Face numbers of centrally symmetric manifolds

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Connection to yesterday's talk – a few surprises

* Know (a bit) more about face numbers of c.s. manifolds than about those of c.s. polytopes

reason: can use Algebra! (and it seems to give right bounds)

* While the neighborliness of cs polytopes is very restricted, there do exist $\lfloor \frac{d}{2} \rfloor$-neighborly cs spheres

[Without the cs assumption, no $(d-1)$-sphere can be more neighborly than $C_d(n)$.]

Simplicial complexes

Def $\Gamma \subseteq 2^V$ is a simplicial complex on V if $F \in \Gamma,\ G \subseteq F \Rightarrow G \in \Gamma$.

V — vertex set

elements of Γ — faces

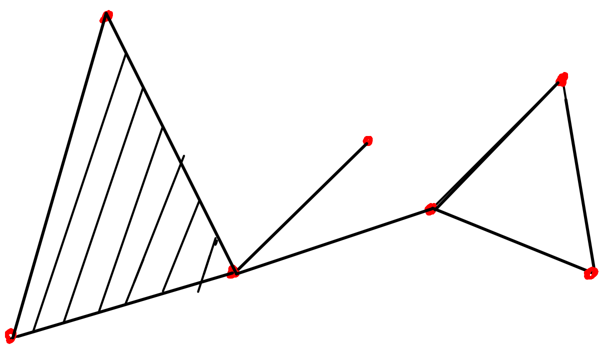
Geometric realization

$V = \{1, 2, \dots, n\}$ $\quad e_1, e_2, \dots, e_n \in \mathbb{R}^n$

$F = \{i_1, \dots, i_k\} \in \Gamma \rightsquigarrow \sigma_F := \operatorname{conv}\{e_{i_1}, \dots, e_{i_k}\}$

$$|\Gamma| := \bigcup_{F \in \Gamma} \sigma_F$$

e.g.

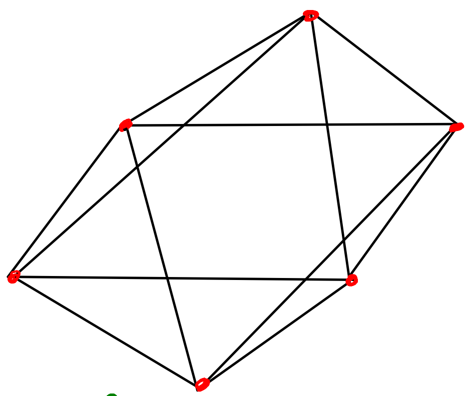


Important Examples

* Simplicial polytopes

  (boundary – simplicial complex)

  e.g.



* Simplicial spheres: $|\Gamma| \cong S^{d-1}$

[Many simplicial spheres are NOT polytopal]

* Simplicial manifolds

Combinatorial and topological invariants

F - simplex $\quad \dim F := |F| - 1$

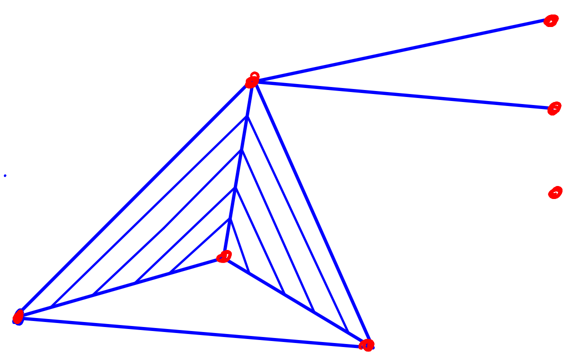
dimension: $\dim \Gamma = \max \{\dim F : F \in \Gamma\}$

f-numbers: $f_i(\Gamma) = \#\, i\text{-dim faces}$

$$f(\Gamma) = (f_{-1}, f_0, f_1, \dots, f_{\dim \Gamma})$$

(reduced) Betti numbers: $\beta_i = \dim \widetilde{H}_i(\Gamma; k)$

e.g.



