Gorenstein rings through face rings of manifolds.

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Abstract

The face ring of a homology manifold (without boundary) modulo a generic system of parameters is studied. Its socle is computed and it is verified that a particular quotient of this ring is Gorenstein. This fact is used to prove that the algebraic g-conjecture for spheres implies all enumerative consequences of its far reaching generalization (due to Kalai) to manifolds. A special case of Kalai's conjecture is established for homology manifolds that have a codimension-two face whose link contains many vertices.

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1   Introduction

In 1980 Stanley proved the necessity of McMullen's conjectured description of f-vectors of boundaries of simplicial convex polytopes [14]. At about the same time, Billera and Lee demonstrated that McMullen's conditions were sufficient [1]. Since then, one of the central problems in the field of face numbers of simplicial complexes is the possible extension of McMullen's g-conjecture to other spheres. In its most optimistic form it states that, just as in the case of polytope boundaries, the face ring of a homology sphere modulo a generic system of parameters has a Lefschetz element. We call this the *algebraic g-conjecture*. In the

middle 90's Kalai suggested a far reaching generalization of this conjecture to all homology manifolds [11, Section 7]. It is a remarkable fact that all of the enumerative consequences of Kalai's conjecture are implied by the apparently weaker algebraic g-conjecture. This follows from our main result, Theorem 1.4, that a particular quotient of the face ring of a homology manifold is Gorenstein. We also verify a special case of Kalai's conjecture when the complex has a codimension-two face whose link contains many vertices.

The main objects we will consider are Buchsbaum complexes and more specifically homology manifolds. Historically, Buchsbaum complexes were defined algebraically. Here we adopt the following theorem of Schenzel [13] as our definition. Let $\mathbf{k}$ be an infinite field of an arbitrary characteristic, let Δ be a simplicial complex, and let $\tilde{H}_i(\Delta)$ be the i-th reduced simplicial homology of Δ with coefficients in $\mathbf{k}$. Recall that if τ is a face of Δ, then its link is

$$\mathrm{lk}\,\tau = \{\sigma \in \Delta : \sigma \cap \tau = \emptyset,\ \sigma \cup \tau \in \Delta\}.$$

Definition 1.1 *A $(d-1)$-dimensional simplicial complex Δ is called* **Buchsbaum** *(over $\mathbf{k}$) if it is pure and for every non empty face $\tau \in \Delta$, $\tilde{H}_i(\mathrm{lk}\,\tau) = 0$ for all $i < d - |\tau| - 1$.*

We say that Δ is a **homology manifold** over $\mathbf{k}$ (without boundary) if it is Buchsbaum and in addition $\tilde{H}_{d-|\tau|-1}(\mathrm{lk}\,\tau) \cong \mathbf{k}$ for all $\emptyset \neq \tau \in \Delta$. A $\mathbf{k}$-*homology sphere* is a complex Δ such that for all $\tau \in \Delta$, including $\tau = \emptyset$,

$$\tilde{H}_i(\mathrm{lk}\,\tau) = \begin{cases} 0 & \text{if } i < d - |\tau| - 1, \\ \mathbf{k} & \text{if } i = d - |\tau| - 1. \end{cases}$$

In particular, a $\mathbf{k}$-homology sphere is a $\mathbf{k}$-homology manifold, and a triangulation of a topological sphere (topological manifold, resp.) is a $\mathbf{k}$-homology sphere ($\mathbf{k}$-homology manifold, resp.) for any field $\mathbf{k}$.

If Δ is a simplicial complex on $[n] := \{1, 2, \ldots, n\}$, then its *face ring* (or the *Stanley-Reisner ring*) is

$$\mathbf{k}[\Delta] := \mathbf{k}[x_1, \ldots, x_n]/I_\Delta, \quad \text{where } I_\Delta = (x_{i_1} x_{i_2} \cdots x_{i_k} : \{i_1 < i_2 < \cdots < i_k\} \notin \Delta).$$

Various combinatorial and topological invariants of Δ are encoded in the algebraic invariants of $\mathbf{k}[\Delta]$ and vice versa. For instance, if Δ is a $(d-1)$-dimensional complex, then the Krull dimension of $\mathbf{k}[\Delta]$, $\dim \mathbf{k}[\Delta]$, is equal to d. In this case, a set of d linear forms $\theta_1, \ldots, \theta_d \in \mathbf{k}[\Delta]$ is called a *linear system of parameters* (abbreviated, l.s.o.p.) if

$$\mathbf{k}(\Delta, \Theta) := \mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_d)$$

has Krull dimension zero (equivalently, $\mathbf{k}(\Delta, \Theta)$ is finite-dimensional as a $\mathbf{k}$-vector space). Assuming $\mathbf{k}$ is infinite, an l.s.o.p. always exists: a generic choice of $\theta_1, \ldots, \theta_d$ does the job. Although the algebraic structure of $\mathbf{k}(\Delta, \Theta)$ generally depends on $\theta_1, \ldots, \theta_d$, in the following we somewhat abuse notation and write $\mathbf{k}(\Delta)$ instead of $\mathbf{k}(\Delta, \Theta)$. Unless it is specifically stated that Θ is generic, our results apply to an arbitrary l.s.o.p.

