] gave a positive answer for $d = 4$: he showed that for *every* value of $n \geq 4$, there is a cs simplicial 3-sphere with $2n$ vertices that is cs-2-neighborly. A few years later, for each $d \leq 7$, Lutz [54] found (by a computer search) several cs simplicial $(d-1)$-spheres with $2(d+2)$ vertices that are cs-$\lfloor d/2 \rfloor$-neighborly. Recently, building on the work of Jockusch [39], Novik and Zheng [75] provided a complete answer: for *all* values of $d \geq 4$ and $n \geq d$, there exists a cs simplicial $(d-1)$-sphere with $2n$ vertices, Δ_n^{d-1}, that is cs-$\lfloor d/2 \rfloor$-neighborly. Combined with the work of Adin and Stanley, this result completely resolved the upper bound problem for cs simplicial spheres.

The construction of Δ_n^{d-1} is quite involved and uses induction on both $d \geq 2$ and $n \geq d$. One key idea of the construction is for all $d, n \geq d$, and $i \leq \lfloor d/2 \rfloor - 1$, to define (by triple induction) an auxiliary simplicial $(d-1)$-ball, $B_n^{d-1,i} \subset \partial C_n^*$, on the vertex set $\{\pm 1, \ldots, \pm n\}$, that is both i-stacked and cs-i-neighborly. (Recall that i-stacked balls were defined at the end of Section 7. While $B_n^{d-1,i} \subset \partial C_n^*$ is not a cs complex, it is a subcomplex of one; hence the definition of cs-i-neighborliness still makes sense.) In the case of $d = 4$, the construction reduces to Jockusch's construction. A curious property of Δ_n^{d-1} worth mentioning is that the link of $\{n-1, n\}$ in Δ_n^{2k+1} is the complex Δ_{n-2}^{2k-1}, while the link of $\{n-2, n-1, n\}$ in Δ_n^{2k+2} is Δ_{n-3}^{2k-1}. Another property worth mentioning is that in addition to being cs-k-neighborly, the sphere Δ_n^{2k-1} is also *k-stacked*, i.e., it is the boundary of a k-stacked ball.

By results of McMullen and Shephard [62], for $d \geq 4$ and $n \geq d+2$, the complex Δ_n^{d-1} is not isomorphic to the boundary complex of a *cs* polytope. This still leaves open the question of whether Δ_n^{d-1} can be realized as the boundary complex of some non-cs polytope. The answer was recently provided by Pfeifle [80] who proved that for all $d \geq 4$ and $n \geq d+1$ (including $n = d+1$), the complex Δ_n^{d-1} is *not* polytopal! This is quite a remarkable achievement, because while most of simplicial spheres are not polytopal (see Section 7), determining whether a particular simplicial sphere is polytopal or not is very hard. For his proof, Pfeifle introduced a new method for finding a non-realizability certificate of a simplicial sphere; these certificates involve combinations of Plücker relations.

Now that the existence of cs $(d-1)$-spheres with arbitrarily many vertices that are cs-$\lfloor d/2 \rfloor$-neighborly is established, a new tantalizing question is: for a fixed $d \geq 4$, how many pairwise non-isomorphic such cs spheres with $2n$ vertices are there? In light of Section 7, it is very tempting to conjecture that there are at least $2^{\Omega(n^{\lfloor (d-1)/2 \rfloor})}$ of them. This is wide open at present. Indeed, our current knowledge on this subject is as follows [76]: while for $d = 4$ and 5, there are at least $\Omega(2^n)$ such non-isomorphic cs spheres, for $d = 2k > 4$ and $n \gg 0$, only two non-isomorphic constructions are available at present; they are the edge links of $\{n+1, n+2\}$ and $\{1, 2\}$ in Δ_{n+2}^{2k+1}. For $d = 2k+1$, there are three such constructions: the suspensions of the two complexes just mentioned (but with $2(n-1)$ vertices) and Δ_n^{2k}.

Another natural question is whether a cs analog of Klee's UBC holds. Specifically, is it true that in the class of all *cs* Eulerian simplicial complexes of dimension $d-1$ with $2n$ vertices, the complex Δ_n^{d-1} simultaneously maximizes all the face numbers?

