Face numbers of centrally symmetric manifolds

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Connection to yesterday’s talk—
a few surprises

* Know (a bit) more about face numbers of
c.s. manifolds than about those of c.s. polytopes

reason: can use Algebra!
(and it seems to give tight bounds)

* While the neighborliness of c.s
polytopes is very restricted, there
do exist \( L^d \)-neighborly c.s. spheres

[Without the c.s assumption, no \((d-1)\)-sphere
  can be more neighborly than \( G_d(n) \).]
**Simplicial complexes**

**Def** \( \Gamma \subseteq 2^V \) is a simplicial complex on \( V \) if \( F \in \Gamma \), \( G \subseteq F \Rightarrow G \in \Gamma \).

\( V \) — vertex set
\( \Gamma \) — faces

**Geometric realization**

\[ V = \{1, 2, ..., n\} \quad e_1, e_2, ..., e_n \in \mathbb{R}^n \]

\[ \Gamma = \{i_1, ..., i_k\} \in \Gamma \quad \implies \quad \varphi_F := \text{conv} \{e_{i_1}, ..., e_{i_k}\} \]

\[ |\Gamma| := \bigcup_{F \in \Gamma} \varphi_F \]

**e.g.**

![Diagram of a simplicial complex](image)
Important Examples

* Simplicial polytopes
  (boundary—simplicial complex)

  e.g.

* Simplicial spheres: $|\Gamma| \cong S^{d-1}$

[Many simplicial spheres are NOT polytopal]

* Simplicial manifolds
Combinatorial and topological invariants

F-simplex \( \dim F = |F|-1 \)

dimension: \( \dim \Gamma = \max \{ \dim F : F \in \Gamma \} \)

\( f \)-numbers: \( f_i(\Gamma) = \# \text{ i-dim faces} \)
\( f(\Gamma) = (f_{-1}, f_0, f_1, \ldots, f_{\dim \Gamma}) \)

(reduced) Betti numbers: \( b_i = \dim \tilde{H}_i(\Gamma; k) \)

e.g. \( \dim \Gamma = 2 \)
\( f(\Gamma) = (1, 7, 8, 2) \)
\( b_0 = 1, b_1 = 1, b_2 = 0 \)
Main problem

Characterize f-vectors of various classes of simplicial complexes

Characterizations are known for:

(1) f-numbers of all simplicial complexes
    (Kruskal-Katona, 1963)

(2) f-numbers of all Cohen-Macaulay complexes
    (Stanley, 1975)

(3) f-numbers of all simplicial polytopes
    (Billera, Lee, Stanley, 1980)

(4) pairs \((f, \beta)\) of simplicial complexes
    (Björner-Kalai, 1988)
Will discuss necessary conditions on \((f, g)\) for
- simplicial manifolds, and
- simplicial manifolds with symmetry generalizing those of (2) and (4).

Main method: study combinatorics of the Stanley-Reisner rings.
**Stanley–Reisner ring**

\( \Gamma \)-simplic. compl.
\( V = \{1, 2, \ldots, n \} \)

\( k \)-field

ideal \( \frac{I}{\Gamma} \) in \( k[x] = k[x_1, \ldots, x_n] \)

\[ I : = \langle x = x_{i_1} x_{i_2} \cdots x_{i_k} : G = \{ i_1, i_2, \ldots, i_k \} \not\in \Gamma \rangle \]

Stanley–Reisner ideal of \( \Gamma \)

\( k[x] / \frac{I}{\Gamma} \)-Stanley–Reisner (face) ring of \( \Gamma \)

\[ I : = \langle x_1 x_4, x_2 x_3 x_4 \rangle \]

e.g.

\[ k[x] / \frac{I}{\Gamma} = \bigoplus_{i=0}^{\infty} (k[x] / \frac{I}{\Gamma})_i \]

