A Discussion of Thurston's Geometrization Conjecture

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June 3, 2014

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Acknowledgments

I would like to thank Professor James Morrow for his patience, instruction, and support over a number of years. Professor Morrow's guidance has been invaluable to my mathematical development. I would also like to thank my family and friends who have supported me during this time.

1 Introduction and Background

Here, we will give an exposition discussing Thurston's Geometrization Conjecture, notably the background results as well as some of the methods that come up in Perelman's proof of this. We will tackle (1) Kneser's Theorem on the existence and uniqueness of a prime decomposition, (2) The geometric structures Thurston claims are sufficient to look at, and (3) Hamilton's Ricci flow equation. It it beyond our technical purview to discuss the main technical aspect of exactly how Hamilton's Ricci flow equation works in the proof of the Geometrization Conjecture, but some full proofs of Perelman's Theorem 7.4 (the Geometrization Conjecture) have now been published, such as [2].

Let us first state Thurston's original Conjecture.

Theorem 1.1 (Thurston Geometrization Conjecture) *Let M be a closed, orientable prime 3-manifold. Then there exists an embedding of a disjoint union of 2-tori and Klein mottles in M such that every component of their complement admits a locally homogenous Riemannian metric of finite volume.*

Ostensibly, this seems to only be a classification for so-called prime manifolds. However, in fact this is motivated by a 1929 Theorem of Kneser.

Theorem 1.2 (Kneser's Theorem) *Every closed, oriented 3-manifold admits a decomposition as a connected sum of oriented prime 3-manifolds, called the prime factors, and orientation-preserving diffeomorphisms thereof. This decomposition is unique.*

By this result, we see that it suffices to simply examine prime manifolds instead of general closed, oriented 3-manifolds. Much of our treatment and other results dealing with general 3-manifolds comes from [4], [5] and similar sources.

The so-called locally homogenous Riemannian metrics of finite volume are also known as geometric structures. In [9] a detailed discussion of the eight possble geometric structures are given. We give the list below. Later, we will tackle and describe the individual geometric structures.

It is a very nontrivial result to show that these eight geometric structures (listed in section 4.2) are the only geometric structures worth considering. However, armed with this knowledge, if M has a locally homogenous Riemannian metric of finite volume, then we can classify its geometric structures by looking at the fundamental group. We will not discuss this issue in depth.

In the proof of the Geometrization Conjecture, we rely on Hamilton's Ricci Flow equation. We will not be able to completely trace the proof here. Our basic treatment of the Ricci flow equation is taken from [8].

Let us examine Riemannian metrics satisfying some given parabolic evolution equation

$$g'(t) = F(g(t))$$

. Its solution will be a one-parameter family of metrics whose derivative is $F(g(t))$. Now Hamilton introduced his Ricci flow equation

$$\frac{d}{dt} g_{ij}(t) = -2R_{ij}$$

, and we see that it admits a scale invariance in the sense that for any λ, $h(t) = \lambda^2 g(\lambda^{-2}(t))$ satisfies the equation if and only if $g(t)$ also satisfies the equation.

Hamilton then proved several important results about this evolution equation, the most fundamental being the following result.

Theorem 1.3 (Short-Time Existence and Uniqueness) *Let M be a Riemannian three-manifold.*

1. *If g_0 is smooth on M, then there exists some ϵ dependent on g_0 such that there is a unique solution g to the Ricci flow equation on $[0, \epsilon)$ with $g(0) = g_0$.*
2. *There is a so-called 'curvature characterization ' of singularity information, i.e. if there is a unique solution to the Ricci flow equation on $[0, T)$ but not on any larger interval, then exists x such that the Riemann curvature tensor $R(x, t)$ of $g(t)$ is unbounded as $t \to T$.*

The Ricci flow equations play an extremely important role in Perelman's proof of the Poincare Conjecture.

2 Discussion of Kneser's Theorem

2.1 Definitions

Remark Call two manifolds M and M' isomorphic if there is a piecewise linear orientation-preserving homeomorphism between the two of them.

Definition We define the connected sum $M \# M'$ of two manifolds M and M' to be the manifold obtained by removing the interior of a 3-cell from each of them and matching the resulting boundaries with an orientation-reversing homeomorphism. These operations are well-defined up to isomorphism; S^3 serves as an identity for the operation $\#$.

Definition A prime manifold M is one such that if $M = M_1 \# M_2$, then either M_1 or M_2 is S^3.

Definition A manifold M is irreducible if each 2-sphere bounds a ball in M.

Here, we will only be working with PL-manifolds, rather than the smooth manifolds that we are more interested in thinking about.

2.2 First Steps

We first give a basic theorem that will be pertinent to the discussion of irreducible and prime manifolds, or rather in identifying the differences between them.

Theorem 2.1 (Alexander's Theorem) *Every embedded 2-sphere in* $\mathbb{R}^3$ *bounds an embedded 3-ball.*

Lemma 2.2 *The only prime manifolds that are not irreducible are those isomorphic to* S^3 *or* $S^1 \times S^2$.

Proof Write $M = M_1 \# M_2$ with both nontrivial manifolds. The 2-sphere separating the two summands in M cannot bound a cell.

Then let $S \subset M$ be a 2-sphere not bounding a cell, as if S separates M then M can be expressed as a nontrivial sum. If it separates, M is not prime and there is little to discuss. If it does not separate, then we can cut along S and paste in 3-cells to eliminate the boundary obtained from cutting along S to obtain a new connected manifold, say M'.

Clearly, the original manifold M is obtained from M' by adding the handle, so $M = M' \# (S^1 \times S^2)$. Hence M is either not prime or isomorphic to $S^1 \times S^2$. This proves the requisite assertion.

Lemma 2.3 $S^1 \times S^2$ *is prime.*

Proof Every bounding 2-sphere in $S^1 \times S^2$ bounds a cell.

2.3 Proving Kneser's Theorem

The treatment of Kneser's Theorem here is due to [4] and [7].

Theorem 2.4 (Kneser's Theorem) *Every 3-manifold* M *admits a unique decomposition into primes,* $M = M_1 \# \cdots \# M_n$.

Technically, the theorem should be credited to both Kneser (who proved the existence in 1929) and John Milnor (who proved the uniqueness in a 1958 paper).

2.3.1 Part 1: Existence of a Prime Decomposition

Existence Portion: For existence, if M has a non-separating S^2, there exists a decomposition $M = N\#(S^1 \times S^2)$. This process must end after finitely many steps because each $S^1 \times S^2$ summand gives a summand of $\mathbb{Z}$ for the fundamental group of M. However, the fundamental group of a compact manifold is finitely generated, so this process terminates after finitely many steps.

We therefore reduce to the case of proving a prime decomposition exists when all 2-spheres separate M. Each 2-sphere component of ∂M corresponds to a S^3-summand of M, so WLOG we assume that ∂M as no 2-spheres.

We prove the following assertion, which will therefore imply the exist of prime decompositions.

Claim : There exists a bound on the number of spheres in the system S of disjoint spheres with no component of M-S a punctured 3-sphere.

Proof First observe that if the system S has that condition, we can perform surgery in the following way. On a sphere S_i of S with disk $D \subset M$, $D \cap S = \partial D \subset S_i$, then at least 1 of S' and S'' obtained by replacing S_i with S_i' or S_i'' from the surgery satisfies the requirement. To see this, first perturb S_i' and S_i'' to be disjoint from S_i and each other so that the three together bound a 3-punctured sphere.

On the other side of S_i from P we have a component A of $M|S$ while sphere S_i' and S_i'' split the component of $M|S$ containing P into pieces B', B'', P. If both B' and B'' are punctured spheres, then $B' \cup B'' \cup P$, a component of $M|S$ would be a punctured sphere, contrary to hypothesis. WLOG B' is not a punctured sphere. If $A \cup P \cup B''$ is a punctured sphere, this would force A also to be a punctured sphere, again contrary to hypothesis.. Hence no component of $M|S'$ adjacent to S_i' is a punctured sphere andthe sphere system S' satisfies the condition.

Now we prove that S can only have a finite number of spheres. Let $\mathfrak{T}$ be a smooth triangulation of M. Since M is compact this has finitely many simplices. The given system S can be perturbed to be transverse to all the simplices of $\mathfrak{T}$. The perturbation can be done inductively over dimensions (first make S disjoint from vertices, then transverse to edges, then transverse to 2-simplices).