$\dim \Gamma = 2$

$f(\Gamma) = (1, 7, 8, 2)$

$\beta_0 = 1, \ \beta_1 = 1, \ \beta_2 = 0$

Main problem

Characterize f-vectors of various classes of simplic. complexes

Characterizations are known for:

(1) f-numbers of all simplic. complexes (Kruskal-Katona, 1963)

(2) f-numbers of all Cohen-Macaulay compl. (Stanley, 1975)

(3) f-numbers of all simplic. polytopes (Billera, Lee, Stanley, 1980)

(4) pairs (f, β) of simplicial complexes (Björner-Kalai, 1988)

Will discuss

necessary conditions on (f, β) for

- simplicial manifolds, and

- simplic. manifolds with symmetry

generalizing those of (2) and (4)

Main method: study combinatorics of the Stanley-Reisner rings.

Stanley-Reisner ring

Γ - simplic. compl.
$V = \{1, 2, \dots, n\}$
k - field

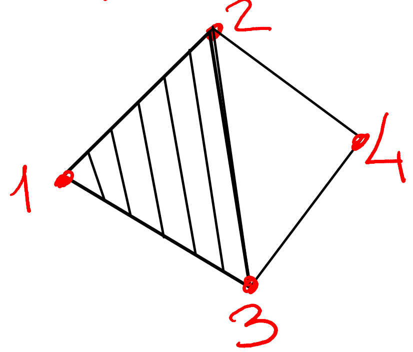
$\rightsquigarrow$ ideal I_Γ in $k[\underline{x}] := k[x_1, \dots, x_n]$

$$I_\Gamma := \langle x^G = x_{i_1} x_{i_2} \cdots x_{i_k} : G = \{i_1 < i_2 < \dots < i_k\} \notin \Gamma \rangle$$

Stanley-Reisner ideal of Γ

$k[\underline{x}]/I_\Gamma$ - Stanley-Reisner (face) ring of Γ

e.g.



$$I_\Gamma = \langle x_1 x_4, x_2 x_3 x_4 \rangle$$

$$k[\underline{x}]/I_\Gamma = \bigoplus_{i=0}^{\infty} \underbrace{(k[x]/I_\Gamma)_i}_{\text{i-th component}}$$

h-vector

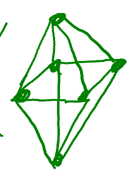
$$k[\underline{x}]/I_\Gamma = \bigoplus_{F\in\Gamma} x^F \cdot k[x_j : j\in F] \Longrightarrow$$

$$\sum_{i=0}^{\infty} \dim_k \left(k[\underline{x}]/I_\Gamma\right)_i \cdot t^i = \sum_{F\in\Gamma} \frac{t^{|F|}}{(1-t)^{|F|}} = \sum_{s=0}^{\dim\Gamma+1} \frac{f_{s-1}\cdot t^s}{(1-t)^s}$$

$$= \frac{\sum h_i(\Gamma)\cdot t^i}{(1-t)^{\dim\Gamma+1}}$$

where $h(\Gamma) := (h_0, h_1, \dots, h_d)$ – h-vector of Γ

$$\sum_{i=0}^{d} h_i \cdot t^{d-i} = \sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} \quad (d=\dim\Gamma+1)$$

e.g. $h\Big($  $\Big) = 1\cdot(t-1)^3 + 6(t-1)^2 + 12(t-1) + 8$

$$1\cdot t^3 + 3\cdot t^2 + 3\cdot t + 1$$

Cohen-Macaulay complexes

Def Γ is CM (over k) if $k[\underline{x}]/I_\Gamma$ is a CM ring, i.e. for every set of $\theta_1,\dots,\theta_d \in \left(k[\underline{x}]/I_\Gamma\right)_1$ if $k[\underline{x}]/I_\Gamma+\langle\theta_1,\dots,\theta_d\rangle$ is a finite-dim vector space, then

$$\dim\left(k[\underline{x}]/I_\Gamma+\langle\theta_1,\dots,\theta_d\rangle\right)_i = h_i(\Gamma)$$

* Reisner, Munkres: Being CM is a topological invariant of $|\Gamma|$:

Γ is CM if $\widetilde{H}_i(\Gamma)=H_i(|\Gamma|,|\Gamma|-p)=0$ $\forall p\in|\Gamma|$ and $\forall i<\dim\Gamma$.