One invariant that measures how far Δ is from being a homology sphere is the socle of $\mathbf{k}(\Delta)$, $\operatorname{Soc} \mathbf{k}(\Delta)$, where for a $\mathbf{k}[x_1, \ldots, x_n]$- or $\mathbf{k}[\Delta]$-module M,

$$\operatorname{Soc} M := \{y \in M \: : \: x_i \cdot y = 0 \text{ for all } i = 1, \ldots, n\}.$$

When Δ is a homology sphere, $\operatorname{Soc} \mathbf{k}(\Delta)$ is a 1-dimensional $\mathbf{k}$-space. Since $\mathbf{k}[\Delta]$ is a graded $\mathbf{k}$-algebra for any Δ, the ring $\mathbf{k}(\Delta)$ and its ideal $\operatorname{Soc} \mathbf{k}(\Delta)$ are graded as well. We denote by $\mathbf{k}(\Delta)_i$ and $(\operatorname{Soc} \mathbf{k}(\Delta))_i$ their i-th homogeneous components. It is well known (for instance, see [15, Lemma III.2.4(b)]) that for any $(d-1)$-dimensional Δ, $\mathbf{k}(\Delta)_i$, and hence also $(\operatorname{Soc} \mathbf{k}(\Delta))_i$, vanish for all $i > d$. If Δ is a Buchsbaum complex, then for $i \leq d$, $(\operatorname{Soc} (\mathbf{k}(\Delta)))_i$ can be expressed in terms of the local cohomology modules of $\mathbf{k}[\Delta]$ with respect to the irrelevant ideal, $H^j(\mathbf{k}[\Delta])$, as follows, see [12, Theorem 2.2].

Theorem 1.2 *Let Δ be a $(d-1)$-dimensional simplicial complex. If Δ is Buchsbaum, then for all $0 \leq i \leq d$,*

$$(\operatorname{Soc} \mathbf{k}(\Delta))_i \cong \left(\bigoplus_{j=0}^{d-1} \binom{d}{j} H^j(\mathbf{k}[\Delta])_{i-j} \right) \bigoplus \mathcal{S}_{i-d},$$

where $\mathcal{S}$ is a graded submodule of $\operatorname{Soc} H^d(\mathbf{k}[\Delta])$ and rM denotes the direct sum of r copies of M.

While Theorem 1.2 identifies a big chunk of the socle of $\mathbf{k}(\Delta)$, its other part, $\mathcal{S}$, remains a mystery. Here we solve this mystery in the special case of a connected *orientable* $\mathbf{k}$-homology manifold without boundary, thus verifying Conjecture 7.2 of [12]. A connected $\mathbf{k}$-homology manifold Δ without boundary is called orientable if $\tilde{H}_{d-1}(\Delta) \cong \mathbf{k}$.

Theorem 1.3 *Let Δ be a $(d-1)$-dimensional connected orientable $\mathbf{k}$-homology manifold without boundary. Then $\dim_{\mathbf{k}} \mathcal{S} = \dim_{\mathbf{k}} \mathcal{S}_0 = 1$. In particular, $\dim_{\mathbf{k}} \operatorname{Soc} \mathbf{k}(\Delta)_i = \binom{d}{i} \beta_{i-1}$, where $\beta_{i-1} := \dim_{\mathbf{k}} \tilde{H}_{i-1}(\Delta)$ is the i-th reduced Betti number of Δ.*

A graded $\mathbf{k}$-algebra of Krull dimension zero is called *Gorenstein* if its socle is a 1-dimensional $\mathbf{k}$-space (see [15, p. 50] for many other equivalent definitions). Let

$$I := \bigoplus_{i=0}^{d-1} \operatorname{Soc} \mathbf{k}(\Delta)_i \quad \text{and} \quad \overline{\mathbf{k}(\Delta)} := \mathbf{k}(\Delta)/I.$$

(Note that the top-dimensional component of the socle is not a part of I). If Δ is a homology sphere, then $I = 0$ and $\overline{\mathbf{k}(\Delta)} = \mathbf{k}(\Delta)$ is a Gorenstein ring [15, Theorem II.5.1]. What if Δ is a homology manifold other than a sphere? How far is $\mathbf{k}(\Delta)$ from being Gorenstein in this case? The answer (that was conjectured in [12, Conjecture 7.3]) turns out to be surprisingly simple:

Theorem 1.4 *Let Δ be a $(d-1)$-dimensional connected simplicial complex. If Δ is an orientable $\mathbf{k}$-homology manifold without boundary, then $\overline{\mathbf{k}(\Delta)}$ is Gorenstein.*

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In [13], Schenzel computed the Hilbert function of $\mathbf{k}(\Delta)$ for a Buchsbaum complex Δ in terms of its face and Betti numbers. It follows from Theorem 1.3, Schenzel's formula, and the Dehn-Sommerville relations [7] that for a connected orientable homology manifold Δ, the Hilbert function of $\overline{\mathbf{k}(\Delta)}$ is symmetric, that is,

$$\dim_{\mathbf{k}} \overline{\mathbf{k}(\Delta)}_i = \dim_{\mathbf{k}} \overline{\mathbf{k}(\Delta)}_{d-i} \quad \text{for all } 0 \le i \le d. \tag{1}$$

As the Hilbert function of a Gorenstein ring of Krull dimension zero is always symmetric [15, p. 50], Theorem 1.4 gives an alternative algebraic proof of (1).

As noted above, when Δ is a $(d-1)$-dimensional homology sphere, $\mathbf{k}(\Delta)$ is a Gorenstein ring which implies that $\dim_{\mathbf{k}} \mathbf{k}(\Delta)_i = \dim_{\mathbf{k}} \mathbf{k}(\Delta)_{d-i}$. We say that Δ satisfies the *algebraic g-conjecture* if, for sufficiently generic linear forms ω and $\Theta = \{\theta_1, \ldots, \theta_d\}$, and $i \le d/2$, multiplication

$$\cdot \omega^{d-2i} : \mathbf{k}(\Delta)_i \longrightarrow \mathbf{k}(\Delta)_{d-i}$$

is an isomorphism. At present the only large classes of spheres for which the algebraic g-conjecture is known to hold are polytopal spheres [14] and edge decomposable spheres [9], [10].

Kalai's far-reaching generalization of the algebraic g-conjecture for homology spheres [11] posits that if Δ is an orientable homology manifold and ω, Θ are sufficiently generic, then

$$\cdot \omega^{d-2i} : \overline{\mathbf{k}(\Delta)}_i \longrightarrow \overline{\mathbf{k}(\Delta)}_{d-i}$$

is still an isomorphism for all $i \le d/2$. Let $h_i'' = \dim_{\mathbf{k}} \overline{\mathbf{k}(\Delta)}_i$. Given a system of parameters and ω which satisfy Kalai's conjecture it is immediate that multiplication $\cdot \omega : \overline{\mathbf{k}(\Delta)}_i \to \overline{\mathbf{k}(\Delta)}_{i+1}$ is an injection for $i < d/2$. So, $h_0'' \le \cdots \le h_{\lfloor d/2 \rfloor}''$, and examination of $\overline{\mathbf{k}(\Delta)}/(\omega)$ shows that the nonnegative integer vector $(h_0'', h_1'' - h_0'', \ldots, h_{\lfloor d/2 \rfloor}'' - h_{\lfloor d/2 \rfloor - 1}'')$ is an M-vector, i.e. satisfies Macaulay's nonlinear arithmetic conditions (see [15, p. 56]) for the Hilbert series of a homogeneous quotient of a polynomial ring. Applying the same reasoning to $I_{[i]} := \oplus_{j=0}^i I_j$, which is also an ideal, and $\mathbf{k}(\Delta, i) := \mathbf{k}(\Delta)/I_{[i]}$, instead of I and $\overline{\mathbf{k}(\Delta)} = \mathbf{k}(\Delta)/I$, the second conclusion can be strengthened to $(h_0'', h_1'' - h_0'', \ldots, h_i'' - h_{i-1}'', h_{i+1}'' - h_i'' + \binom{d}{i+1}\beta_i)$ is an M-vector for every $i < \lfloor d/2 \rfloor$. We call these arithmetic restrictions on the h''-vector the *manifold g-conjecture*. In fact, as the following theorem shows (see Theorem 3.2 for a somewhat stronger version), these two conclusions follow from the algebraic g-conjecture. (Recall that the links of vertices of a homology manifold are homology spheres.)

Theorem 1.5 *Let Δ be a $(d-1)$-dimensional connected orientable homology manifold. If the links of all vertices of Δ satisfy the algebraic g-conjecture, then for generically chosen ω and Θ, $\cdot \omega : \overline{\mathbf{k}(\Delta)}_i \longrightarrow \overline{\mathbf{k}(\Delta)}_{i+1}$ is an injection for all $i < \lfloor d/2 \rfloor$ and is a surjection for all $i > \lceil d/2 \rceil$.*

For Kalai's conjecture we prove this special case.