For a 3-simplex $\mathfrak{t}$ of $\mathfrak{T}$, we make the components of $S \cap \mathfrak{t}$ all disks as follows. Such a component must meet $\partial\mathfrak{t}$ by Alexander's Theorem and the condition. Consider a circle C in $S \cap \partial\mathfrak{t}$ which is innermost in $\partial\mathfrak{t}$. If C bounds a disk component of $S \cap \mathfrak{t}$, we may isotope this disk to lie near $\partial\mathfrak{t}$. If an innermost remaining C does not bound a disk component of $S \cap \mathfrak{t}$, surger S along C using a disk D lying near $\partial\mathfrak{t}$ with $D \cap S = \partial D = C$. This replaces S by a new system S' satisfies the condition, in which either C does bound a disk component of $S' \cap \mathfrak{t}$ or C is eliminated from $S' \cap \mathfrak{t}$. After finitely many such steps we arrive at a system S with the $S \cap \mathfrak{t}$ consisting solely of disks for each $\mathfrak{t}$. In particular this implies that no component of the intersection of S with a 2-simplex of $\mathfrak{T}$ can be a circle, since this would bound disks in adjacent 3-simplies and thus obtain a sphere S bounding a ball in their union, a contradiction. □

For each 2-simplex σ we eliminate arcs α of $S \cap \sigma$ having both endpoints on the same edge of σ. Such an α cuts off from σ a disk D which meets only one edge of σ. We choose α to be edgemost so that D contains no other arcs of $S \cap \sigma$; hence $D \cap S = \alpha$, as circles of $S \cap \sigma$ have been eliminated in the previous step. By an isotopy of S, supported near α, push the intersection α across D, thereby eliminating α and decreasing by two the number of points of intersection of S with the 1-skeleton of $\mathfrak{T}$.

After such an isotopy, we repeat the first step of forcing S to intersect all 3-simplices in disks. As this does not increase the number of intersections with the 1-skeleton, after finitely many steps we arrive at the situation where S meets each 2-simplex only in arcs connecting adjacent sides, and S meets 3-simplices only in disks.

Now consider the intersection of S with a 2-simplex σ. With at most four exceptions the complementary regions of $S \cap \sigma$ in σ are rectangles with two opposite sides on $\partial\sigma$, then all but at most $4t$ of the components of $M|S$ meet all the 2-simplices of $\mathfrak{T}$ only in such rectangles.

If R is a component of $M|S$ meeting all 2-simplices only in rectangles, for a 3-simplex $\mathfrak{t}$, each component of $R \cap \partial\mathfrak{t}$ is an annulus A which is a union of rectangles. The two circles of ∂A bound disks in $\mathfrak{t}$, and A together with these two disks is a sphere bounding a ball in $\mathfrak{t}$, and A together with these two disks is a sphere bounding a ball in $\mathfrak{t}$, and A together with these two disks is a sphere bounding a ball in $\mathfrak{t}$, a component of $R \cap \mathfrak{t}$ which can be written as $D^2 \times I$ with $\partial D^2 \times I = A$. The I-fiberings of all such products $D^2 \times I$ may be assumed to agree on their common intersections, the rectangles, to give R the structure of an I-bundle. Since ∂R consists of sphere components of S, R is either the product $S^2 \times I$ or the twisted I-bundle over $\mathbb{R}P^2$. This is just $\mathbb{R}P^3$ minus a ball, so each I-bundle R gives a connected summand $\mathbb{R}P^3$ of M, hence a $\mathbb{Z}_2$-direct summand of the fundamental group of M.

Thus the number of such components is bounded, and the number of other components bounded by $4t$, this shows the finiteness of the prime decomposition.

2.3.2 Part 2: Uniqueness of a Prime Decomposition

Uniqueness portion: First we prove the existence of the decomposition. If M is not itself prime, then it admits a decomposition $M \approx M_1 \# M_2$ with M_1 and M_2 not S^3. Repeat this process on M_1 and M_2 if they are nonprime. This indicates that a prime decomposition exists, we are left with checking finiteness of the decomposition.

We prove the uniqueness of this decomposition. It suffices to show the following result.

Lemma 2.5 *If $M = M_1 \# M_2$ and $M \approx P_1 \# \cdots \# P_k$, there exists a re-numbering of $P_1, \cdots, P_k$ such that $M_1 \approx P_1 \# \cdots \# P_r$ and $M_2 \approx P_{r+1} \# \cdots \# P_k$.*

Proof Let us consider only manifolds bounded by finitely many 2-spheres. We say that two manifolds are equivalent if one can be obtained from the other by removing the interiors of finitely many disjiont interior 3-cells, or by filling in the interiors of finitely many disjoint 3-cells.

For our decomposition, let us suppose that $P_1, \cdots, P_s$ are irreducible while $P_{s+1}, \cdots P_k \approx S^1 \times S^2$. Let T separate M_1 and M_2. We consider the cases of $s = 0$ and $s > 0$.

1. Case 1: $s > 0$ Let us consider the collection of 2-spheres $\{\Sigma_1, \cdots, \Sigma_{k-1}\}$ with the properties

    (a) $(S \# T) \cap \Sigma_n$ are unions of closed curves.

    (b) Cutting M along Σ_n, we obtain a manifold with boundary with components $B_1 \cdots B_s$ with B_i equivalent to P_i.

    We show that such systems exist. Consider $M = P_1 \# \cdots \# P_s$ along with $k - s$ handles, where $\Sigma_s, \cdots, \Sigma_{k-1}$ cut out such handles. By a so-called general position argument, the first condition is satisfied.

    Suppose now that the union of Σ_n does intersect T. We construct a new system of Σ_i that has fewer intersections. Among the curves $T \cap \Sigma_n$, choose a curve C that bounds a 2-cell $E \subset T$ containing no other intersection curves. E is contained in some manifold, say B_i and let B_j denote the manifold on the "other side" of Σ_h from B_i.

    Since B_i is equivalent to an irreducible P_i, E must cut B_i into two parts B_i' and B_i'', one of which equivalent to S^3. Let E' and E'' denote the corresponding 2-cells Σ_n bounded by C.

    (a) $i \neq j$
    Suppose $B_i'' \approx S^3$. Consider the 2-sphere Σ_n' formed by $E \cap E'$ by deforming slightly into B_i'.Then Σ_n' has fewer intersection curves with T than Σ_n. If Σ_n is replaced by Σ_n', then the effect is to subtract B_i'' from B_i are add it to B_j; but $B_i'' \approx S^3$, so this doesn't change the equivalency classes.

    (b) $i = j$ WLOG B_i' contains the "other side" of Σ_n. Again, let Σ_n' be obtained by deforming $E \cup E'$ slightly into B_i'. If Σ_n gets replaced by Σ_n', then the effect on B_i is to subtract B_i'' from one part of B_i and add it back on in the other part, so the equivalency classes are not changed.

    Thus the two conditions are satisfied and our new collection replacing Σ_n by Σ_n' satisfies the two conditions. This means that we can eliminate all intersection curves and suppose WLOG that $\{\Sigma_1, \cdots \Sigma_{k-1}\}$ is totally disjiont with T.

    Suppose that T lies within B_i and cuts it into B_i' and B_i'' where B_i'' is equivalent to S^3, and furthermore that $B_1 \cdots B_{i-1}, B_i'$ lie on M_1-side while $B_i'', B_{i+1}, \cdots B_s$ lie on M_2 side. Then clear M_1 is isomorphic to $P_1 \# \cdots \# P_i$ with a certain number of handles attached while M_2 is isomorphic to $P_{i+1} \# \cdots \# P_s$ with a certain number of handles attached. This completes the proof for case 1.

2. Case 2: $s = 0$

Here, the proof is the same as case 1, we simply use k disjoint spheres rather than $k-1$ and the following condition that should we cut M along the k spheres, we obtain a connected manifold with boundary equivalent to S^3, in place of condition 2.

Otherwise the argument is the same. This completes the proof of the lemma.

This treatment was taken from [4], [5], and [7].

2.4 A Brief Discussion of the proof of Kneser's Theorem

A 'general position argument' simply consists of performing an isotopy on whatever manifold we're dealing with. This induces a slight local change that allows us to make simplifying local assumptions regarding the manifold. An isotopy is just a small homotopy. So far as the author has been unable to find any precise justification for making this type of argument.

The assertions about the fundamental group for various manifolds are left to the reader. They follow by the Van Kampen's Theorem for Fundamental Groups and Grushko's Theorem.

Theorem 2.6 (Van Kampen) *Let X be a topological space which is the union of two open, path-connected subspaces, U_1 and U_2. If $U_1 \cap U_2$ is path-connected and nonempty, let x_0 be a point in X to be used as the base of all fundamental groups. Then X is path-connected, and its fundamental group is the free product of $\pi_1(U_1)$ with $\pi_1(U_2)$ modulo the amalgamation of $\pi_1(U_1 \cap U_2, x_0)$.*

That is to say, we know that there exists embedding $f : \pi_1(U_1 \cap U_2) \to \pi_1(U_1)$ and $g : \pi_1(U_1 \cap U_2) \to \pi_1(U_2)$, so we start with the free product of $\pi_1(U_1)$ and $\pi_1(U_2)$, and then take the quotient by N, where N is the relation $f(u)g(u)^{-1} = 1$. We then write $(\pi_1(U_1) * \pi_1(U_2))/N = \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi(U_2)$, and this is the fundamental group of X.

We can use this to show multiple things, for example that $S^1 \times S^2$ is the only oriented 3-manifold that is prime but not irreducible.