It is also worth mentioning that parallel to Kühnel's conjecture (see Section 4) is Sparla's conjecture on the Euler characteristic of cs simplicial $2k$-manifolds [92, 93]. This conjecture is still open for manifolds with fewer than $6k+4$ vertices (see [46, 69] for the state of the art). On a related note, there do exist non-Eulerian cs simplicial $2k$-manifolds that are cs-$(k+1)$-neighborly. A construction of such a cs $(4k+4)$-vertex triangulation of $S^k \times S^k$ for each $k \geq 1$ is given in [46].

9. How neighborly can a cs polytope be?

We now arrive at the most mysterious part of the story: trying to understand possible neighborliness of cs polytopes as well as trying to come up with tight upper bounds on face numbers of cs polytopes.

As was mentioned in Section 8, a cs d-polytope with at least $2(d+2)$ vertices cannot be cs-$(\lfloor (d+1)/3 \rfloor + 1)$-neighborly [62]. To prove this result, McMullen and Shephard developed a notion of *cs transforms* of cs polytopes — a cs analog of the celebrated Gale diagrams. A cs transform associates with a cs set $V = \{\pm v_1, \dots, \pm v_m\} \subset \mathbb{R}^d$ a certain cs set $\overline{V} = \{\pm \bar{v}_1, \dots, \pm \bar{v}_m\} \subset \mathbb{R}^{m-d}$ in such a way that given $\overline{V}$, one can check whether it is a cs transform of the vertex set of a cs polytope, and, if this is the case, one can read the vertex sets of the faces of this polytope from $\overline{V}$.

How neighborly can a cs polytope be? We let $k(d,n)$ denote the largest integer k such that there exists a cs d-polytope with $2(d+n)$ vertices that is cs-k-neighborly. In view of their results that $k(d,1) = \lfloor d/2 \rfloor$ and $k(d,2) = \lfloor (d+1)/3 \rfloor$, McMullen and Shephard [62] conjectured that $k(d,n) \leq \lfloor (d+n-1)/(n+1) \rfloor$ for all $n \geq 3$. In particular, according to this conjecture, $k(d,d) = 1$, so if the conjecture holds, a cs d-polytope with $4d$ vertices cannot even be cs-2-neighborly. The conjecture was quickly refuted by Halsey [36] and then by Schneider [88], but only for $d \gg n$. In a positive direction, Burton [18] proved that a cs d-polytope with a sufficiently large number of vertices ($\approx (d/2)^{d/2}$) indeed cannot even be cs-2-neighborly.

The field lay dormant for a few decades, until Donoho et al. [23, 24], see also [85], discovered some amazing connections between cs polytopes with many faces and seemingly unrelated areas of error-correcting codes and sparse signal reconstruction. In particular, Donoho [23] proved that there exists a positive constant ρ such that for large d, $k(d,d) \geq \rho d$. More specifically, he showed that the orthogonal projection of the cross-polytope C^*_{2d} onto a d-dimensional subspace of $\mathbb{R}^{2d}$, chosen uniformly at random, is with high probability at least cs-$\lfloor \rho d \rfloor$-neighborly.

Linial and Novik [53], following the work of Donoho, established the asymptotics of $k(d,n)$. They proved that there exist constants $C_1, C_2 > 0$ independent of d and n such that

$$\frac{C_1 d}{1 + \log((d+n)/d)} \leq k(d,n) \leq \frac{C_2 d}{1 + \log((d+n)/d)}. \tag{9.1}$$

The lower bound $k(d,n) \geq \frac{C_1 d}{1+\log((d+n)/d)}$ was also proved independently and at about the same time by Rudelson and Vershynin [85]. Both proofs of the lower bound relied on probabilistic arguments or, more precisely, on "high-dimensional" paradoxes such as Kašin's theorem [45] and Garnaev–Gluskin's theorem [27]. Consider the Grassmannian manifold $G_{n,d+n}$ endowed with the normed unitary invariant measure. Garnaev–Gluskin's theorem asserts that an n-dimensional subspace L of $\mathbb{R}^{d+n}$, chosen uniformly at random, is with positive probability "almost Euclidean", meaning that for all $x \in L \setminus \{0\}$, the ratio $\|x\|_2/\|x\|_1$ is bounded from above by $\tilde{C}\sqrt{\frac{1+\log((d+n)/d)}{d}}$ for some absolute constant $\tilde{C} > 0$. Via cs transforms, the existence of such a subspace L implies the existence of a cs-k-neighborly d-polytope with $2(d+n)$ vertices where k is given by the left-hand side of (9.1); see [53].

