Face numbers of centrally symmetric polytopes

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joint work with

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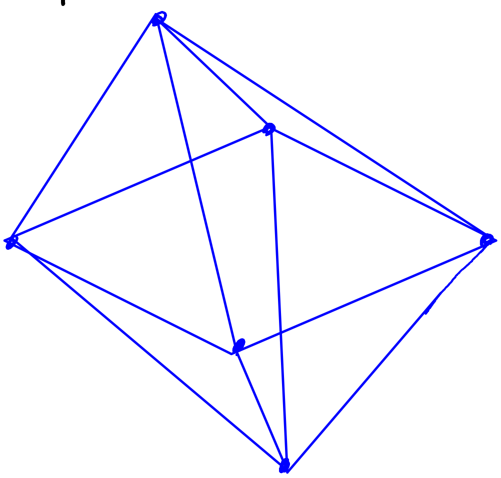
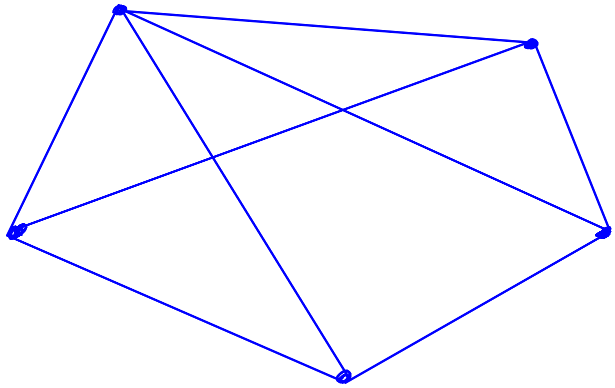
Basic Definitions

- Polytope $\qquad P = \mathrm{conv}\{v_1, \dots, v_n\}$



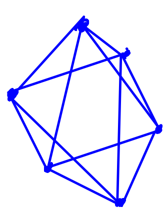


- centrally symmetric (cs) $\qquad P = -P$

$$(x \in P \Longrightarrow -x \in P)$$

- $f_i(P) = \#\, i\text{-dim faces of } P$

$$f(P) = (f_0, f_1, \dots, f_{d-1})$$

e.g. $f\Big($  $\Big) = (6, 12, 8)$

Problem: What is the maximal possible number, $f_{max}(d,n;i)$, of i-faces that a d-dim cs polytope on n vertices can have?

Motivation: If drop cs assumption — the answer is well-known:

Upper Bound Theorem (McMullen, 1970)

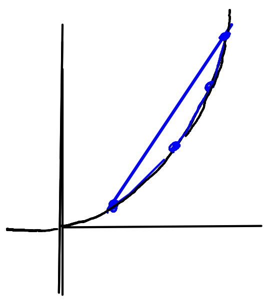
Among all d-polytopes with n vertices the cyclic polytope, $C_d(n)$, maximizes f_i $\forall i$.

Cyclic Polytope (Carathéodory, Gale, Motzkin, ...)

• **Def** $C_d(n) := C_d(t_1, \dots, t_n)$

$$= \operatorname{conv}\{M(t_j) : j=1,\dots,n\}$$

where $M(t) = (t, t^2, \dots, t^d)$ – moment curve



and $t_1 < t_2 < \dots < t_n$

• For $d = 2k$ can also define

$$C_d(n) = \{TM(t_j) : j = 1, \dots, n\}$$ where

$$TM(t) = (\cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos kt, \sin kt)$$

– trigonometric moment curve

Properties of $C_d(n)$

- simplicial d-polytope with n vertices

- combinatorial type is independent of $t_1, t_2 \dots t_n$

- $C_d(n)$ is $\lfloor d/2 \rfloor$-neighborly :

  $\forall K \leq \lfloor d/2 \rfloor$ every K vertices of $C_d(n)$ form the vertex set of a face of $C_d(n)$

$$\left(\Longrightarrow f_{K-1} = \binom{n}{K} \quad \forall\ K \leq \lfloor d/2 \rfloor \right)$$

cs-neighborliness

No two antipodal vertices of a cs polytope form an edge.

Def A cs polytope P is K-neighborly if every set $\mathcal{U} \subset \mathrm{Vert}(P)$, $|\mathcal{U}| = K$ such that

$$v \in \mathcal{U} \Longrightarrow -v \notin \mathcal{U}$$

is the vertex set of a face of P.

Q: * How neighborly can a cs polytope be?

* Is there a cs version of $C_d(n)$?