Grushko's Theorem, on the other hand, allows us to identify ranks of products, which is used to prove the finiteness of decompositions.

Theorem 2.7 (Grushko) *If A and B are finitely generated groups, $A * B$ their free product. Then the rank is additive. In fact,*

$$rank(A * B) = rank(A) + rank(B).$$

Though useful tools, these two theorems are only tengentially related to our topic. As such, we will not go into further detail in discussing them. Their proofs are widely available both online and in book form.

3 Introducing Geometries

Let us first recall some basic definitions surrounding Riemannian manifolds.

Definition A Riemannian manifold is a pair (M, g) where M is a smooth manifold and g consists of a series of inner products g_p at each point $p \in M$ such that for any smooth vector fields x, y on M, the map

$$p \mapsto g_p(x(p), y(p))$$

is smooth.

If a manifold M admits a Riemannian structure, then we can define geometric notions such as angle, length of curves, volume, curvature, and gradient.

Next we define isometries.

Definition If (M, g) and (M', g') are Riemannian manifolds, an isometry is a diffeomorphism $f : M \to M'$ such that $g = f^*g'$, where f^*g' denote the pullback of g' by f. If f is a local diffeomorphism, then say f is a local isometry. The set of isometries from M to itself form a group under composition and is denoted $\text{Isom}(M)$.

3.1 Coverings and Deck Groups

Definition Let $p : E \to B$ be a continuous surjection between topological spaces E, B, say that an open set $U \subset B$ is evenly covered by p if

$$p^{-1}(U) = \cup_\alpha V_\alpha,$$

where the $V_\alpha \subset E$ are disjoint open sets in E such that for each α, $p|_{V_\alpha}$ is a homeomorphism onto U.

If every point b of B has a neighborhood that is evenly covered by p, we call p a covering map and E a covering space of B. Given a covering map $p : E \to B$, the space of automorphisms of E is called the *deck group* or *covering group* and is denoted by $\mathcal{C}(E, p, B)$, such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ B & \xrightarrow{id} & B \end{array}$$

Typically we denote the deck group simply as $\mathcal{C}$, dropping the E, p, B portion.

Definition If X, Y are topological spaces and $h : X \to Y$ with $h(x_0) = y_0$, let us define the map $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*([f]) = [h \circ f]$. Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Define the group $H_0 = p_*(\pi_1(E, e_0))$. Note that H_0 is a subgroup of $\pi_1(B, b_0)$. We say that p is a *regular covering map* if $H_0 \subset \pi_1(B, b_0)$ is a normal subgroup.

Now we move to define geometric structures.

First we will examine geometric structures in two dimensions. Thanks to the Uniformization Theorem, we have a complete understanding of two-dimensional geometric structures. We will use this as our basis for understanding three-dimensional geometric structures.

3.2 Geometric Structures

First we will define geometric structures for 2-manifolds. Here $\mathrm{Isom}(X)$ denotes the isometry group.

Definition Let X be one of the spaces $\mathbb{R}^2$, S^2, or H^2 (hyperbolic 2-space), and $\Gamma \subset \mathrm{Isom}(X)$. If F is a 2-manifold with $F \simeq X/\Gamma$, and the projection $X \to X/\Gamma$ a covering map, we say that F has a geometric structure modelled on X.

Definition A metric on a manifold M is locally homogenous if for all $x, y \in M$, there exist neighborhoods $x \in U$ and $y \in V$ and an isometry $f : U \to V$. We say the metric is homogenous if for all $x, y \in M$ there exists an isometry of M sending x to y. Say that the metric is complete if M is complete as a metric space.

Generally, we say that a manifold M admits a geometric structure if it can be equipped with a complete and locally homogenous metric. It can be shown that the only possible 2-dimensional geometric structures are based off these three spaces.

Theorem 3.1 (Uniformization Theorem) *Every simply connected Riemann Surface is conformally equivalent to either the open unit disk, complex plane, or Riemann sphere.*

Though this is phrased in the complex-analytic term of 'conformally equivalent,' in fact we have an equivalent statement for Riemannian manifolds.

Theorem 3.2 (Uniformization Theorem, Version 2) *Every oriented, compact, connected surface admits a geometric structure modelled on $\mathbb{R}^2$, S^2, or H^2; the plane, Riemann sphere, and hyperbolic plane respectively.*

This is shown essentially by giving a oriented Riemannian manifold a Riemann surface structure as follows. On an oriented surface, a Riemannian metric induces an 'almost complex' structure as follows. For a tangent vector v, we define $J(v)$ as the vector of the same length which is orthongonal to v, such that $(v, J(v))$ is positively oriented. This turns the given surface into a Riemann surface.

This is a differential-geometric version of the Uniformization Theorem, and sometimes also called the Uniformization Theorem.

Lemma 3.3 *If M is a Riemannian manifold and admits a geometric structure; X its universal covering space, then there exists a subgroup $\Gamma \subset \mathrm{Isom}(X)$ such that M is isometric to X/Γ. Specifically, Γ will be the deck group of X.*

Proof Let M be such a manifold and $\tilde{M}$ its covering space. The covering space inherits a pullback metric, such that the projection of $\tilde{M}$ onto M is a local isometry. Thus if M admits a geometric structure, M has a complete, locally homogenous metric, so the inherited metric is also complete and locally homogenous. The inherited metric is thus complete and locally homogenous.

We use the fact that a locally homogenous metric on any simply connected manifold is homogenous. A universal covering space is always simply connected, so the metric that X inherits from M is homogenous (the universal covering space being just the simply connected covering space).

Γ is the isometry group of X, by definition for any $x, y \in X$, there exists a $\gamma \in \Gamma$ such that $\gamma(x) = y$, so Γ acts transitively on X.

Now if $\psi \in \mathcal{C}$, then ψ is a local isometry, and as it is a diffeomorphism, it is a global isometry. Thus the deck group of X is a subgroup of Γ. If we have $p : X \to M$ and X is a universal covering space, then H_0 is trivial, hence normal, and p is a regular covering map, so $M \simeq X/\mathcal{C}$. Thus, given M admitting a geometric structure, we can write M as a quotient of its universal cover.

Definition A geometry is a simply connected homogenous Riemannian manifold X with isometry group Isom(X). M has a geometric structure modelled on X if there exists a subgroup $\Gamma \subset \mathrm{Isom}(X)$ such that M is isometric to X/Γ.

Remark Call two geometries equivalent, $(X, \Gamma) \simeq (X', \Gamma')$ if there exists an isomorphism $\Gamma \to \Gamma'$ and an equivariant map $\phi : X \to X'$, i.e. $\phi(g \cdot x) = g' \cdot \phi(x)$, where g' is the isomorphic image of g in Γ'.

Call a geometry (X, G) maximal if there exists no geometry (X, G') with $G \subsetneq G'$.

One of the key issues in understanding Thurston's Geometrization Conjecture is to study what subgroups of th eisometry group will actually generate a Riemannian manifold modelled on X. The answer is that if a subgroup $\Gamma \subset \mathrm{Isom}(X)$ acts freely and properly discontinuously on X, then X/Γ is Riemannian. We call these subgroups discrete subgroups.

Definition A group G acts properly discontinuously on a space X if for any compact subset $C \subset X$, the set

$$\{g \in G : gC \cap C \neq \emptyset\}$$

is finite.

If G acts properly discontinuously on X and $p \in X$, then Stab(p) if finite.

Definition If G is a group acting on a space X and Stab(p) is trivial for all $p \in X$, then we say that G acts freely on X.

We say a subgroup $G \subset \mathrm{Isom}(X)$ is discrete if it acts freely and properly discontinuously on X.

The following theorem ties these notions together.

Theorem 3.4 *Suppose X is a connected smooth manifold and Γ is a finite or countably infinite group with the discrete topology acting smoothly, freely, and properly discontinuously on X. Then the quotient space X/Γ is a topological manifold and has a unique smooth structure such that $\pi : X \to X/\Gamma$ is a smooth normal covering map.*

Next we move to discuss orbifolds.

3.3 Orbifolds

We will define some basic notions relating to orbifolds and relate them to the present discussion.

First come a collection of basic topological terms.

Definition If G is a discrete group of isometries for a Riemannian manifold M that acts freely on M, taking the quotient M/G and deploying the natural metric generates a Riemannian manifold. On the other hand, if G does not act freely, the quotient space still has a natural metric but it is no longer a Riemannian manifold. One example of this is that if we have $\mathbb{Z}_2$ acting on $\mathbb{R}^3$ where the nontrivial action takes $x \mapsto -x$, then $\mathbb{R}^3/\mathbb{Z}_2$ is homeomorphic to a cone on $\mathbb{P}^2$ but is not a manifold (because of the behavior at the cone point).

An n-manifold without boundary is a Hausdorff, paracompact space, i.e. every open cover has a locally finite refinement. Locally finite means that for every point in the space, there is a neighborhood of it that intersects only finitely many elements in the cover. Further, it is equipped with smooth atlas of charts with smooth intersection.