Buchsbaum complexes

Def (Schenzel)
Γ is Buchsbaum if it is pure

and $\quad H_i(|\Gamma|, |\Gamma| - p) = 0 \quad \forall i < \dim\Gamma$
$\forall p \in |\Gamma|$.

Thus

$$\begin{array}{ccc} \text{CM complexes} & \subset & \text{Buchsbaum compl} \\ \cup & & \cup \\ \text{simplic. spheres} & \subset & \text{simplic. manifolds} \end{array}$$

Group actions

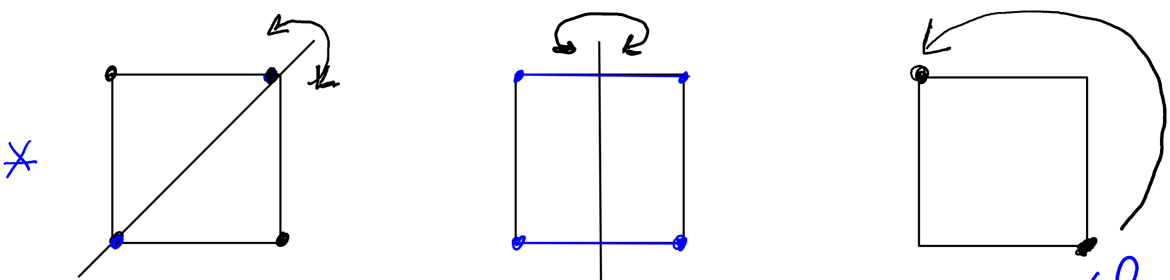
An action of $\mathbb{Z}/p\mathbb{Z}$ on Γ is **proper** if

$\varphi(F)=F$ for some $\emptyset \neq F \in \Gamma$ and $\varphi \in \mathbb{Z}/p\mathbb{Z}$

$$\Downarrow$$

$\varphi(v)=v$ for all $v \in F$

Examples: * trivial action – proper



proper   NOT proper   proper (free)

cs complex = proper $\mathbb{Z}/2\mathbb{Z}$ action, no invariant pts.

Proper action $\Longrightarrow$ $V = \underbrace{* \; * \ldots *}_{\text{invariant}} \; \underbrace{\boxed{p} \ldots \boxed{p}}_{m \text{ "free" orbits}}$

Milticomplexes

Notation: $S(k_1,\dots,k_n) := \{x_1^{a_1}\cdots x_n^{a_n} : 0 \le a_i \le k_i\}$

e.g. $S(\infty,\dots,\infty) =: S(\infty^n)$ — poset of all monomials in $x_1,\dots,x_n$.

$S(1,\dots,1) =: S(1^n)$ — all square-free monomials

Def * $M \subseteq S(\infty^n)$ is a **multicomplex**

if (1) $m \in M$, $m' | m \Longrightarrow m' \in M$

(2) $x_i \in M \quad \forall i = 1,\dots,n$

* M-multicomplex $\rightsquigarrow F(M) = (F_0, F_1, \dots)$

where $F_i := |\{m \in M : \deg m = i\}|$

e.g. multicompl. $M \subseteq S(1^n)$ is simplicial

Stanley's theorem

Γ - simpl. complex $\rightsquigarrow$ $f(\Gamma)=(f_{-1},f_0,\dots,f_{d-1})$

$\dim \Gamma = d-1$ $\qquad h(\Gamma)=(h_0,h_1,\dots,h_d)$

$$\sum_{i=0}^{d} h_i t^{d-i} = \sum_{i=0}^{d} f_{i-1}(t-1)^{d-i}$$

(h-numbers may, in general, be negative)

Thm (Stanley, 1975)

A sequence $h=(h_0,\dots,h_d)$ is the h-vector of a $(d-1)$-dim Cohen-Macaulay complex Γ on n vertices <u>iff</u> $h=F(M)$ for some multicomplex $M \subseteq S(\infty^{n-d})$.

