Mathematical Finance

Option Pricing under the Risk-Neutral Measure

Cory Barnes

Department of Mathematics
University of Washington

June 11, 2013

Outline

We work in $(\Omega, \mathcal{F}, P)$, with a 1-d Brownian Motion W_t, on time interval $[0, T]$.

- $W_0 = 0$ a.s.
- For partition $0 = t_0 < t_1, \ldots, < t_k = T$, increments $W_{t_j} - W_{t_{j-1}}$ independent.
- For $s < t$, increment $W_t - W_s \sim N(0, t - s)$.
- For $\omega \in \Omega$, the mapping $t \mapsto W_t(\omega)$ is continuous a.s.

Definition

A filtration of $\mathcal{F}$ is a an increasing sequence of sub σ-algebras of $\mathcal{F}$, i.e. $(\mathcal{F}_t)$ where $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$.

We take $(\mathcal{F}_t)$ to be the filtration generated by W_t.

All the information available at time t is the data that W_s attained for $0 \leq s \leq t$.

Intuitively, we can make choices on the past and present, but we don't know the future.

Definition

A filtration of $\mathcal{F}$ is a an increasing sequence of sub σ-algebras of $\mathcal{F}$, i.e. $(\mathcal{F}_t)$ where $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$.

We take $(\mathcal{F}_t)$ to be the filtration generated by W_t.

All the information available at time t is the data that W_s attained for $0 \leq s \leq t$.

Intuitively, we can make choices on the past and present, but we don't know the future.

Definition

A stochastic process is a family (X_t) of real valued random variables indexed by time (we take $t \in [0, T]$). It is continuous if $t \mapsto X_t(\omega)$ is continuous almost surely.

We will construct various stochastic processes from (W_t)

Definition

A stochastic process (X_t) is adapted to the filtration $(\mathcal{F}_t)$ if for every $s \in [0, T]$, the random variable X_s is $\mathcal{F}_s$ measurable.

Definition

A stochastic process is a family (X_t) of real valued random variables indexed by time (we take $t \in [0, T]$). It is continuous if $t \mapsto X_t(\omega)$ is continuous almost surely.

We will construct various stochastic processes from (W_t)

Definition

A stochastic process (X_t) is adapted to the filtration $(\mathcal{F}_t)$ if for every $s \in [0, T]$, the random variable X_s is $\mathcal{F}_s$ measurable.

Definition

An $\mathcal{F}_t$-adapted stochastic process (X_t) is a martingale with respect to $\mathcal{F}_t$ if for all $0 \leq s \leq t$,

$$E[X_t|\mathcal{F}_s] = X_s.$$

As an example, take (W_t) on $[0, T]$, w.r.t. $\mathcal{F}_t$. Verify

$$\begin{aligned} E[W_t|\mathcal{F}_s] &= E[W_t - W_s + W_s|\mathcal{F}_s] \\ &= E[W_t - W_s|\mathcal{F}_s] + E[W_s|\mathcal{F}_s] \\ &= W_s. \end{aligned}$$

We wish to make sense of the following object:

$$dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dW_t.$$

- (S_t) stock price as an adapted continuous stochastic process
- Deterministic growth $\mu_t S_t$, proportional to current stock price
- Random Brownian noise, volatility $\sigma_t S_t$, proportional to current stock price
- Both (μ_t) and (σ_t) adapted continuous processes (constants in simple case)

We wish to make sense of the following object:

$$dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dW_t.$$

- (S_t) stock price as an adapted continuous stochastic process
- Deterministic growth $\mu_t S_t$, proportional to current stock price
- Random Brownian noise, volatility $\sigma_t S_t$, proportional to current stock price
- Both (μ_t) and (σ_t) adapted continuous processes (constants in simple case)

How does dW_t behave? Look at quadratic variation. For partition $s_0 < s_1 < \ldots < s_n$ of interval $s_n - s_0 = t$,

$$Q_n = \sum_{j=0}^{n-1} \left(W_{s_{j+1}} - W_{s_j}\right)^2 \to t.$$

The convergence here happens in mean-square, since $E(Q_n) \to t$ and $Var(Q_n) \to 0$. Thus we write the following 'formal' multiplication rule:

$$dW \cdot dW = dt.$$

Similarly, we can show that as $n \to \infty$,

$$\sum_{j=0}^{n-1} \left(W_{s_{j+1}} - W_{s_j}\right)\left(s_{j+1} - s_j\right) \to 0,$$

$$\sum_{j=0}^{n-1} \left(s_{j+1} - s_j\right)^2 \to 0.$$

Thus we add the following to our multiplication rules:

$$dW \cdot dt = 0, \qquad dt \cdot dt = 0.$$

We make sense of the stochastic differential equation

$$dX_t = a_t\,dt + b_t\,dW_t,$$

where a_t and b_t are adapted to the Brownian filtration, by

$$X_t = X_0 + \int_0^t a_s\,ds + \int_0^t b_s\,dW_s.$$

But what does this last term mean? Turns out it will be a martingale.