These allow us to define orbifolds.

Definition An n-orbifold to be a Hausdorff, paracompact space that is locally homeomorphic to $\mathbb{R}^m/G$, along with a covering $\{U_i\}$ of open sets which are closed under taking finite intersections. To each U_i we associate a Γ_i, an action of Γ_i on $\overline{U_i} \subset \mathbb{R}^n$, and a homeomorphism $\phi_i : \overline{U_i}/\Gamma_i \to U_i$ such that if $U_i \subset U_j$, there exists an injection $f_{ij} : \Gamma_i \to \Gamma_j$ and $\bar{\phi_{ij}} : \overline{U_i} \to \overline{U_j}$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\overline{U_i} & \xrightarrow{\overline{\phi_{ij}}} & \overline{U_j} \\
\downarrow & & \downarrow \\
\overline{U_i}/\Gamma_i & \xrightarrow{\phi_{ij}} & \overline{U_j}/(f_{ij}\Gamma_j) \\
\downarrow & & \downarrow \\
U_i & \xrightarrow{i} & U_j
\end{array}
$$

While the collections $\{U_i\}$ are not considered an intrinsic part of the "orbifold" structure, we typically take the $\{U_i\}$ to be the maximal collection.

If G acts properly discontinuously on M, then M/G is a smooth orbifold. In dimension 2, orbifold and manifold are the same concept.

3.4 Basic Facts about Fiber Bundles, Seifert Fiber Spaces

Many of our manifolds with geometric structures can be described as Fiber bundles or Seifert fibered spaces. It is therefore important to get a handle on these.

We start with defining fiber bundles.

Definition Let E, B, and F be topological spaces and $\pi : E \to B$ a continuous surjection. If π satisfies a triviality condition (we will touch on this later), we say that the collection (E, B, π, F) is a fiber bundle. The triviality condition is as follows; for all $x \in B$, there exists a open neighborhood U of x and a homeomorphism

$$\phi : \pi^{-1}(U) \to U \times F$$

such that $\text{proj}_U \cdot \phi = \pi_{\pi^{-1}(U)}$.

This captures the idea that E is locally homeomorphic to $B \times F$. If E is globally homeomorphic to $B \times F$, we say that E is a trivial bundle over B.

We also have the following terminology.

Remark An I-bundle is a fiber bundle where the fiber (corresponding to B) is an interval. If this interval is $\mathbb{R}$, we call this a line bundle.

Now we define Seifert fibered spaces.

Definition Let $i = [0, 1]$, D^2 the unit disk, and $D^2 \times I$ be the solid fibered cylinder with fibers $x \times I$ (where $x \in D^2$). Then a fibered solid torus is obtained by identifying $D^2 \times \{1\}$ with $D^2 \times \{0\}$, and by rotating $D^2 \times \{1\}$ in the process.

Regardless of the rotation, the fiber corresponding to $(0, 0) \times I$ is unchanged. This is called the middle fiber.

Definition A fiber-preserving map is a homeomorphism between two fiber bundles that map fibers to fibers.

Given these definitions, we now define a Seifert fibered space.

Definition A Seifert fiber space is a 3-manifold M that is a disjoint union of fibers such that each fiber H has a fiber neighborhood, that is, a subset of fibers containing H that can be mapped under a fiber-preserving map onto the solid fibered torus, with H mapped to the middle fiber.

Given these definitions, we can think of a Seifert space as a fiber bundle over a base space where the fibers are circles.

One important example of a Seifert fiber space is S^3 with the Hopf fibration. To define this, consider $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$, i.e.

$$S^3 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}.$$

Then we can think of S^3 as equivalent to $\mathbb{C}P^1$. However, we can also think of S^2 as $\mathbb{C}P^1$ (we have a stereographic projection $S^2 \to \mathbb{C}^*$).

Definition The Hopf map $h : S^3 \to S^2$ is given by $h(z_1, z_2) = [z_1, z_2] \in \mathbb{C}P^1$.

If we think of S^3 as lying in $\mathbb{C}^2$ and $S^2 \cong \mathbb{C}^*$, and define $\tilde{h}(z_1, z_2) = \frac{z_1}{z_2}$, then $\tilde{h}^{-1}(\lambda)$ is the circle in S^3 given by $z_1 = \lambda z_2$ (with $\lambda \in \mathbb{C}^*$).

Theorem 3.5 *(S^3, S^2, h, S^1) is a fiber bundle.*

Proof We first show that the map h is a surjection. If $[z_1, z_2] \in S^2$ (considering $S^2 \cong \mathbb{C}P^1$), Normalize this by taking $\lambda = (|z_1|^2 + |z_2|^2)^{\frac{-1}{2}}$ and choose the representative $[\lambda z_1, \lambda z_2]$. This lies in S^3 and $h(\lambda z_1, \lambda z_2) = [z_1, z_2]$. Hence h is surjective.

To show the triviality condition, take a point $x \in S^2$ and let $\bar{x}$ be its antipodal point. Take $U = S^2 - \{\bar{x}\}$; this is an open neighborhood of x in S^2. Let us define the map

$$\phi : h^{-1}(U) \to U \times S^1$$

by

$$(z_1, z_2) \mapsto ([1, \frac{z_2}{z_1}], \frac{z_2}{z_1}).$$

Then we have $\text{proj}_U \circ \phi = h|_{h^{-1}(U)}$, giving the required triviality condition.

Now consider $h^{-1}([z_1, z_2])$. This maps to all the points $(\lambda z_1, \lambda z_2)$ in S^3 with $|\lambda| = 1$, precisely a great circle in S^3. Thus the fiber associated with this fiber bundle is indeed a copy of S^1.

Here the definitions and results are taken from [3] and [9].

4 Geometric Structures in two and three dimensions

We will try to get a feel for working with geometric structures by first tackling the two-dimensional geometries. We have a complete classification via the Uniformization Theorem.

4.1 Two-Dimensional Geometries

There are three possible bases for a two-dimensional geometry; $\mathbb{R}^2$-Euclidean 2-space, equipped with the usual Euclidean metric, S^2-the 2-sphere, equipped with the Euclidean metric as we think of S^2 being embedded in $\mathbb{R}^3$, and H^2-hyperbolic 2-space. We think of H^2 as the set $\{x + iy : y > 0\} \subset \mathbb{C}$, with the metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$.

To prove that all two-dimensional geometric structures are based off these, we use some facts relating to orbifolds, in particular that in dimension 2 there is no distinction between manifolds and orbifolds.

4.1.1 The case of $\mathbb{R}^2$

Here, the isometries of $\mathbb{R}^2$ consist of translations, reflections, rotations, and glide reflections, and we may express any isometry of $\mathbb{R}^2$ as $x \mapsto Ax+b$, where A is orthogonal and b is a vector in $\mathbb{R}^2$. One can verify that there is a group homomorphism $\text{Isom}(\mathbb{R}^2) \to O(2)$ given by $(x \mapsto Ax + b) \mapsto A$. This is surjective and the kernel of this map is the set of all translations on $\mathbb{R}^2$.

As we are interested in the discrete subgroups of the isometry group, noting that any rotations and reflections will have fixed points, the discrete subgroups are generated possibly by (1) a single translation, (2) two translations, (3) one glide reflection, and (4) one glide reflection and one translation.

Examining the respective quotients, we obtain respectively (1) cylinder, (2) torus, (3) Mobius strip, and (4) Klein bottle (viewing as the connected sum of two projective planes).

4.1.2 The case of S^2

We have a natural embedding of S^2 into $\mathbb{R}^3$. Note that any isometry of $\mathbb{R}^3$ fixing the origin restricts to an isometry of S^2, and that any isometry of S^2 extends naturally to an isometry of $\mathbb{R}^3$ which fixes the origin. Hence we have that $\text{Isom}(S^2) \equiv O(3)$.

Any orientation-preserving isometry fixes either a line or a great circle on S^2, so the only isometry with no fixed points is the antipodal map $x \mapsto -x$. Hence the two possible quotients are S^2 or $S^2/\{\pm 1\} \equiv \mathbb{R}P^2$.

4.1.3 The case of H^2

This is possibly the most interesting case. One can verify that the geodesics in H^2 are lines and semi-circles with center on the real axis. The orientation-preserving isometries are the LFTs on the upper half-plane, i.e. the group

$$G = \{z \mapsto \frac{az+b}{cz+d} : ad - bc > 0\}$$

where multiplication in G corresponds to composition. The orientation-reversing ones are precisely the composition with the conjugation operation. Note of course that we can think of $G \equiv PGL_2(\mathbb{R})$, and then in examining the group structure the relevant results follow through.

We consider the two subgroups generated by $z \to \lambda z$ and $z \to \lambda \bar{z}$. These generate the annulus and Mobius strip respectively.

These are the only three two-dimensional geometries. The above thus constitutes a full characterization of two-dimensional geometric structures.