While proofs using probabilistic arguments are very beautiful, some disadvantages of such proofs are that they only show existence rather than provide explicit constructions and they only produce asymptotic bounds rather than exact values. Recently some progress has been made on understanding the maximum possible number of vertices that a cs-2-neighborly d-polytope can have. Although we still do not know the exact value, we now know it up to a factor of two: *for $d \geq 2$, there exists a cs d-polytope with $2^{d-1}+2$ vertices that is cs-2-neighborly* [70]; *on the other hand, for $d \geq 3$, no cs polytope with 2^d or more vertices can be cs-2-neighborly* [53]. Written in terms of $k(d,n)$, this result says that, for $d \geq 3$, $k(d,n) \geq 2$ for all $n \leq 2^{d-2}+1-d$ while $k(d,n) = 1$ for all $n \geq 2^{d-1}-d$.

The result that for $d \geq 3$, no cs polytope with 2^d or more vertices can be cs-2-neighborly is due to Linial and Novik [53]. The proof has two ingredients. The first one is a simple observation that the vertex set $V \subset \mathbb{R}^d$ of a cs-2-neighborly polytope is *antipodal*, i.e., for every two vertices $x, y \in V$ $(x \neq y)$, there exist two distinct parallel hyperplanes H_x and H_y such that $x \in H_x$, $y \in H_y$, and all elements of V lie in the closed strip defined by H_x and H_y. The second ingredient is a celebrated theorem of Danzer and Grünbaum [22] (see also [6, Chapter 17]) asserting that an antipodal set in $\mathbb{R}^d$ has at most 2^d points, and it has exactly 2^d points if and only if it is the vertex set of a parallelotope (which is not cs-2-neighborly, unless $d = 2$).

The existence of a cs d-polytope with $2^{d-1}+2$ vertices that is cs-2-neighborly was established by Novik [70]. The construction of such a polytope is a modification of a recent construction due to Gerencsér and Harangi [28] of an acute set in $\mathbb{R}^d$ of size $2^{d-1}+1$. Informally, the description is as follows: start with the vertex set V of the $(d-1)$-cube $[-1,1]^{d-1}$ embedded in the coordinate hyperplane $\mathbb{R}^{d-1} \times \{0\}$. Then use the extra dimension to perturb the vertices in such a way that the resulting set V' is "almost acute", i.e., V' is cs and for every $v, u, w \in V'$ with $v \neq -w$, the angle $\angle vuw$ is acute. Adding to V' a pair of antipodes of the form $\pm(0,0,\ldots,0,c)$, where $0 < c \in \mathbb{R}$ is sufficiently large, creates the vertex set of a cs d-polytope that is cs-2-neighborly and has a desired number of vertices; see [70] for details.

To summarize our discussion, for $d \geq 3$, the maximum number of vertices, m_d, that a cs-2-neighborly d-polytope can have lies in the interval $[2^{d-1}+2, 2^d-2]$. The value of m_3 is $6 = 2^2+2$ as the only cs-2-neighborly 3-polytope is an octahedron. The value of m_4 is $10 = 2^3+2$: this is a consequence of Grünbaum's result that a cs 4-polytope with 12 vertices cannot be cs-2-neighborly. The exact values of m_d for $d \geq 5$ are unknown at present.

10. Towards an Upper Bound Conjecture for cs polytopes

What is the largest number $\text{fmax}(d, N; i)$ of i-faces that a cs d-polytope with $N = 2n$ vertices can have? The discussion in the previous section indicates that at present we are very far from even being able to pose a plausible conjecture, even in the case of $i = 1$. Instead, we can try to ask for asymptotic bounds on $\text{fmax}(d, N; i)$.

For the case of $i = 1$, the best to-date bounds are:

$$\frac{3}{4} \cdot \frac{N^2}{2} - O(N) \leq \text{fmax}(4, N; 1) \leq \frac{15}{16} \cdot \frac{N^2}{2}, \tag{10.1}$$

and for any even $d = 2k > 4$,

$$\left(1 - \frac{1}{3}(\sqrt{3})^{-d}\right) \cdot \binom{N}{2} \leq \text{fmax}(d, N; 1) \leq \left(1 - 2^{-d}\right) \cdot \frac{N^2}{2}. \tag{10.2}$$

The upper bounds are due to Barvinok and Novik [11]. Their proof involves a careful use of a volume trick similar to the one utilized in the proof of the Danzer–Grünbaum theorem [22]. The lower bounds were established by Barvinok, Lee, and Novik [9]; they rely on a construction that we sketch below.