Known results

$$\kappa(d,n) := \max\left\{ K : \begin{array}{c} \exists \text{ a } K\text{-neighborly} \\ \text{cs } d\text{-polytope with} \\ 2(n+d) \text{ vertices} \end{array} \right\}$$

* Thm (Grünbaum, 1967) $\kappa(4,2)=1$

* Thm (McMullen-Shephard, 1968)

$$\kappa(d,0)=d$$

$$\kappa(d,1)=\lfloor d/2 \rfloor$$

$$\kappa(d,n) \leq \kappa(d,2) = \lfloor (d+1)/3 \rfloor \quad \forall n \geq 2$$

* Conjecture (McMullen-Shephard)

$$\kappa(d,n) \leq \left\lfloor \frac{d+n-1}{n+1} \right\rfloor \quad \forall n \geq 3$$

In particular, $\kappa(d,d-2)=1$.

Extreme cases

* Halsey, 1972:

McMullen-Shephard's conjecture fails for $d \gg n$

* Thm (Schneider, 1975)

$$\liminf_{d \to \infty} \frac{K(d,n)}{d+n} \geq 0.2390$$

However,

* Thm (Burton, 1991)

$$K(d,n) = 1 \quad \text{if} \quad n \gg d \quad \left(n \sim (d/2)^{d/2}\right)$$

[a cs d-polytope with sufficiently many vertices can NOT be even 2-neighborly]

New results

Thm 1 (Linial, N 2006) Let $m := n+d$

$$\frac{C_1 d}{1+\log\frac{m}{d}} \overset{*}{\leq} K(d,n) \leq 1 + \frac{C_2 d}{1+\log\frac{m}{d}} \quad \forall d,n,$$

where $C_1, C_2 \geq 0$ are absolute constants.

[* also due Rudelson-Vershynin, 2005]

Cor (1) There is a cs d-polytope with $4d$ vertices that is $\overset{**}{\geq} \frac{d}{400}$-neighborly.

(2) There is a 2-neighborly cs d-polytope with $e^{\Theta(d)}$ vertices.

[** Donoho, 2005: $\geq 0.089d$-neighborly]

Case of $K(d,n)=1$

Problem Determine $n_0(d) := \min\{n : K(d,n)=1\}$

McMullen-Shephard conj $\Longrightarrow$ $n = d-2$ (false)

Burton's result $\Longrightarrow$ $n_0(d)$ is finite

Thm 1 $\Longrightarrow$ $n_0 = e^{\Theta(d)}$

Thm 2 (Linial-N) $n_0 \leq 2^{d-1}+1-d$, i.e.
a 2-neighborly cs d-polytope has $\leq 2^d$ vertices

Ideas of Proofs

Lower bound of Thm 1:

* McMullen-Shephard's cs transform
* Kashin's and Garnaev-Gluskin's thms [geometry of Banach spaces]

Upshot: a random set $\{\bar{u}_1, \dots, \bar{u}_{n+d}, -\bar{u}_1, \dots, -\bar{u}_{n+d}\} \subseteq \mathbb{R}^n$ is a cs transform of a "highly"-neighborly cs d-polytope on $n+d$ vertices.

Thm 2 + upper bound of Thm 1

* volume trick of Danzer-Grünbaum thm

From non-neighborliness to f-numbers

Thm 2 $\Longrightarrow$ If P is a cs d-polytope with $f_0 > 2^d$, then

$$f_1(P) < \binom{f_0}{2} - \frac{f_0}{2}.$$

Q1: How big is the gap between $\binom{n}{2} - \frac{n}{2}$ and $f_{max}(d,n;1) =$

$$\max\{f_1(P) : P \text{ is a cs } d\text{-polytope}, f_0(P) = n\} ?$$

Q2: Is $f_{max}(d,n;j-1) = \Theta(n^j)$ for $j \leq \frac{d}{2}$?

Main results

Thm 3 (Barvinok–N)

If $d=2k$ is fixed and $n \to \infty$, then

① $$1-\frac{1}{d-1}+o(1) \le \frac{f_{max}(d,n;1)}{\binom{n}{2}} \le 1-\frac{1}{2^d}+o(1)$$