Thm 1 (N, 2005)

Let Γ be a $(d-1)$-dim CM complex on n vertices. If Γ is endowed with a proper $\mathbb{Z}/p\mathbb{Z}$-action, and has m free orbits of vertices, then

$$h(\Gamma) = F(M)$$

for some multicomplex $M \subseteq S((p-1)^m, \infty^{n-d-m})$

* $\underline{m=0}$ – trivial action – Stanley's thm
* $\underline{\text{cs complex}}$: $p=2$, $m=\frac{n}{2}$ get

$$h(\Gamma) = F(M) \text{ for } M \subseteq S(1^m, \infty^{m-d}).$$

How to decide if $F = (F_0, F_1, \dots)$ is the F-vector of a multicomplex?

Thm (Clements–Lindström, 1969)

For every poset $S = S(k_1, \dots, k_n)$ there exists a set of explicit functions $\partial_{S,j}$ such that $F = (F_0, F_1, \dots)$ is the F-vector of a multicomplex $M \subseteq S$ iff

$$F_0 = 1,\quad F_1 = n,\quad \text{and}\quad \boxed{\partial_{S,j}(F_{j+1}) \leq F_j}\quad \forall j \geq 1$$

* $k_1 = \dots = k_n = \infty$ - case — due to Macaulay (1927)

* $k_1 = \dots = k_n = 1$ - case — Kruskal-Katona (1963)

h'-vectors of Buchsbaum complexes

Γ – $(d-1)$-dim manifold (or Buchsbaum)

$$h'_j(\Gamma) := h_j(\Gamma) + \binom{d}{j}\left(\beta_{j-2} - \beta_{j-3} + \dots \pm \beta_0\right)$$

Thm 2 (N) Let Γ be a $(d-1)$-dim simplicial manifold on n vertices. Assume Γ is endowed with a proper $\mathbb{Z}/p\mathbb{Z}$-action, and has m free orbits. Then $h'_0 = 1$, $h'_1 = n-d$, and

$$\boxed{\partial_{S,j}\left(h'_{j+1}\right) \leq h'_j - \binom{d-1}{j}\beta_{j-1}} \quad \forall j \geq 1$$

where $S = S\left((p-1)^m, \infty^{n-d-m}\right)$.

Applications

$$\chi(\Gamma) := f_0 - f_1 + f_2 - \dots + (-1)^{d-1} f_{d-1}$$

Conjecture (Kühnel)

If Γ is a $2k$-dim manifold, $f_0(\Gamma) = n$,

then $$(-1)^k \cdot \binom{2k+1}{k} \left(\chi(\Gamma) - 2\right) \leq \binom{n-k+2}{k+1}.$$

Conjecture (Sparla)

If Γ is a $2k$-dim cs manifold, $f_0(\Gamma) = 2m$,

then $$(-1)^k \cdot \binom{2k+1}{k} \cdot \left(\chi(\Gamma) - 2\right) \leq 4^{k+1} \binom{\frac{1}{2}(m-1)}{k+1}.$$

Kühnel's Bound $= F_{K+1}\left(S\left(\infty^{n-2K-1}\right)\right) - F_K\left(S\left(\infty^{n-2K-1}\right)\right)$

Sparla's Bound $= F_{K+1}\left(S\left(1^m, \infty^{m-2K-1}\right)\right) - F_K\left(S\left(1^m, \infty^{m-2K-1}\right)\right)$

Thm 3 (N)

Assume Γ – $2K$-dim manifold,

$\mathbb{Z}/p\mathbb{Z}$ acts properly, # free orbits $= m$.