\( i \)-th component
\[ h - \text{vector} \]

\[
\frac{k[x]}{\Gamma} = \bigoplus_{\Gamma \in \Gamma} x^F \cdot k[x_j : j \in \Gamma] \implies \\
\sum_{i=0}^{\infty} \dim (\frac{k[x]}{\Gamma}) \cdot t^i \\
= \sum_{\Gamma \in \Gamma} \frac{\dim \Gamma + 1}{(1-t)} = \sum_{s=0}^{\dim \Gamma + 1} \frac{1}{(1-t)^{s+1}} \\
= \frac{\sum h_i(\Gamma) \cdot t^i}{(1-t)^{\dim \Gamma + 1}}
\]

where \( h(\Gamma) := (h_0, h_1, \ldots, h_d) \) - h-vector of \( \Gamma \)

\[
\sum_{i=0}^{d} h_i \cdot t^{d-i} = \sum_{i=0}^{d} \frac{d}{i} (t-1)^{d-i} \quad (d = \dim \Gamma + 1)
\]

e.g. \( h(\text{tetrahedron}) = 1 \cdot (t-1)^3 + 6 \cdot (t-1)^2 + 12 \cdot (t-1) + 8 \)

\[ 1 \cdot t^3 + 3 \cdot t^2 + 3 \cdot t + 1 \]
Cohen-Macaulay complexes

Def \( \Gamma \) is CM (over \( k \)) if \( k[x]/I_\Gamma \) is a CM ring, i.e. for every set of \( \theta_1, \ldots, \theta_d \in (k[x]/I_\Gamma) \), if \( \frac{k[x]}{I_\Gamma + \langle \theta_1, \ldots, \theta_d \rangle} \) is a finite-dim vector space, then

\[
\dim \left( \frac{k[x]}{I_\Gamma + \langle \theta_1, \ldots, \theta_d \rangle} \right) = h_i(\Gamma)
\]

* Reisner, Munkres: being CM is a topological invariant of \( \Gamma \):

\( \Gamma \) is CM if \( \widetilde{H}_i(\Gamma) = H_i(\Gamma, I_\Gamma - p) = 0 \) for all \( p \in \Gamma \) and all \( i < \dim \Gamma \).
Buchsbaum complexes

**Def (Schenzel)**

Γ is Buchsbaum if it is pure

and \( H_i(\Gamma^1, \Gamma^1 - p) = 0 \) \( \forall i < \dim \Gamma \)

\( \forall p \in \Gamma \).

Thus

CM complexes \( \subset \) Buchsbaum compl \( \cup \)

simplex, spheres \( \subset \) simplex, manifolds
Group actions

An action of \( \mathbb{Z}/p\mathbb{Z} \) on \( \Gamma \) is proper if

\[ y(F) = F \quad \text{for some} \quad \phi \neq F \in \Gamma \quad \text{and} \quad y \in \mathbb{Z}/p\mathbb{Z} \]

\[ \Downarrow \]

\[ y(v) = v \quad \text{for all} \quad v \in F \]

Examples: * trivial action - proper

\[ \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} \]

proper  NOT proper  proper (free)

CS complex = proper \( \mathbb{Z}/2\mathbb{Z} \) action, no invariant pts.

Proper action \( \Rightarrow \quad V = \begin{array}{c}
\ast \ast \ldots \ast \\
\scriptsize{\text{invariant}} \\
\end{array}
\begin{array}{c}
P \\
\ldots \\
P \\
\scriptsize{\text{m "free" orbits}} \\
\end{array} \)
Multicomplexes

Notation: $S(k_1, \ldots, k_n) := \{x_1^{a_1} \cdots x_n^{a_n} : 0 \leq a_i \leq k_i\}$

e.g. $S(\infty, \ldots, \infty) =: S(\infty^n)$ — poset of all monomials in $x_1, \ldots, x_n$.