[e.g. $d=4$: $\frac{2}{3} \le$ — " — $\le \frac{15}{16}$],

② and for $j \le k$

$$c_j(d)+o(1) \le \frac{f_{max}(d,n;j-1)}{\binom{n}{j}} \le 1-\frac{1}{2^d}+o(1)$$

where $c_j(d)$ is a positive constant

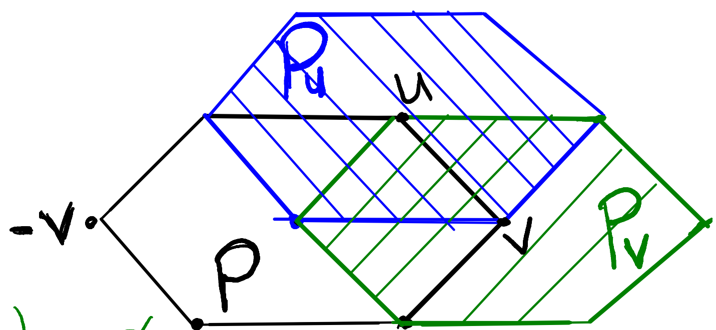
Upper Bound – Sketch of the proof

Proposition If P is a cs d-polytope with $f_0(P) = n$, then

$$f_1(P) \le \frac{n^2}{2} \cdot \left(1 - 2^{-d}\right)$$

proof: $\mathcal{F}(P) = \{ P_v = P + v : v \in \mathrm{Vert}(P) \}$

*** Key observation:**



if P is cs and

$\mathrm{Int}(P_u) \cap \mathrm{Int}(P_v) \neq \emptyset$

$\Longrightarrow$ $(u, -v)$ is not an edge

Proof continued

- $P_u = P + u \subseteq 2P \Longrightarrow Vol(\bigcup P_u) \leq 2^d \cdot Vol(P)$
- But $\underbrace{\sum_{u \in V} Vol(P_u) = n \cdot Vol(P)}$

$\Downarrow$

a pt of $2P$ is in $\geq \frac{n}{\lfloor 2^d \rfloor}$ polytopes P_u on ave

$\Downarrow$ (key observation)

ave degree of a vertex of $P \leq n - \frac{n}{2^d} = n\left(1 - \frac{1}{2^d}\right)$

To prove observation

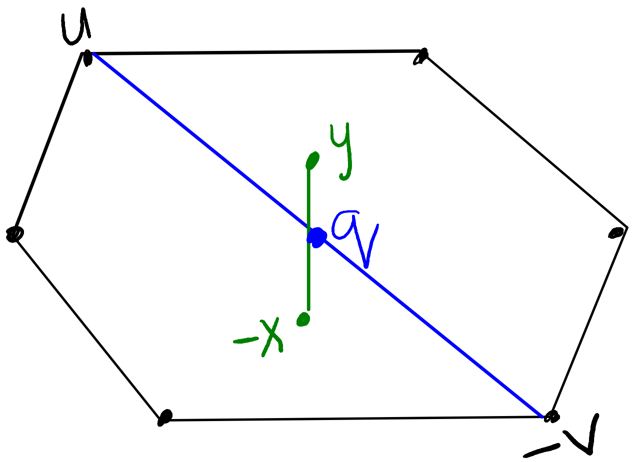
if $x, y \in \operatorname{Int}(P)$ are such that

$x + u = y + v$ then

$$\frac{u-v}{2} = \frac{y-x}{2} =: q \in \operatorname{Int}(P)$$

$$\Downarrow$$

$$(u, -v) \notin E$$



Lower Bound

Is there a cs version of $C_d(n)$?

$C_{2K}(n) = \text{conv}\{TM(t_1), \dots, TM(t_n)\}$, $t_1 < \dots < t_n$,

$TM_{2K}(t) = (\cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos Kt, \sin Kt)$

Consider Symmetric Moment Curve

$SM_{2K}(t) = (\cos t, \sin t, \cos 3t, \sin 3t, \dots,$
$\cos(2K-1)t, \sin(2K-1)t)$

- $SM(t+2\pi) = SM(t) \Rightarrow SM_{2K}: S^1 \to \mathbb{R}^{2K}$
- $SM(t+\pi) = -SM(t)$

Bicyclic polytopes

Main definition:

- $\mathcal{B}_{2k} := \operatorname{conv}(SM_{2k})$

  [k=2, Smilansky, 1985]

- If $X \subset S^1$ is a finite cs set, define $\mathcal{B}_{2k}(X) := \operatorname{conv}\{SM_{2k}(x) : x \in X\}$ – bicyclic polytope.

  It is c.s.!