An idea for such a construction arose from trying to come up with a cs analog of the cyclic polytope. Recall that the cyclic polytope is the convex hull of n points on the d-th moment curve. It is a result of Gale [26] that for $d = 2k$, the convex hull of n points on the *trigonometric moment curve* $\mathcal{T}_k = (\cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos kt, \sin kt)$ has the same combinatorial type as the cyclic polytope. The curve $\mathcal{T}_k$ does not suit our purpose since it is not symmetric, but this is easily rectified if one considers only odd multiples of t.

Specifically, consider the *symmetric moment curve*

$$U_k(t) = (\cos t, \sin t, \cos 3t, \sin 3t, \ldots, \cos(2k-1)t, \sin(2k-1)t) \quad \text{for } t \in \mathbb{R},$$

or one of its variants such as

$$\Phi_k(t) = \left(\cos t, \sin t, \cos 3t, \sin 3t, \cos 3^2 t, \sin 3^2 t, \ldots, \cos 3^{k-1} t, \sin 3^{k-1} t\right) \quad \text{for } t \in \mathbb{R}.$$

Since $U_k(t + 2\pi) = U_k(t)$ and similarly for Φ_k, from this point on, we consider both curves as defined on the unit circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Furthermore, since t and $t + \pi$ form a pair of opposite points on S^1 (for all $t \in S^1$) and since $U_k(t + \pi) = -U_k(t)$, it follows that for each choice of $0 \leq t_1 < \cdots < t_n < \pi$, the convex hull of $\{U_k(t_i), U(t_i + \pi) \mid i \in [n]\}$ is a cs polytope; a similar statement holds for Φ_k. Polytopes with vertices on $U_2 = \Phi_2$ (among certain more general 4-polytopes) were introduced and analyzed by Smilansky [91]. Polytopes with vertices on U_k, for general k, were studied in [10,11,100]; they are known as *bicyclic polytopes*. Polytopes with vertices on Φ_k were defined in [9]. One property of bicyclic $2k$-polytopes worth mentioning is that they are *locally k-neighborly*: if the set $\{t_{i_1}, \ldots, t_{i_k}\}$ is contained in an arc of S^1 of length $\pi/2$, then $\{U_k(t_{i_1}), \ldots, U_k(t_{i_k})\}$ is the vertex set of a face. For some applications of bicyclic polytopes to topology, see [1].

To obtain the lower bound on $\text{fmax}(4, N; 1)$ promised in (10.1), take a cs subset X of S^1 consisting of four clusters of points, each of size $N/4$, with the j-th cluster lying on a small arc containing $j\pi/2$. The cs 4-polytope $\text{conv}(U_2(x) \mid x \in X)$ has at least $\frac{1}{2} \cdot$

$N\left(\frac{3}{4}N-1\right) \approx \frac{3}{4}\binom{N}{2}$ edges. Similarly, for $k > 2$, consider $A = 2\left(3^{k-1}-1\right)$ equally spaced points $p_1, \ldots, p_A$ on S^1. Replace each p_j with a cluster of N/A points lying on a small arc containing p_j in such a way that the resulting set V is cs. The convex hull of $\Phi_k(V)$ is then a cs $2k$-polytope that verifies the lower bound of (10.2); see [9] for details.

For $i > 2$, the gap between the current best upper and lower bounds on $\mathrm{fmax}(d, N; i-1)$ is so much worse than the gap for the number of edges that instead of stating the bounds here, we merely refer the reader to [9] for the lower bound and to [11] for the upper bound.

To conclude, we want to emphasize once again that in sharp contrast with the situation for cs spheres, at the moment we are nowhere near having a good handle on the upper bound type results for cs polytopes, and not for the lack of effort. We do not even know what is the largest number of edges that a cs 4-polytope with $N = 2n$ vertices can have. In fact, for $d \geq 6$ and $N \geq 2(d+2)$, we do not even know if in the class of cs d-polytopes with N vertices, there is a polytope that simultaneously maximizes all the f-numbers. What we seem to be lacking is new constructions (either explicit or probabilistic) of cs polytopes, and, in particular, constructions that may improve the lower bounds given in (10.1) and (10.2): in light of the main result of [70], we believe that $\mathrm{fmax}(d, N; 1)$ is closer to the right-hand side of (10.2) than to the left one.

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