$S(1, \ldots, 1) =: S(1^n)$ — all square-free monomials

Def \[ M \subseteq S(\infty^n) \text{ is a multicomplex if} \]

1. $m \in M$, $m' | m \Rightarrow m' \in M$
2. $x_i \in M \quad \forall i = 1, \ldots, n$

* $M$-multicomplex $\mapsto F(M) = (F_0, F_1, \ldots)$ where $F_i := \left\{ m \in M : \deg m = i \right\}$

e.g. multicompl. $M \subseteq S(1^n)$ is simplicial
Stanley's theorem

$\Gamma$-simpl. complex $\dim \Gamma = d-1$

$$f(\Gamma) = (f_{-1}, f_{0}, \ldots, f_{d-1})$$
$$h(\Gamma) = (h_0, h_1, \ldots, h_d)$$

$$\sum_{i=0}^{d} h_i t^{d-i} = \sum_{i=0}^{d} f_{i-1} (t-1)^{d-i}$$

(h-numbers may, in general, be negative)

Thm (Stanley, 1975)

A sequence $h = (h_0, \ldots, h_d)$ is the $h$-vector of a $(d-1)$-dim Cohen-Macaulay complex $\Gamma$ on $n$ vertices iff $h = F(M)$ for some multicomplex $M \subseteq S(\infty^{n-d})$. 
Let $\Gamma$ be a $d=1$-dim CM complex on $n$ vertices. If $\Gamma$ is endowed with a proper $\mathbb{Z}/p\mathbb{Z}$-action, and has $m$ free orbits of vertices, then $$h(\Gamma) = F(M)$$ for some multicomplex $M \subseteq S(p-1, \infty, n-d-m)$.

* $m=0$ - trivial action - Stanley's thm
* cs complex: $p=2$, $m=\frac{n}{2}$ get $$h(\Gamma) = F(M) \text{ for } M \subseteq S(1, \infty, n-d)$$.
How to decide if $F=(F_0, F_1, \ldots)$ is the F-vector of a multicompex?

Thm (Clements-Lindström, 1969)

For every poset $S=S(k_1, \ldots, k_n)$ there exists a set of explicit functions $\partial_{S,j}$ such that $F=(F_0, F_1, \ldots)$ is the F-vector of a multicompex $M \subseteq S$ iff

$F_0=1$, $F_1=n$, and $\partial_{S,j}(F_{j+1}) \leq F_j \forall j \geq 1$

* $k_1=\ldots=k_n=\infty$ - case due to Macaulay (1927)

* $k_1=\ldots=k_n=1$ - case Kruskal-Katona (1963)
$h'$-vectors of Buchsbaum complexes

$(d-1)$-dim manifold (or Buchsbaum)

$$h'_j(\Gamma) = h_j(\Gamma) + \binom{d}{j} \sum_{k=2}^{j-3} \beta_{j-k} - \sum_{k=3}^{j-3} \beta_{j-k}$$

Thm 2 (N) Let $\Gamma$ be a $(d-1)$-dim simplicial manifold on $n$ vertices. Assume $\Gamma$ is endowed with a proper $\mathbb{Z}/p\mathbb{Z}$-action, and has $m$ free orbits. Then $h'_0 = 1$, $h'_1 = n-d$, and

$$\partial_{S,j} (h'_{j+1}) \leq h'_j - \binom{d-1}{j} \beta_{j-1} \quad \forall j \geq 1$$

where $S = \mathbb{S}((p-1)^m, n-d-m)$. 

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Applications

\[ \chi(\Gamma) := \frac{f_0}{f_1} - \frac{f_2}{f_1} - \ldots - (-1)^{d-1} \frac{f_{d-1}}{f_1} \]

Conjecture (Kühnel)

If \( \Gamma \) is a 2\( K \)-dim manifold, \( f_0(\Gamma) = n \), then

\[ (-1)^K \binom{2K+1}{K} (\chi(\Gamma) - 2) \leq \binom{n-K+2}{K+1} \]

Conjecture (Sparla)

If \( \Gamma \) is a 2\( K \)-dim cs manifold, \( f_0(\Gamma) = 2m \), then

\[ (-1)^K \binom{2K+1}{K} (\chi(\Gamma) - 2) \leq 4^{K+1} \binom{\frac{1}{2}(m-1)}{K+1} \]
Kühnel’s bound $= F_{K+1}(S(\infty_{n-2K-1})) - F_K(S(\infty_{n-2K-1}))$

Sparla’s bound $= F_{K+1}(S(1,\infty_{m-2K-1})) - F_K(S(1,\infty_{m-2K-1}))$

**Thm 3 (N)**

Assume $\Gamma$ is a $2K$-dim manifold, $\mathbb{Z}/p\mathbb{Z}$ acts properly, $\# $ free orbits $= m$.