If $n \geq 6K+3$, then

$$(-1)^K \cdot \binom{2K+1}{K} \cdot \left(\chi(\Gamma) - 2\right) \leq F_{K+1}(S) - F_K(S),$$

where $S = S\left((p-1)^m, \infty^{n-2K-1-m}\right)$.

In particular Kühnel's and Sparla's conjectures hold for manifolds with $f_0 \geq 6K+4$!

Upper Bounds on the h-numbers

Thm 1 $\Longrightarrow$ Corollary (Adin, 1991)

Γ – $(d-1)$-dim CM complex, $f_0(\Gamma)=n$,
$\mathbb{Z}/p\mathbb{Z}$ acts properly, # free orbits $= m$.
Then $h_j(\Gamma) \le F_j\left(S\left((p-1)^m, \infty^{n-d-m}\right)\right)$.

Thm 2 + computations
$\Downarrow$
same holds for $(d-1)$-dim Buchsbaum complex on $n \geq 3d-2$ vertices.

[closely related to the UBT for manifolds]

proofs — variant of Kalai's algebraic shifting

Γ – $(d-1)$-dim complex $\longrightarrow$ $k[\underline{x}]/I_\Gamma$

topol. prop. of Γ (e.g. Γ – manifold) $\longrightarrow$ algebr. proper. of $k[\underline{x}]/I_\Gamma$ (e.g. Buchsb.)

$g \in GL_n(k)$

first d columns of g^{-1} – "generic"

$$g \cdot x_i = \sum g_{ji} x_j$$

$$k[\underline{x}]/gI_\Gamma$$

$$k[\underline{x}]/\mathrm{in}_{\mathrm{revlex}}(gI_\Gamma)$$

combin. prop. of

$$B = B_k(g,\Gamma) := S(\infty^n) - \mathrm{in}(gI_\Gamma)$$

Example Γ – cs CM complex on $V = \{x_1, \ldots, x_{2m}\}$

Assume: $\mathrm{Orb}(x_i) = \{x_i, x_{m+i}\}$ $\forall i \leq m$

Then
$$g = \begin{array}{c} m \\ \\ m \end{array}\left[\begin{array}{c|c} \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} \\ \hline 0 & Y^{-1}\end{array}\right] \in GL_{2m}(k)$$

(where Y is generic)

is "generic enough", and

$$M := \left\{ \begin{array}{l} \text{all monomials in } B(g\Gamma) \text{ that} \\ \text{do not involve last } d \text{ vars} \end{array} \right\}$$

satisfies

$M \subseteq S(1^m, \infty^{m-d})$ and $F(M) = h(\Gamma)$. $\square$

Open Problems

I Manifolds — upper bounds

- Are Kühnel's and Sparla's conj. sharp?

Equality $\iff$ Γ is $(k+1)$-neighb. $2k$-manif. / cs $(k+1)$-neighb. manif., resp.

Existence of such manifolds?

$k=1$: in both cases there exist ∞-families

"Heawood conjecture" (Jungerman-Ringel)

$k\geq 2$: only a few examples are known.

(e.g. 9-vertex triangulation of $\mathbb{CP}^2$;
cs 12-vertex triangul. of $S^2\times S^2$)

reference — Frank Lutz's manifold page

Problem: construct infinite families of $(k+1)$-neighborly $2k$-manifolds ; $\lfloor \frac{d}{2} \rfloor$-neighborly <u>cs</u> $(d-1)$-spheres, and $(k+1)$-neighborly cs $2k$-manifolds.

II Manifolds – Lower Bounds

Stanley 1987: Γ – cs $(d-1)$-dim CM complex, then

$$h_i(\Gamma) \geq \binom{d}{i} \quad i=0,1,\dots,d$$

Problem: Are there similar bounds for cs Buchsbaum complexes?

III Other groups

Conjecture: Γ – manifold with a free $(\mathbb{Z}/2\mathbb{Z})^s$ – action, then

$$\sum \beta_i(\Gamma) \geq 2^s$$

IV general simplicial complexes

Problem: characterize f-numbers of CS complexes with prescribed Betti #s

Babson-N : some necessary conditions
2005