4.2 Three-Dimensional Geometries

Recall that the eight three-dimensional geometries are given as below

1. $\mathbb{R}^3$, with the usual Euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$.

2. S^3, with the Euclidean metric on $\mathbb{R}^4$, $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$.

3. H^3; consider H^3 as a subset of $\mathbb{R}^3$,

$$\{(x, y, z) : z > 0\}$$

and being endowed with the metric $ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2)$ rather than the normal Euclidean metric.

4. $S^2 \times \mathbb{R}$; this admits the product metric of S^2 with $\mathbb{R}$ (the Euclidean metric). In this case, there are actually only 7 manifolds without boundary that have a structure modelled on this. We explore this at a later point in time.

5. $H^2 \times \mathbb{R}$; this admits the product metric of H^2 with $\mathbb{R}$. There are infinitely many manifolds modelled on this.

6. $\overline{SL_2(\mathbb{R})}$; this is the universal covering space of the Lie group of $SL_2(\mathbb{R})$. The metric is given by taking the tangent bundle of H^2, which is isomorphic to $PSL_2(\mathbb{R})$. Now $PSL_2(\mathbb{R})$ is coverred by $SL_2(\mathbb{R})$, and the metric on H^2 pulls back to induce a metric on $SL_2(\mathbb{R})$.

7. (Nil): This is the three-dimensional Lie group of all real 3 by 3 upper-triangular matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

The Lie group is nilpotent, can be identified with $\mathbb{R}^3$, and induces the metric $ds^2 = dx^2 + dy^2 + (dz - ydx)^2$. We can think of this as $\mathbb{R}^3$ with the multiplication $(x, y, z)(x', y', z') = (x + x', y' + xz' + y', z + z')$.

8. (Sol): This can be thought of $\mathbb{R}^3$ with the multiplication

$$(x, y, z) \cdot (x', y', z') = (x + e^{-z}x', y + e^{z}y', z + z'),$$

which induces the metric

$$ds^2 = e^{2x}dx^2 + e^{-2z}dy^2 + dz^2.$$

This is called *Sol* because it is a solvable group.

We will not prove that these are the only three-dimensional geometries worth considering. This is a very deep result that was one of Thurston's main motivations for the Geometrization Conjecture. Recall that Thurston's Geometrization Conjecture then states the following

Theorem 4.1 *Every oriented, closed, prime 3-manifold can be cut along tori so that the interior of the resulting manifolds admits one of the above 8 geometric structures.*

We have seen that to find the various manifolds having geometric structure modelled on one of those eight mentioned above, it suffices to examine their isometry groups and find their discrete subgroups.

5 Classifying Geometric Structures

Generally speaking, there are three constant-curvature Riemannian structure.

Theorem 5.1 (Classification of Constant Curvature Metrics, [1], Introduction) *If M is a complete, simply connected Riemannian manifold with constant sectional curvature, then $M \equiv S^n, \mathbb{R}^n, \mathbb{H}^n$.*

These have curvature $+1, 0, -1$ respectively. This was a theorem proved by Hopf in the 1920s.

Two other manifolds, $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, are products of Riemannian manifolds, which we see to be given a canonical Riemannian metric structure. We define the product Riemannian metric as follows.

Definition If (M_1, g^1) and (M_2, g^2) are two Riemannian manifolds. We identify

$$T_{(p_1,p_2)}(M_1 \times M_2) \equiv T_{p_1}M_1 \oplus T_{p_2}M_2,$$

and there exists a canonical Riemannian metric

$$g = g^1 \oplus g^2$$

given by

$$g_{(p_1,p_2)}(u_1, u_2, v_1, v_2) = g^1_{p_1}(u_1, u_2) + g^2_{p_2}(v_1, v_2)$$

We then have to physically derive the cases of $\overline{SL_2(\mathbb{R})}$, Nil, and Sol. This process is too difficult to include here.

We begin with a classification of $S^2 \times \mathbb{R}$. This is the simplest manifold to work with as it has only seven distinct quotients. Recall that we want to compute the discrete subgroups of the isometry group $Isom(S^2 \times \mathbb{R})$.

5.1 Geometric Structures on $S^2 \times \mathbb{R}$

We consider the case of $S^2 \times \mathbb{R}$.We make the natural identification of Isom$(S^2 \times \mathbb{R}) \equiv$ Isom$(S^2) \times$ Isom$(\mathbb{R})$ based on considerations of the product metric.

This result follows from noting that S^2 has positive Ricci curvature and that $\mathbb{R}$ has zero Ricci curvature. A variety of other results that look at the isometries of S^2 and $\mathbb{R}$

will imply this result. We will not go into more detail regarding this issue as it is only tangential.

Isom(S^2) is just $O(3)$ and Iso$m$($\mathbb{R}$) is just $O(1)$. The first is given by the identity/antipodal maps as well as rotations/reflections, while Isom($\mathbb{R}$) is just given by translations/reflections. There are only a few ways to form a discrete subgroup of Isom(S^2) $\times$ Isom($\mathbb{R}$).

Consider an isometry (α, β) and let G be the group generated by this element. We proceed by cases.

If α is the identity, then β acts freely if and only if β is a translation; then $\mathbb{R}/(\beta) = S^1$, so $(S^2 \times \mathbb{R})/G = S^2 \times S^1$.

If β is the identity, then α acts freely if and only if α is the antipodal map; hence $S^2/(\alpha) = \mathbb{R}P^2$, and the quotient $(S^2 \times \mathbb{R})/G = \mathbb{R}P^2 \times \mathbb{R}$.

Thus if at least one of α, β is the identity, the possible quotients are $S^2 \times \mathbb{R}$, $\mathbb{R}P^2 \times \mathbb{R}$, and $S^2 \times S^1$.

In parallel with these cases, if $G = \langle \alpha, \beta \rangle$ where α is the antipodal map and β a translation, the quotient can still be described as "$S^2 \times S^1$". We attempt to describe $(S^2 \times \mathbb{R})/G$. If $x \in S^1$ and $x \in U$ an open neighborhood of x, we have the map

$$\pi_G : (S^2 \times \mathbb{R})/G \to U$$

by π_G given by $(a, b)G \mapsto [b]$, the equivalence class of $[b]$ in $\mathbb{R}$ (which is a point in S^1). So we have that

$$((S^2 \times \mathbb{R})/G, S^1, \pi_G, S^2)$$

satisfies the definition of a fiber bundle. This is not the trivial S^2-bundle over S^1, however.

Let α be the antipodal map and β a reflection of $\mathbb{R}$. Let $H = \langle \alpha, \beta \rangle$. We can define the map

$$\pi_H : (S^2 \times \mathbb{R})/H \to \mathbb{R}P^2$$

by $\pi_H((a, b)H) = [a] \in \mathbb{R}P^2$. If $x \in \mathbb{R}P^2$ and $x \in U$ an open neighborhood of x. Define the map $\phi : (S^2 \times \mathbb{R})/H \to U \times \mathbb{R}$ by $\phi((a, b)H) = ([a], [b])$. Then the composition of the projection and ϕ is π_H. So this is a nontrivial line bundle over $\mathbb{R}P^2$.

This concludes the discussion of all subgroups generated by a single element.

Now consider subgroups of the isometry group generated by two elements.

Suppose α_1 is the antipodal map, β_1 the identity, α_2 the identity, and β a translation on their respective domains. Let G be the group generated by (α_1, β_1) and (α_2, β_2). Then $(S^2 \times \mathbb{R})/G \cong \mathbb{R}P^2 \times S^1$.

If α_1 and α_2 are both antipodal maps, β_1 and β_2 are distinct reflections in $\mathbb{R}$ with H generated by (α_1, β_1), (α_2, β_2). Then $(S^2 \times \mathbb{R})/H$ is the union of two nontrivial I-bundles over $\mathbb{R}P^2$.

This completes the classifications of geometric structures over $S^2 \times \mathbb{R}$. There are seven of them.

We have also described each quotient as something endowed with a fiber bundle structure, which was part of our original goal.

5.2 Geometric Structures on S^3

Another class of geometric structures to consider are those modeled on S^3. Working in S^3 has the advantage that it is compact, hence a discrete subgroup of the isometry group is finite.

The main result that we obtain here is that every manifold with a geometric structure modeled on S^3 is a Seifert fiber space. Our analysis is aided by the fact that S^3 is compact, so any discrete subgroup will be finite. Here we will alternately consider our groups both as groups and manifolds.

We will alternately take one of the possible embeddings for S^3 at our convenience, depending on the situation

1. $S^3 \subset \mathbb{R}^4$
2. $S^3 \subset \mathbb{C}^2$
3. Consider S^3 as the unit quaternion group $\mathbb{H}^\times$, by taking the isomorphism $(z_1, z_2) \mapsto z_1 + z_2 j$, where $z_1, z_2, \in \mathbb{C}$

We see that S^3 inherits the Euclidean metric from $\mathbb{R}^4$, and its geodesics are the great circles. Furthermore, we know that its isometry group is $O(4)$. As usual, we want to find the discrete subgroups of $O(4)$.