Theorem 1.6 *Let Δ be a $(d-1)$-dimensional orientable $\mathbf{k}$-homology manifold with $d \ge 3$. If Δ has a $(d-3)$-dimensional face τ whose link contains all of the vertices of Δ that are not in τ, then for generic choices of ω and Θ, $\cdot \omega^{d-2} : \overline{\mathbf{k}(\Delta)}_1 \longrightarrow \overline{\mathbf{k}(\Delta)}_{d-1}$ is an isomorphism.*

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The condition that the link of τ contains all of the vertices of Δ that are not in τ is equivalent to saying that every vertex of Δ is in the star of τ, $\operatorname{st}\tau := \{\sigma \in \Delta : \sigma \cup \tau \in \Delta\}$. This condition is not as restrictive on the homeomorphism type of Δ as one might think: the results of [16, Section 5] imply that every connected homology manifold M without boundary that has a triangulation, always has a triangulation Δ satisfying this condition.

The outline of the paper is as follows. We verify Theorem 1.3 in Section 2. The main ingredient in the proof is Gräbe's explicit description of $H^d(\mathbf{k}[\Delta])$ as a $\mathbf{k}[\Delta]$-module in terms of the simplicial (co)homology of the links of faces of Δ and maps between them [4]. Theorem 1.4 is proved in Section 3 and is used to explore the relationship between the algebraic g-conjecture and Kalai's conjecture (such as Theorem 1.5). Section 4 is devoted to the proof of Theorem 1.6. The proofs of Theorems 1.4 and 1.6 rely heavily on a result from [16] relating $\mathbf{k}(\operatorname{lk} v)$ (for a vertex v of Δ) to the principal ideal $(x_v) \subset \mathbf{k}(\Delta)$.

2   Socles of homology manifolds

The goal of this section is to verify Theorem 1.3. To do so we analyze $\operatorname{Soc} H^d(\mathbf{k}[\Delta])$ and prove the following.

Theorem 2.1 *If Δ is a connected orientable $(d-1)$-dimensional $\mathbf{k}$-homology manifold without boundary, then $\operatorname{Soc} H^d(\mathbf{k}[\Delta])_i = 0$ for all $i \neq 0$.*

The proof relies on results from [4]. We denote by $|\Delta|$ the geometric realization of Δ. For a face $\tau \in \Delta$, let $\operatorname{cost}\tau := \{\sigma \in \Delta : \sigma \not\supseteq \tau\}$ be the contrastar of τ, let $H^i(\Delta, \operatorname{cost}\tau)$ be the simplicial i-th cohomology of a pair (with coefficients in $\mathbf{k}$), and for $\tau \subset \sigma \in \Delta$, let ι^* be the map $H^i(\Delta, \operatorname{cost}\sigma) \to H^i(\Delta, \operatorname{cost}\tau)$ induced by inclusion $\iota : \operatorname{cost}\tau \to \operatorname{cost}\sigma$. Also, if $\emptyset \neq \tau \in \Delta$, let $\hat{\tau}$ be the barycenter of τ. Finally, for a vector $U = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, let $s(U) := \{l : u_l \neq 0\} \subseteq [n]$ be the support of U, let $\{e_l\}_{l=1}^n$ be the standard basis for $\mathbb{Z}^n$, and let $\mathbb{N}$ denote the set of nonnegative integers.

We consider the $\mathbb{Z}^n$-grading of $\mathbf{k}[x_1, \ldots, x_n]$ obtained by declaring x_l to be of degree e_l. This grading refines the usual Z-grading and induces a $\mathbb{Z}^n$-grading of $\mathbf{k}[\Delta]$ and its local cohomology modules. Thus, $H^i(\mathbf{k}[\Delta])_j = \bigoplus H^i(\mathbf{k}[\Delta])_U$ where the sum is over all $U = (u_1, \ldots, u_n) \in \mathbb{Z}^n$ with $\sum u_l = j$, and multiplication by x_l is a linear map from $H^i(\mathbf{k}[\Delta])_U$ to $H^i(\mathbf{k}[\Delta])_{U+e_l}$ for all $U \in \mathbb{Z}^n$.