  (But combin. type depends on X)

Faces of $\mathcal{B}_{2K}$

$$\text{Face} = \underbrace{\text{supp hyperplane}} \cap \mathcal{B}_{2K}$$

non-negative trig poly

$$T(t) = c + \sum_{j=1}^{K} a_{2j-1}\cos\big((2j-1)t\big) + \sum_{j=1}^{K} b_{2j-1}\cdot\sin\big((2j-1)t\big)$$

$z := e^{it}$

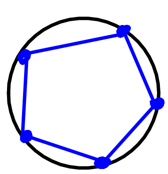
$$D(z) = \sum_{j=0}^{4K-2} c_j z^j : \qquad c_j = \overline{c_{4K-2-j}}$$

and $\qquad c_{2j-1} = 0 \quad \forall j < K$

"raked self-inversive poly" of deg $4K-2$

Lemma: Faces of $\mathcal{B}_{2K}$ are defined by raked self-inversive poly of degree $\le 4K-2$ all of whose roots of modulus 1 have even multiplicity. If $D(z)$ is such a poly and $\{e^{it_1}, \dots, e^{it_s}\}$ is the set of its roots of modulus 1, then $F = \mathrm{conv}\{SM(t_1), \dots, SM(T_s)\}$ is a face of $\mathcal{B}_{2K}$.

Example



$D(z) = (z^{2K-1} - 1)^2$

$\{z : z^{2K-1} = 1\} \longleftrightarrow (2K-2)$-dim face

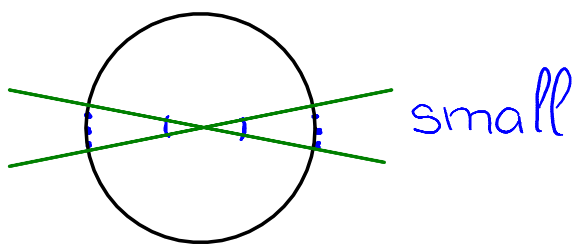
Lower Bounds on f_{max}

Studying zeros of raked self-inv poly:

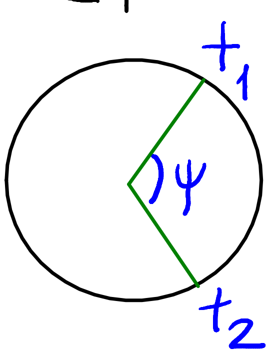
* Locally $\mathcal{B}_{2k} = \text{conv}(\text{full Symm Moment curve})$ behaves as $\text{conv}(\text{full Moment curve})$

$$\Downarrow$$

$$f_{max}(2k, n; j-1) \geq 2 \cdot \binom{n/2}{j}, \quad j \leq k$$



* If $\psi < \frac{2k-2}{2k-1} \cdot \pi$, then



$[SM(t_1), SM(t_2)]$ – edge of $\mathcal{B}_{2k}$.

Open Problems

I Upper Bounds – edges

Know:

$$\frac{2}{3} + o(1) \leq \frac{f\max(4,n;1)}{\binom{n}{2}} \leq \frac{15}{16} + o(1)$$

* What is the actual value?

* (More) exact formulas?

Upper Bounds — higher-dim faces

* A characterization of k-vertex faces of $\mathcal{B}_{2k}$?

* Do $\mathcal{B}_{2k}(X_n)$,
  X_n = n equally-spaced pts on S^1,
  (asymptotically) maximize f_j among cs $2k$-polytopes ?

cs – neighborliness

Our construction of neighborly cs polytopes is probabilistic.

* Explicit constructions?
(J. Pfeifle: some results for dim $= d$, $f_0 = 4d$)

* Exact values of $\kappa(d,n)$?

* At least, determine
$$n_0(d) = \min\{n : \kappa(d,n) = 1\}$$

cs polytopes vs cs spheres

While neighborliness of cs polytopes is very restricted, there do exist $\lfloor d/2 \rfloor$-neighborly cs $(d-1)$-dim spheres

$d=4$ $\exists$ infinite family (Jockusch)

$d>4$ some values of f_0 (Lutz)

* Construct infinite families of $\lfloor d/2 \rfloor$-neighborly cs $(d-1)$-spheres $\forall d \geq 5$

Lower Bounds

d-simplex $f_0 = f_{d-1} = d+1 \leftarrow$ both small

can NOT happen for cs polytopes:

Thm (Fiegel-Lindenstrauss-Milman, 1977)

$\exists\, \alpha > 0$ s.t. $\forall$ cs d-polytope P

$$\log f_0(P) \cdot \log f_{d-1}(P) \geq \alpha d$$

Conjecture (Kalai)

Every cs d-polytope has $\geq 3^d$ faces

* e.g $\displaystyle\sum_{i=0}^{d} f_i(d\text{-cube}) = 3^d$

* Stanley, 1987: conj. holds for simplicial polyt.
* Werner-Ziegler, 2007: conj. holds for $d=4$