Consider S^3 as the unit quaternion group. Then we have the following result.

Lemma 5.2 *For all $x, q_1, q_2, \in S^3$, $x \mapsto q_1 x q_2^{-1}$ is an isometry of S^3.*

This induces a group homomorphism $\phi : S^3 \times S^3 \to SO(4)$ by mapping $(q_1, q_2) \mapsto (x \mapsto q_1 x q_2^{-1})$. This homomorphism is surjective and its kernel is $\{(1,1), (-1,-1)\}$, hence every orientation-preserving isometry has the above form also.

Proof $x \mapsto q_1 x q_2^{-1}$ is certainly an isometry. We look at ϕ. Clearly ϕ is an homomorphism whose image is in $O(4)$.

Since S^3 is connected, so is $S^3 \times S^3$. Its image under ϕ is still connected in $O(4)$ and it contains the identity, so it lies in $SO(4)$. Now $S^3 \times S^3$ is six-dimensional, so its image ϕ is also six-dimensional. But $SO(4)$ is six-dimensional itself, so ϕ has image the entirety of $SO(4)$.

Now we examine the kernel. If $x \mapsto q_1 x q_2^{-1}$ is the identity, take $x = 1$, then we must have $q_1 = q_2$ thus $q_1 x = x q_1$ for all $x \in S^3$, which occurs only for $q_1 = \pm 1$. Hence the kernel is precisely $\{(1,1), (-1,-1)\}$.

Thus $S^3 \times S^3$ is a double cover of $SO(4)$.

Lemma 5.3 *(1) Any isometry $\alpha : S^3 \to S^3$ of the form $\alpha(x) = xq$ for $q \in S^3$ preserves the Hopf fibration.*

(2) Any isometry $\alpha : S^3 \to S^3$ of the form $\alpha(x) = qx$ where $q = (w_1, 0)$ or $(0, w_2 j)$ for $w_1, w_2 \in S^1$ preserves the Hopf fibration.

Proof We simply verify that the fibers of the Hopf fibration are taken to other fibers.

Lemma 5.4 *If α is an isometry of the form $\alpha(x) = q_1 x q_2^{-1}$, then α has a fixed point if and only if q_1, q_2 are conjugate.*

Proof This is clear.

Lemma 5.5 *If $G \subset SO(4)$ is a subgroup that acts freely on S^3 of order 2, then $G = \{\pm I\}$. Hence a finite subgroup of SO(4) acting freely on S^3 can have at most one element of order 2.*

Proof If such a map $\alpha(x) = q_1 x q_2^{-1}$ is of order 2, then $xq_2^2 = q_1^2 x$ for all x. At $x = 1$ this gives $q_1^2 = q_2^2$, or equivalently that both commute with all $x \in S^3$, so $q_1^2 = q_2^2 = \pm 1$. If they are equal to -1, the unit quaternions have fixed points, so $q_1^2 = q_2^2 = 1$.

We can now consider $SO(3)$ as a quotient of S^3. We have the following.

Lemma 5.6 *For any $q \in S^3$, we let $\psi(q) = \alpha_q$, where α_q is the map $\alpha_q(x) = qxq^{-1}$. Then ψ is a map $S^3 \to SO(3)$ with kernel ± 1, and it gives a map $p : SO(4) \to SO(3) \times SO(3)$ as ϕ and $\psi \times \psi$ has the same kernel.*

We will investigate this map p.

Lemma 5.7 *If $G \subset SO(4)$ acts freely on S^3, then $p(G)$ is a subgroup of $SO(3) \times SO(3)$ that acts freely on $\mathbb{R}P^3$.*

Proof First we want to show that $p(G) \subset \text{Isom}(\mathbb{R}P^3)$. It will suffice to show that $\mathbb{R}P^3 \cong S^3/\{\pm I\}$. Furthermore w eknow by the first isomorphism theorem, etc, that $S^3/\{\pm I\} \cong SO(3)$ also, hence $SO(3)$ and $\mathbb{R}P^3$ are diffeomorphic.

We may take $\alpha : \mathbb{R}P^3 \to \mathbb{R}P^3$ by $x \mapsto u_1 x u_2^{-1}$ for $u_1, u_2 \in \mathbb{R}P^3$; this is still an isometry, for example by considering the quotient $\mathbb{R}P^3 \cong S^3/\{\pm I\}$. Hence every element of $SO(3) \times SO(3)$ is an isometry of $\mathbb{R}P^3$. This gives that p(G) is also a subgroup of $\text{Isom}(S^3)$.

From now on, we will examine $\text{Isom}(\mathbb{R}P^3)$ rather than $\text{Isom}(S^3)$. If $G \subset SO(4)$ and contains an element of order 2, then we obtain S^3/G by first modding out by $\pm I$, letting $p(G)$ operate freely on $\mathbb{R}P^3$, and considering the behavior on $\mathbb{R}P^3$. Either way, we can pass the quotient of G down to $\text{Isom}(\mathbb{R}P^3)$ to work on this group. If G acts freely on S^3 and $\bar{G}$ doesn't, then there exists $g \in G$ and $x \in S^3$ such that $g(x) = -x$ or $g^2(x) = x$, which is a contradiction as G is odd order.

This shows that if G acts freely on $\mathbb{R}P^3$, then its preimage $\bar{G}$ acts freely on S^3.

Corollary 5.8 *If $p(G) \subset SO(3) \times SO(3) \cong \mathbb{R}P^3 \times \mathbb{R}P^3$ acts freely on $\mathbb{R}P^3$, then $p(G)$ does not containa nontrivial element (u_1, u_2) with u_1, u_2 conjugate in $SO(3)$. In particular, this is not of order 2, as if they were then they would be conjugate.*

Lemma 5.9 *If $H \subset SO(3) \times SO(3)$ is a finite subgroup and H_1, H_2 the projections of H onto the first and second summands. If H acts freely on $SO(3)$, then at least one of H_1 or H_2 is cyclic.*

Proof Finite subgroups of $SO(3)$ are either cyclic, dihedral, or take the form of S_4, A_4 or A_5. Each of these subgroups has even order apart from the odd cyclic groups.

Let $H_1 = \{(x, I) : (x, y) \in H \subset SO(3) \times SO(3)\}$ and $H_2 = \{(I, y) : (x, y) \in H \subset SO(3) \times SO(3)\}$. Let $H_i' = H \cap H_i$. I claim that $H_i' \subset H_i$ is normal. This is clear from doing computations with the definition of H_1 and H_2. Furthermore, we have that $H_1/H_1' \cong H_2/H_2'$.

We have $H \supset H_1' \times H_2'$, hence one of H_1' or H_2' must have odd order and be cyclic. WLOG suppose that group is H_1 and that $b \in H_2$ has an element of order 2. Then there exists $a \in H_1$ with $(a, b) \in H$ or $a^2 \in H_1'$. If a^2 has order 2, then $(a^r)^2 = 1$, so $a^r = 1$, or has order 2. If the latter case occurs we generate the obvious contradiction. Hence $a^r = 1$ and H_2' contains an element of order 2 in H_2.

Now either $H_2' = H_2$, in which case $H_1 = H_1'$ and H_1 will be cyclic, or H_2 is not generated by an element of order 2, so it is either either cyclic or isomorphic to A_4. If $H_2 = A_4$, then $H_2' \equiv \mathbb{Z}_2 \times \mathbb{Z}_2$, and H_2/H_2', H_1/H_1' both have order 3. This implies H_1 is cyclic.

This gives rise to the following theorem.

Theorem 5.10 *Let $\Gamma_1 = \phi(S^1 \times S^3)$, $\Gamma_2 = \phi(S^3 \times S^1)$. If $G \subset SO(4)$ a finite subgroup that acts freely on S^3, then G is conjugate in $SO(4)$ to a subgroup of either Γ_1 or Γ_2.*

Proof If G acts freely on S_3, then recalling our definitions of H_1, H_2, we know that $p(G)$ is a subgroup of $H_1 \times H_2$. At least one of H_1 or H_2 is cyclic, WLOG H_1. Then we have $\tilde{G} = \phi^{-1}(p^{-1}(H_1 \times H_2)) \subset S^3 \times S^3$, or $\tilde{G} = \psi^{-1}(H_1) \times \psi^{-1}(H_2)$ where $\psi : S^3 \to SO(3)$ was defined previesouly. We want to show that $\tilde{G}$ is conjugate to a subgroup of $S^1 \times S^3$. Here think of $S^1 \cong \partial\mathbb{D} \subset \mathbb{C}$.

Let $q \in S^3$ be such that $\psi(q)$ is a generator for H_1. Consider $\ker(\psi) = \{\pm 1\}$ as a subgroup of S^1 to consider $\psi^{-1}(H_1)$ as a subgroup of $S^1 \times S^3$.