Theorem 2.2 [Gräbe] *The following is an isomorphism of $\mathbb{Z}^n$-graded $\mathbf{k}[\Delta]$-modules*

$$H^i(\mathbf{k}[\Delta]) \cong \bigoplus_{\substack{-U \,\in\, \mathbb{N}^n \\ s(U) \,\in\, \Delta}} \mathcal{M}_U^i, \quad where \quad \mathcal{M}_U^i = H^{i-1}(\Delta, \operatorname{cost} s(U)) \tag{2}$$

and the $\mathbf{k}[\Delta]$-structure on the U-th component of the right-hand side is given by

$$\cdot x_l = \begin{cases} 0\text{-}map, & if\ l \notin s(U) \\ identity\ map, & if\ l \in s(U)\ and\ l \in s(U + e_l) \\ \iota^* : H^{i-1}(\Delta, \operatorname{cost} s(U)) \to H^{i-1}(\Delta, \operatorname{cost} s(U + e_l)), & otherwise. \end{cases}$$

We remark that the isomorphism of (2) on the level of vector spaces (rather than $\mathbf{k}[\Delta]$-modules) is due to Hochster, see [15, Section II.4], and that $H^i(\mathbf{k}[\Delta])_0 \cong \mathcal{M}^i_{(0,\dots,0)} = H^{i-1}(\Delta, \emptyset)$.

Proof of Theorem 2.1: In view of Theorem 2.2, if $-U = (-u_1, \dots, -u_n) \in \mathbb{N}^n$ is such that $-u_l \geq 2$ for some $l \in [n]$, then multiplication $\cdot x_l : H^d(\mathbf{k}[\Delta])_U \to H^d(\mathbf{k}[\Delta])_{U+e_l}$ is an isomorphism, and hence no non-zero element of $H^d(\mathbf{k}[\Delta])_U$ is in the socle. Thus (using Theorem 2.2 once again), to prove that $\mathrm{Soc}\, H^d(\mathbf{k}[\Delta])_i = 0$ for all $i \neq 0$, it is enough to show that for every $\emptyset \neq \tau \in \Delta$ and $l \in \tau$, the map $\iota^* : H^{d-1}(\Delta, \mathrm{cost}\, \tau) \to H^{d-1}(\Delta, \mathrm{cost}\, \sigma)$, where $\sigma = \tau - \{l\}$, is an isomorphism. Assume first that $\sigma \neq \emptyset$. Consider the following diagram.

$$
\begin{array}{ccccc}
H^{d-1}(|\Delta|) & \xrightarrow{(j^\star)^{-1}} & H^{d-1}(|\Delta|, |\Delta| - \hat{\sigma}) & \xrightarrow{f^\star} & H^{d-1}(\Delta, \mathrm{cost}\, \sigma) \\
\| & & & & \iota^\star \uparrow \\
H^{d-1}(|\Delta|) & \xrightarrow{(j^\star)^{-1}} & H^{d-1}(|\Delta|, |\Delta| - \hat{\tau}) & \xrightarrow{f^\star} & H^{d-1}(\Delta, \mathrm{cost}\, \tau)
\end{array}
$$

The two $f^\star$ maps are induced by inclusion and are isomorphisms by the usual deformation retractions. $j^\star$ is also induced by inclusion. Since Δ is connected and orientable, all of the spaces are one-dimensional and $j^\star$ is an isomorphism, so that $(j^\star)^{-1}$ is well-defined and is an isomomorphism as well. Hence the compositions $f^\star \circ (j^\star)^{-1}$ are isomorphisms. The naturality of $j^\star$ implies that the diagram is commutative. It follows that $\iota^\star$ is an isomorphism. If $\sigma = \emptyset$, replace $H^{d-1}(|\Delta|, |\Delta| - \hat{\sigma})$ with $H^{d-1}(|\Delta|)$ in the above diagram. The same reasoning applies. $\square$

We are now in a position to complete the proof of Theorem 1.3.

Proof of Theorem 1.3: Since $\mathcal{S}_i$ is a subspace of $\mathrm{Soc}\, H^d(\mathbf{k}[\Delta])_i$ (Theorem 1.2), and since the later space is the zero-space whenever $i \neq 0$ (see Theorem 2.1), it follows that $\mathcal{S}_i = 0$ for all $i \neq 0$. For $i = 0$, we have

$$
\dim_{\mathbf{k}} \mathcal{S}_0 = \dim_{\mathbf{k}} \mathrm{Soc}\, \mathbf{k}(\Delta)_d = \dim_{\mathbf{k}} \mathbf{k}(\Delta)_d = \beta_{d-1}(\Delta) = 1.
$$

Here, the first step is by Theorem 1.2, the second step is a consequence of $\mathbf{k}(\Delta)_d$ being the last non vanishing homogeneous component of $\mathbf{k}(\Delta)$, and the third step is by Schenzel's formula [13] (see also [15, Theorem II.8.2]). The "In particular"-part then follows from Theorem 1.2, isomorphism (2), and the standard fact that $H^{i-1}(\Delta, \mathrm{cost}\, \tau) \cong \tilde{H}_{i-|\tau|-1}(\mathrm{lk}\, \tau)$, see e.g. [4, Lemma 1.3]. $\square$