If $n \geq 6K + 3$, then

$(-1)^K \binom{2K+1}{K} \chi(\Gamma) - 2 \leq F_{K+1}(S) - F_K(S),$

where $S = S((p-1)^m, \infty_{n-2K-1-m})$.

In particular, Kühnel’s and Sparla’s conjectures hold for manifolds with $f \geq 6K+4$.
Upper Bounds on the $h$-numbers

Thm 1 \[ \Rightarrow \text{Corollary (Adin, 1991)} \]

$\Gamma$- $(d-1)$-dim CM complex, $f_0(\Gamma) = n$, $\mathbb{Z}/p\mathbb{Z}$ acts properly, \# free orbits = $m$.

Then \[ h_j(\Gamma) \leq F_j \left( S((p-1)^m, \infty^{n-d-m}) \right). \]

Thm 2 + computations

\[ \downarrow \]

same holds for $(d-1)$-dim Buchsbaum complex on $n \geq 3d-2$ vertices.

[closely related to the UBT for manifolds]
Proofs of Kalai’s algebraic shifting

\[ \Gamma - (d-1) \text{-dim complex} \rightarrow k[x]/I_\Gamma \]

Topological properties of \( \Gamma \)
(e.g. \( \Gamma \)-manifold)

Algebraic properties of \( k[x]/I_\Gamma \)
(e.g. Buchsbaum)

\[ g \in GL_n(k) \]
First \( d \) columns of \( g^{-1} \) — “generic”

\[ g \cdot x_i = \sum g_{ij} x_j \]

\[ k[x]/g I_\Gamma \]

Combinatorial properties of
\[ B = B_k(g, \Gamma) = S(\infty^n) - \text{in}(g I_\Gamma) \]

\[ k[x]/\text{in}(g I_\Gamma) \text{ revlex} \]
Example: \( \Gamma \) - cs CM complex
on \( V = \{ x_1, \ldots, x_{2m} \} \)

Assume: \( \text{Orb}(x_i) = \{ x_i, x_{m+i} \} \quad \forall i \leq m \)

Then \( g = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & Y^{-1} & 0 \end{bmatrix} \in \text{GL}_{2m}(k) \)

(where \( Y \) is generic)

is "generic enough," and

\[ M := \{ \text{all monomials in } B(\mathfrak{g}) \text{ that do not involve last } d \text{ vars} \} \]

satisfies

\[ M \leq S(1^m, 1^{m-d}) \text{ and } F(M) = h(\Gamma). \]
Open Problems

I Manifolds — upper Bounds

• Are Kühnel's and Sparla's conj. sharp?

Equality $\iff \Gamma$ is (k+1)-neighb. 2k-manif.
    $\cong$ (k+1)-neighb. manif., resp.

Existence of such manifolds?

$k = 1$: in both cases there exist $\infty$-families
    "Heawood conjecture" (Jungerman-Ringel)

$k = 2$: only a few examples are known.
    (e.g. 9-vertex triangulation of $\mathbb{CP}^2$;
    12-vertex triangul. of $S^2 \times S^2$)

reference — Frank Lütz's manifold page
**Problem**: construct infinite families of 
$(k+1)$-neighborly $2k$-manifolds; 
$L_{d-1}^d$-neighborly cs $(d-1)$-spheres, 
and $(k+1)$-neighborly cs $2k$-manifolds.
Manifolds - Lower Bounds

Stanley: If $\Gamma \subseteq (d-1)$-dim CM complex, then

$$h_i(\Gamma) \geq \binom{d}{i} \quad i=0,1,\ldots,d$$

Problem: Are there similar bounds for CS Buchsbaum complexes?

Other groups

Conjecture: If manifold with a free $(\mathbb{Z}/2\mathbb{Z})^S$-action, then

$$\sum \beta_i(\Gamma) \geq 2^S$$
IV general simplicial complexes

Problem: characterize $f$-numbers of CS complexes with prescribed Betti #s

Babson-N: some necessary conditions
2005