Hence $\tilde{G}$ is conjugate to a subgroup of $S^1 \times S^3$, as is $p(G)$. So G is conjugate in $SO(4)$ to a subgroup of Γ_1.

While we will be unable to compute the individual manifolds (there are infinitely many of them corresponding to the finite subgroups of $SO(4)$, of course), what we have shown is that every manifold with an S^3-geometric structure is a Seifert fiber space.

Similarly for the other possible geometric structures we will unable to give the explicit constructions as for $S^2 \times \mathbb{R}$. Rather we will generally only be able to give a classification of the discrete subgroups.

5.3 Geometric Structures on $\mathbb{R}^3$

Any isometry in $\mathbb{R}^3$ is of the form

$$x \mapsto Ax + b,$$

where $A \in O(3)$. This generates the short exact sequence

$$0 \to \mathbb{R}^3 \to \mathrm{Isom}(\mathbb{R}^3) \to O(3) \to 1.$$

In [9], Scott notes the following theorem of Bieberbach: A subgroup $G \subset \mathrm{Isom}(\mathbb{R}^3)$ is discrete if and only if (a) G is a finite $\mathbb{Z}$-module or(b) the translation group of G is of finite index in G (recall that the translation group is just the projection from $x \mapsto Ax + b$ to b). We are interested in a discrete subgroup $G \subset \mathrm{Isom}(\mathbb{R}^3)$ which acts freely, in particular, it must be torsion-free as a $\mathbb{Z}$-module.

Lemma 5.11 *Let G be a discrete group of orientation-prserving isometries of $\mathbb{R}^3$ with translation group T isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, and with G/T isomorphic to A_4. Then G contains an element of order 3.*

We consider the first case that $G/\mathbb{Z}$ is finite; in this case, G is infinite cyclic, and $\mathbb{R}^3/G$ is the interior of a solid torus or Klein bottle, hence admits a natural structure of a Seifert fiber space.

Otherwise, let us consider the translation subgroup $T \subset G$. If $T \cong \mathbb{Z} \times \mathbb{Z}$, then $G \cong \mathbb{Z} \times \mathbb{Z}$ or is isomorphic to the Klein bottle group.

We have the following theorem

Theorem 5.12 (Theorem 4.3, [9]) *Let G be a non-cyclic discrete subgroup of isometries of $\mathbb{R}^3$ that acts freely. Then G leaves invariant some family of parallel straight lines in $\mathbb{R}^3$ and $\mathbb{R}^3/G$ is Seifert fibered by circles that are the images of these lines.*

We can describe the structure of $\mathbb{R}^3/G$ in some more detail. This implies that $\mathbb{R}^3$ has a product structure $\mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R}^1$ which is preserved by G. If we consider the natural projection $\mathbb{R}^3 \to \mathbb{R}^2$, the action of G on $\mathbb{R}^3$ descends to an orthogonal $\mathbb{R}^2$-action, whereby we obtain the base orbifold X or $\mathbb{R}^3/G$.

5.4 Geometric Structures on H^3

We make the following identification of

$$H^3 = \{(x, y, z) : z > 0, ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2)\}.$$

Then ds^2 defines the metric. The geodesics are vertical straight lines and great circles of spheres which intersect the xy-plane orthogonally. The isometries are defined by inversions of $\mathbb{R}^3$ by a sphere with center on the xy-plane.

Let us make the identification $(x, y, z) \mapsto (x + yi + zj)$ where i, j are quaternions. Then we have an identification os $PSL_2(\mathbb{C})$ with the isometry group of H^3, by taking $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)=\{w \mapsto \frac{aw+b}{cw+d}\}$.

Now we discuss when isometries in H^3 commute. Let α be some isometry of H^3. Let us examine the elements fixed by α.

Call α a hyperbolic isometry if it fixes x, y and ∞.

If α fixes only a single point, call it hyperbolic. We can conjugate it to make the fixed point ∞.

Lemma 5.13 *If α,β are nontrivial isometries, then they commute if and only if they fix the same points.*

If α is nontrivial, let $C(\alpha)$ denote the set of all isometries that commute with α. Then this group is abelian and isomorphic to either $\mathbb{R}^2$ or $S^1 \times \mathbb{R}$.

An immediate consequence that we get is

Corollary 5.14 *Let M be some 3-manifold with a hyperbolic structure. Then $\pi_1(M)$ cannot contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

5.5 Geometric Structures on $H^2 \times \mathbb{R}$

By examining the product metric, we can make a natural idenfitication of $\text{Isom}(H^2 \times \mathbb{R})$ with $\text{Isom}(H^2) \times \text{Isom}(\mathbb{R})$ by using the characteristic of the product Riemannian metric.

We have a natural foliation of $H^2 \times \mathbb{R}$ by $\{x\} \times \mathbb{R}$, which is then left invariant. Taking a quotient means that said foliation descends to a foliation of lines or circles. In most cases, these are circles, so we get a Seifert fibration.

This gives rise to the following theorem.

Theorem 5.15 *Let G be a discrete subgroup of Isom($H^2 \times \mathbb{R}$) with quotient M, then one of these three conditions holds.*

1. *The natural foliation of $H^2 \times \mathbb{R}$ by lines descends to be a Seifert bundle structure on M.*
2. *The natural foliation of $H^2 \times \mathbb{R}$ by lines gives M the structure of a line bundle over some hyperbolic surface.*
3. *The natural foliation of $H^2 \times \mathbb{R}$ by lines descends to a foliation of M by lines in which each line has non-closed image in M. In this case, G must be isomorphic to one of $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, or the Klein bottle group.*

In the second case, G is isomorphic to the fundamental group of some hyperbolic surface. This includes all countable free groups and the trivial groups.

In the first case that M is closed, M then admits a natural Seifert fibration. In the second and third cases, M cannot be closed.

5.6 Geometric Structures on $\widetilde{SL_2}$

SL_2 is just $SL_2(\mathbb{R})$, i.e. all two by two matrices with real entries and determinant 1. $\widetilde{SL_2}$ denotes the universal covering. It also has a Lie group structure. We have the following theorem.

Theorem 5.16 ([9], Theorem 4.15) *Let G be a discrete group of symmetries for $\widetilde{SL_2}$, acting freely with quotient M. The foliation of $\widetilde{SL_2}$ by vertical lines descends to a foliation of M wtih one of the following cases occurring.*

1. *The foliation gives M the structure of a line bundle over a non-closed surface.*
2. *The foliation is a Seifert fibration.*
3. *The foliation of M is by lines whose image in M is not closed. In this case, G is isomorphic to $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, or the Klein bottle group.*

5.7 Geometric Structures on Nil

Nil is the 3-dimensional Lie group consisting of all 3 by 3 real upper triangular matrices with diagonal entries 1, under multiplication, i.e. matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

with $x, y, z \in \mathbb{R}$. Then Nil admits a natural homeomorphism with $\mathbb{R}^3$ by taking such a matrix to (x, y, z), and admits the natural multiplication

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy'),$$

which is induced from the multiplication on the matrices. The formula for ds^2 is then

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$

This is known as the Heisenberg group and it is nilpotent, with a short exact sequence

$$0 \to \mathbb{R} \to \mathrm{Nil} \to \mathbb{R}^2 \to 0.$$

As Nil is a Lie group, it has a metric, which is invariant under left multiplication, Then it is also a line bundle over $\mathbb{R}^2$, something we see as being induced from the multiplication/metric structure on Nil. We call the fibers vertical. We now obtain an exact sequence

$$0 \to \mathbb{R} \to \mathrm{Isom}(Nil) \to \mathrm{Isom}(\mathbb{R}^2) \to 1.$$

This means that $\mathrm{Isom}(Nil)$ can be decomposed as the direct sum $\mathbb{R} \oplus \mathrm{Isom}(\mathbb{R}^2)$.

We have the following structure Theorem.

Theorem 5.17 ([9],Theorem 4.16) *Let G be a discrete group of isometries of Nil acting freely and with quotient M. The foliation of Nil by vertical lines descends to a foliation of M and exactly one of the following occurs.*

1. *The foliation gives M the structure of a line bundle over a non-closed surface.*
2. *The foliation of M is a Seifert fibration.*
3. *The foliation of M is by lines whose image in M is not closed. In this case, G must be isomorphic to one of $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, or the Klein bottle group.*

This means that a closed 3-manifold with a geometric structure modeled on Nil must inherit the structure of a Seifert bundle from the foliation of Nil by vertical lines. Furthermore, the base orbifold X is a quotient of $\mathbb{R}^2$.

5.8 Geometric Structures on Sol

We may identify Sol as a split extension of $\mathbb{R}^2$ by $\mathbb{R}$, i.e. we have an exact sequence

$$0 \to \mathbb{R}^2 \to \mathrm{Sol} \to \mathbb{R} \to 0,$$

$t \in \mathbb{R}$ acts on $\mathbb{R}^2$ by sending (x, y) to $(e^t x, e^{-t} y)$. For fixed t, this is a linear map on $\mathbb{R}^2$ with the determinant of the corresponding matrix being 1, as well as positive eigenvalues. Such maps are called hyperbolic isomorphisms on $\mathbb{R}^2$.