3   Gorenstein property

In this section we verify Theorem 1.4 and use it to discuss connections between various g-conjectures. To prove that $\overline{\mathbf{k}(\Delta)} = \mathbf{k}(\Delta)/I$ is Gorenstein, where Δ is a $(d-1)$-dimensional connected orientable homology manifold and $I = \oplus_{i=0}^{d-1} \mathrm{Soc}\, \mathbf{k}(\Delta)_i$, we have to check that the

operation of moding out by I does not introduce new socle elements. This turns out to be a simple application of [16, Proposition 4.24], which we review now.

Let $\Theta = \{\theta_1, \ldots, \theta_d\}$ be an l.s.o.p. for $\mathbf{k}[\Delta]$ and let v be a vertex of Δ. Fix a facet $\tau = \{v = v_1, v_2, \ldots, v_d\}$ that contains v. By using Gaussian elimination on the $d \times n$-matrix whose (i, j)-th entry is the coefficient of x_j in θ_i we can assume without loss of generality that $\theta_i = x_{v_i} + \sum_{j \notin \tau} \theta_{i,j} x_j$. Denote by θ_i' the linear form obtained from θ_i by removing all summands involving x_j for $\{j\} \notin \mathrm{lk}\, v$. Then $\Theta' := \{\theta_2', \ldots, \theta_d'\}$ can be considered as a subset of $\mathbf{k}[\mathrm{lk}\, v]_1$. Moreover, it is easy to check, say, using [15, Lemma III.2.4(a)], that Θ' forms an l.s.o.p. for $\mathbf{k}[\mathrm{lk}\, v]$. The ring $\mathbf{k}(\mathrm{lk}\, v) := \mathbf{k}[\mathrm{lk}\, v]/(\Theta')$ has a natural $\mathbf{k}[x_1, \ldots, \hat{x}_v, \ldots, x_n]$-module structure (if $j \neq v$ is not in the link of v, then multiplication by x_j is the zero map), and defining

$$x_v \cdot y := -\theta_1' \cdot y \quad \text{for } y \in \mathbf{k}(\mathrm{lk}\, v)$$

extends it to a $\mathbf{k}[x_1, \ldots, x_n]$-module structure. Proposition 4.24 of [16] asserts the following.

Theorem 3.1 *Let Δ be an orientable homology manifold. The map*

$$\phi : \mathbf{k}[\mathrm{lk}\, v]/(\Theta') \to (x_v)\, (\mathbf{k}[\Delta]/(\Theta)) \quad \text{given by } z \mapsto x_v \cdot z,$$

is well-defined and is an isomorphism (of degree 1) of $\mathbf{k}[x_1, \ldots, x_n]$-modules. Its inverse, $x_v \cdot z \mapsto \overline{z}$, is given by replacing each occurrence of x_v in z with $-\theta_1'$ and setting all x_j for $j \neq v$ not in the link of v to zero.

Proof of Theorem 1.4: To prove the theorem it is enough to show that the socle of $\overline{\mathbf{k}(\Delta)} = \mathbf{k}(\Delta)/I$, where $I = \bigoplus_{j=0}^{d-1} \mathrm{Soc}\, \mathbf{k}(\Delta)_j$, vanishes in all degrees $j \neq d$. This is clear for $j = d - 1$. For $j \leq d - 2$, consider any element $y \in \mathbf{k}(\Delta)_j$ such that $x_v \cdot y \in \mathrm{Soc}\, \mathbf{k}(\Delta)$ for all $v \in [n]$. We have to check that $y \in \mathrm{Soc}\, \mathbf{k}(\Delta)$. And indeed, the isomorphism of Theorem 3.1 implies that $\overline{y} := \phi^{-1}(x_v \cdot y) \in \mathbf{k}(\mathrm{lk}\, v)_j$ is in the socle of $\mathbf{k}(\mathrm{lk}\, v)$. Since $\mathrm{lk}\, v$ is a $(d - 2)$-dimensional homology sphere, $\mathbf{k}(\mathrm{lk}\, v)$ is Gorenstein, and hence its socle vanishes in all degrees except $(d - 1)$-st one. Therefore, $\overline{y} = 0$, and hence $x_v \cdot y = \phi(\overline{y}) = 0$ in $\mathbf{k}(\Delta)$. Since this happens for all $v \in [n]$, it follows that $y \in \mathrm{Soc}\, \mathbf{k}(\Delta)$. $\square$

We now turn to the connection between the various g-conjectures. Evidently, Kalai's conjecture implies the algebraic g-conjecture and the manifold g-conjecture. Since homology spheres are homology manifolds, an immediate consequence of the manifold g-conjecture is that all homology spheres satisfy the enumerative restrictions of McMullen's g-conjecture for polytopes. The surprise is that the algebraic g-conjecture implies the manifold g-conjecture. In fact, as the following strengthening of Theorem 1.5 shows, only the algebraic g-conjecture in the middle dimension is needed.