If we identify Sol with $\mathbb{R}^3$, then multiplication in Sol is given by

$$(x, y, z) \cdot (x', y', z') = (x + e^{-z} x', y + e^z y', z + z').$$

Then $(0, 0, 0)$ is the identity and the xy-plane is a normal subgroup isomorphic to $\mathbb{R}^2$ with associated conjugation action from Sol. The associated metric is given as

$$ds^2 = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2.$$

With this information, we have the following theorem.

Theorem 5.18 ([9], Theorem 4.17) *Let G be a discrete subgroup of Isom(Sol) acting freely on Sol with quotient M. Then the natural 2-dimensional foliation of M gives M the strcture of a bundle over a one-dimensional manifold.*

We have collected some structure theorems about the discrete isometry groups of the eight manifolds, and this allows us to classify the classes of quotients, which function as the 'building blocks' of our individual prime manifolds, per the conclusion of Thurston's Geometrization Conjecture.

6 Ricci Flow

6.1 Definitions and Terminology

Let us recall the following definitions about vector fields on manifolds. Throughout we assume that the manifold M is smooth and admits a Riemannian structure.

Definition Let $C^\infty(M, TM)$ denote the space of vector fields on M. An affine connection of M is a bilinear map

$$C^\infty(M, TM) \times C^\infty(M, TM) \to C^\infty(M, TM)$$

taking $(X, Y) \mapsto \nabla_X Y$, such that for all smooth $f \in C^\infty(M, \mathbb{R})$, we have the following two conditions

1. $\nabla_{f(X)}(Y) = f(\nabla_X Y)$, i.e. ∇ is $C^\infty(M, \mathbb{R})$-linear in the first variable
2. $\nabla_X(f(Y)) = df(X)Y + f\nabla_X Y$, i.e. ∇ satisfies the Lebesgue rule in the second variable.

An affine connection then has the following properties.

1. The value of $\nabla_X Y$ at $x \in M$ depends only on the value of X at x.
2. The value of $\nabla_X Y$ depends on the value of y in a neighborhood of x.
3. If ∇^1, ∇^2 are affine connections at x, we can write

$$\Gamma_x(X_x, Y_x) = \nabla^1_x Y - \nabla^2_x Y,$$

   where $\Gamma_x : T_xM \times T_xM \to T_xM$ is bilinear and smooth with respect to x.
4. If $M \subset \mathbb{R}^n$, the tangent bundle of M is the trivial bundle $M \times \mathbb{R}^n$. There is a 'canonical' definition d on M, any vector field Y is given by a smooth function $V : M \to \mathbb{R}^n$, and $d_x Y$ is the vector corresponding to $dV(x) = \partial_x Y : M \to \mathbb{R}^n$.Any other affine connection ∇ can be written as $d + \Gamma$ where Γ is a smooth bilinear bundle homomorphism.

Recall that a bundle homomorphism $f : E \to F$ is given as follows.

Definition If E and F are fiber bundles over a space M with projections π_E, π_F, then a bundle homomorphism $f : E \to F$ is given by a map such that $\pi_F \circ f = \pi_E$, i.e. the induced and original bundle structures coincide.

In lieu of this, we will also say that $f : E \to F$ is a bundle isomorphism if its inverse $f^{-1} : F \to E$ is a bundle homomorphism. If f is also a diffeomorphism, it is called a smooth bundle isomorphism. In this case E and F are said to be smoothly isomorphic.

Definition If $(M.g)$ is a Riemannian manifold, the unique torsion-free connection ∇ on the tangent bundle TM compatible with the metric is known as the Levi-Civita metric. The existence of such a connection is known as the Fundamental Theorem of Riemannian geometry.

It has the property that if X, Y, Z are vector fields and $\langle , \rangle$ is the metric, let $[X, Y]$ denote their commutators. The Levi-Civita connection is torsion-free, i.e. we have

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

We can then define the Riemannian curvature tensor in terms of the Levi-Civita connection, being

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[v,u]} w.$$

In this case, we may define the Ricci curvature tensor in local coordinates as

$$Ric = R_{ij} dx^i \otimes dx^j,$$

where

$$R_{ij} = R^k_{ikj}.$$

The so-called Ricci flow is given by a 'parabolic evolution equation'

$$\frac{d}{dt} g_{ij}(t) = -2R_{ij},$$

where R_{ij} is the Ricci curvature tensor and g_{ij} is the Riemann curvature tensor.

Definition A metric tensor takes a pair of tangent vectors v, w at points $p \in M$ and producing a scalar $g_p(v, w)$ with the following conditions

1. g_p is bilinear., i.e. $g_p(av_1 + bv_2, w) = ag_p(v_1, w) + bg_p(v_2, w)$, and $g_p(v, aw_1 + bw_2) = ag_p(v, w_1) + bg_p(v, w_2)$.

2. g_p is symmetric, i.e. $g_p(v, w) = g_p(w, v)$.

3. g_p is nondegenerate, i.e. for every $v \neq 0$, there exists a corresponding w such that $g_p(v, w) \neq 0$.

Remark The Riemannian metric structure gives an example of a metric tensor, as does the Ricci curvature tensor.

More generally, metric tensors allow us to define geometric notions such as length and angle on manifolds.

6.2 Two Theorems on Ricci Flow

The study of Ricci flow involves looking at the time-evolution of Ricci curvature on vector fields.

Below are two of Hamilton's foundational results on Ricci flow, we will simply state them without proof. Then we will give a brief explanation as to how these are useful in proving the Geometrization Conjecture.

Theorem 6.1 (Short-Time Existence and Uniqueness) *1. If g_0 is smooth on M, then there exists some ϵ dependent on g_0 such that there is a unique solution g to the Ricci flow equation on $[0, \epsilon)$ with $g(0) = g_0$.*

2. There is a so-called 'curvature characterization ' of singularity information, i.e. if there is a unique solution to the Ricci flow equation on $[0, T)$ but not on any larger interval, then exists x for which the norm of the curvature tensor $R(x, t)$ of $g(t)$ is unbounded as $t \to T$.

This proves that in a neighborhood of 0, the Ricci flow equation has a solution. This will become important, as Perelman constructs ways to remove singularities in the time-evolution of the solution.

Theorem 6.2 (Non-Negative Ricci Curvature) *Let X be a compact connected 3-manifold with positive semidefinite Ricci curvature. Then one of these situations occurs.*

1. *If the Ricci curvature is positive definite for all t small, it develops a singularity in finite time, i.e. the family of metrics is no longer defined on the entirety of X from a certain time onwards. As the singularity develops, the diameter of the manifold goes to 0. Rescaling the family of metrics so that all diameters are 1 leads to a family of metrics converging smoothly to a metric of constant positive definite curvature. In particular, the manifold is diffeomorphic to a spherical space-form.*

2. *There exists a finite cover of the Riemannian manifold which, with the induced metric, is a metric product of a compact surface of positive sectional curvature with S^1. This is true for all Riemannian metrics $g(t)$, which develop singularities in real time. Here the manifold in question is either $S^2 \times S^1$ or the connected sum of two copies of $\mathbb{R}P^3$.*

3. *The metric is flat and both sides of the evolution equation are constant.*

In particular, all manifolds described here do satisfy the conclusions of the Geometrization Conjecture.

Results and definitions are taken from [6] and [8].

6.3 Perelman's Argument

This section is based strongly off of [8].

Perelman's attack on the Geometrization Conjecture involves looking at the finite-time singularities that arise from solving the Ricci flow equation. Hamilton proved that the Ricci flow equation has a unique solution in a neighborhood of 0.

Now there are two types of finite-time singularity on the manifold. The first type is classified as those components of the manifold whose metrics are shrinking in a controlled way. The second type consists of a long, thin tube diffeomorphic to $S^2 \times [a, b]$ or the union of such a tube with a cap of positive curvature on the end. Perelman's surgery process then consists of going to this singular time, removing all regions of the first type (i.e. the components of the manifolds) and perform surgery near the 'large' ends of the long thin tubes to replace them with more standard metrics on the disk.

The combined topological effect of this procedure is to remove some components known to satisfy Thurston's Geometrization Conjecture, as well as performing surgery on the other components, which is equivalent to some form of direct sum decomposition. It then follows that if the manifold post-surgery satisfies Thurston's Geometrization Conjecture, then so does that manifold pre-surgery. Then one can take the result post-surgery and work the Ricci flow on that.

Perelman now claims that based on time-evolution as $t \to \infty$, there is an analoguous theorem valid for Ricci flow with surgery. As manifolds associated to large time satisfy Thurston's Geometrization Conjecture, we see that the original manifold does also.

Now for the sole purposes of proving the Poincare Conjecture, there is no need for this last part, as the Ricci flow with surgery vanishes after a finite time.

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