Theorem 3.2 *Let Δ be a $(d - 1)$-dimensional connected orientable homology manifold. If for at least $(n - d)$ of the vertices v of Δ and generically chosen ω and Θ' in $\mathbf{k}[\mathrm{lk}\, v]_1$, the map $\cdot\omega : \mathbf{k}(\mathrm{lk}\, v)_{\lfloor (d-1)/2 \rfloor} \to \mathbf{k}(\mathrm{lk}\, v)_{\lfloor (d-1)/2 \rfloor + 1}$ is surjective, then Δ satisfies the manifold g-conjecture.*

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Proof: The condition on the links implies by [16, Theorem 4.26] that for generic choices of ω and Θ in $\mathbf{k}[\Delta]_1$, the map $\cdot\omega : \mathbf{k}(\Delta)_{\lceil d/2\rceil} \to \mathbf{k}(\Delta)_{\lceil d/2\rceil+1}$ is surjective. Hence the map $\cdot\omega : \overline{\mathbf{k}(\Delta)}_{\lceil d/2\rceil} \to \overline{\mathbf{k}(\Delta)}_{\lceil d/2\rceil+1}$ is surjective. Thus, $\overline{\mathbf{k}(\Delta)}/(\omega)_i$ vanishes for $i = \lceil d/2\rceil + 1$, and hence also for all $i > \lceil d/2\rceil + 1$, and we infer that $\cdot\omega : \overline{\mathbf{k}(\Delta)}_i \to \overline{\mathbf{k}(\Delta)}_{i+1}$ is surjective for $i \geq \lceil d/2\rceil$. This in turn yields that the dual map $\cdot\omega : \mathrm{Hom}_{\mathbf{k}}(\overline{\mathbf{k}(\Delta)}_{i+1}, \mathbf{k}) \to \mathrm{Hom}_{\mathbf{k}}(\overline{\mathbf{k}(\Delta)}_i, \mathbf{k})$ is injective for all $i \geq \lceil d/2\rceil$. Since by Theorem 1.4 $\overline{\mathbf{k}(\Delta)}$ is Gorenstein, $\mathrm{Hom}_{\mathbf{k}}(\overline{\mathbf{k}(\Delta)}_i, \mathbf{k})$ is naturally isomorphic to $\overline{\mathbf{k}(\Delta)}_{d-i}$ (see Theorems I.12.5 and I.12.10 in [15]). Therefore, $\cdot\omega : \overline{\mathbf{k}(\Delta)}_j \to \overline{\mathbf{k}(\Delta)}_{j+1}$ is injective for $j < \lfloor d/2\rfloor$. $\qquad\square$

Theorem 3.2 combined with Stanley's g-theorem for polytopes [14], implies that every $(d-1)$-dimensional connected orientable $\mathbb{Q}$-homology manifold all of whose vertex links are polytopal spheres satisfies the manifold g-conjecture.

4   A special case of the g-theorem

In this section we prove Theorem 1.6. As in the proof of Theorem 1.4 we will rely on Theorem 3.1 and the notation introduced there. Since the set of all (ω, Θ) for which $\cdot\omega^{d-2} : \overline{\mathbf{k}(\Delta)}_1 \to \overline{\mathbf{k}(\Delta)}_{d-1}$ is an isomorphism is a Zariski open set (see [16, Section 4]), it is enough to find one ω that does the job. Surprisingly, the ω we find is "very non-generic": as we will see, $w = x_v$ where $v \in \tau$ does the job.

Proof of Theorem 1.6:   We use induction on d starting with $d = 3$. In this case, the face τ is simply a vertex, say v. Let $\sigma = \{v = v_1, v_2, v_3\}$ be a facet containing v. Let Θ be a generic l.s.o.p. for $\mathbf{k}[\Delta]$, and as in Section 3 assume that $\theta_i = x_{v_i} + \sum_{j\notin\sigma} \theta_{i,j}x_j$. Since $\dim_{\mathbf{k}} \overline{\mathbf{k}(\Delta)}_1 = \dim_{\mathbf{k}} \overline{\mathbf{k}(\Delta)}_2$ (see Eq. (1)) and since $\dim_{\mathbf{k}} \mathrm{Soc}\,\mathbf{k}(\Delta)_1 = 0$ (e.g. by Theorem 1.3), we will be done if we check that the map $\cdot x_v : \mathbf{k}(\Delta)_1 \to \overline{\mathbf{k}(\Delta)}_2$ is injective. And indeed, if $x_v \cdot z \in \mathrm{Soc}\,\mathbf{k}(\Delta)_2$ for some $z \in \mathbf{k}(\Delta)_1$, then by the isomorphism of Theorem 3.1, $\overline{z} = \phi^{-1}(x_v \cdot z)$ is in $\mathrm{Soc}\,\mathbf{k}(\mathrm{lk}\,v)_1$. But the later space is the zero space, so $\overline{z} = 0$ in $\mathbf{k}(\mathrm{lk}\,v)_1$, that is, $\overline{z} \in \mathrm{Span}\,(\theta_2', \theta_3')$. Since $\mathrm{lk}\,v$ contains all the vertices of Δ except v, it follows that $\theta_2' = \theta_2$, $\theta_3' = \theta_3$, and $z - \overline{z}$ is a multiple of θ_1, and hence that $z = 0$ in $\mathbf{k}(\Delta)$.

The proof of the induction step goes along the same lines: let $\tau = \{v_1, \ldots, v_{d-2}\}$, let $\sigma = \tau \cup \{v_{d-1}, v_d\}$ be any facet containing τ, and let Θ be a generic ls.o.p. for $\mathbf{k}[\Delta]$. We show that for $v \in \tau$, say $v = v_1$, the map $\cdot x_v^{d-2} : \mathbf{k}(\Delta)_1 \to \overline{\mathbf{k}(\Delta)}_{d-1}$ is an injection. If $x_v^{d-2} \cdot z \in \mathrm{Soc}\,\mathbf{k}(\Delta)_{d-1}$ for some $z \in \mathbf{k}(\Delta)_1$, then by Theorem 3.1,

$$x_v^{d-3} \cdot \overline{z} = (-\theta_1')^{d-3} \cdot \overline{z} \in \mathrm{Soc}\,\mathbf{k}(\mathrm{lk}\,v)_{d-2}.$$

However, $\mathrm{lk}\,v$ is a $(d-2)$-dimensional homology sphere satisfying the same assumptions as Δ: the star of the face $\tau - \{v\}$ contains all the vertices of $\mathrm{lk}\,v$. Hence by inductive hypothesis applied to $\mathrm{lk}\,v$ with $\omega = -\theta_1'$ and $\Theta' = (\theta_2', \ldots, \theta_d')$, the map $\cdot(-\theta_1')^{d-3} : \mathbf{k}(\mathrm{lk}\,v)_1 \to \overline{\mathbf{k}(\mathrm{lk}\,v)}_{d-2}$ is an injection. Thus $\overline{z} = 0$ in $\mathbf{k}(\mathrm{lk}\,v)_1$, and so $z = 0$ in $\mathbf{k}(\Delta)_1$. $\qquad\square$

Remark   Barnette's lower bound theorem [5] asserts that for every d and n, the f-vector of any $(d-1)$-dimensional connected triangulated manifold with n vertices is minimized

componentwise by the f-vector of (the boundary of) a stacked d-polytope with n vertices, $S(d, n)$. There is a conjectural algebraic strengthening of this result based on Kalai's notion of *algebraic shifting*, an operation that was introduced in the mid-eighties as a tool for studying f-numbers of simplicial complexes (see e.g. [2] and a survey paper [6]). This conjecture posits that the algebraic shifting of every connected orientable $(d-1)$-dimensional manifold on n vertices contains as a subset $\Delta(S(d, n))$ — the algebraic shifting of $S(d, n)$. The complex $\Delta(S(d, n))$ was recently computed by Nevo [10, Example 2.18] and independently by Murai [8]. Using their result, together with standard facts on rev-lex generic initial ideals [3, Prop. 15.12] and an algebraic definition of Buchsbaum complexes [13, Def. and Thm. 3.1(ii)], it is easy to show that in the case of symmetric algebraic shifting, a $(d-1)$-dimensional connected orientable $\mathbb{Q}$-homology manifold Δ satisfies the above conjecture if and only if for generically chosen $\theta_1, \ldots, \theta_d$ and ω, the map $\cdot\omega^{d-2} : \overline{\mathbf{k}(\Delta)}_1 \to \overline{\mathbf{k}(\Delta)}_{d-1}$ is a $\mathbb{Q}$-isomorphism. In particular, it follows that this conjecture holds for every $(d-1)$-dimensional orientable $\mathbb{Q}$-homology manifold satisfying assumptions of Theorem 1.6.

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