As long as $E\left[\int_0^T b_t^2\, dt\right] < \infty$, we can approximate the integral with elementary functions. Take

$$\phi_t = \sum_{j=0}^{n-1} e_j \cdot \mathbf{1}_{[t_j, t_{j+1})}(t),$$

where e_j is $\mathcal{F}_{t_j}$ measurable and ϕ has finite squared expectation. Define

$$\int_0^t \phi_s\, dW_s = \sum_{j=0}^{n-1} e_j \cdot (W_{t_{j+1}} - W_{t_j}).$$

Our SDE is a little bit more complicated:

$$dS_t = \mu_t S_t \, dt + \sigma_t S_t \cdot dW_t.$$

Need some sort of chain rule....

Theorem (Ito's Formula)

Let $f(t, x)$ have continuous second partial derivatives. Then

$$f(t, W_t) = f(0, W_0) + \int_0^t \frac{\partial f}{\partial t}(s, W_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, W_s)\, dW_s + \\ + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, W_s)\, ds.$$

The SDE $dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dW_t$ has solution

$$S_t = S_0 \cdot exp(X(t)), \quad X_t = \int_0^t \sigma_s \, dW_s + \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2) \, ds.$$

To verify this we differentiate $f(X_t)$ where $f(x) = S_0 e^x$ using Ito's formula.

$$
\begin{aligned}
dS_t &= df(X_t) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, dX_t \cdot dX_t \\
&= S_t\, dX_t + \frac{1}{2} S_t\, dX_t \cdot\ dX_t \\
&= S_t \left(\sigma_t\, dW_t + (\mu_t - \frac{1}{2}\sigma_t^2)\, dt \right) + \\
&+ \frac{1}{2} S_t \left(\sigma_t\, dW_t + (\mu_t - \frac{1}{2}\sigma_t^2)\, dt \right)^2 \\
&= \mu_t S_t\, dt + \sigma_t S_t\, dW_t.
\end{aligned}
$$

Black Scholes for European Call/Put Options

European style call option is the right to purchase S for K at T

- Strike Price K
- Time to maturity T
- Underlying stock price at $t = 0$, denoted S_0
- Payoff $(S_T - K)^+ = \max\{S_T - K, 0\}$

Put option is the right to sell, payoff $(K - S_T)^+$

Market assumptions/characterizations:

- Single underlying stock with geometric brownian motion
- Stock has expected growth μ_t and volatility σ_t
- No taxes, transaction costs or bid-ask spread
- Can borrow/lend at risk free rate r_f
- No arbitrage

The goal is to construct $f(t, S_t)$ that measures the value of the option at time t, based on the stock price path. We do this with a hedge.

Purchase some amount, Δ, of the underlying stock and sell one option. The value of this portfolio at time t is then

$$\Pi_t = \Delta \cdot S_t - f(t, S_t).$$

Over an infinitesimal time window dt, the portfolio's value changes

$$d\Pi_t = \Delta \cdot dS_t - df(t, S_t).$$

Use Ito's formula to track this change.

The goal is to construct $f(t, S_t)$ that measures the value of the option at time t, based on the stock price path. We do this with a hedge.

Purchase some amount, Δ, of the underlying stock and sell one option. The value of this portfolio at time t is then

$$\Pi_t = \Delta \cdot S_t - f(t, S_t).$$

Over an infinitesimal time window dt, the portfolio's value changes

$$d\Pi_t = \Delta \cdot dS_t - df(t, S_t).$$

Use Ito's formula to track this change.

The goal is to construct $f(t, S_t)$ that measures the value of the option at time t, based on the stock price path. We do this with a hedge.

Purchase some amount, Δ, of the underlying stock and sell one option. The value of this portfolio at time t is then

$$\Pi_t = \Delta \cdot S_t - f(t, S_t).$$

Over an infinitesimal time window dt, the portfolio's value changes

$$d\Pi_t = \Delta \cdot dS_t - df(t, S_t).$$

Use Ito's formula to track this change.

$$
\begin{aligned}
d\Pi_t &= \Delta \cdot dS_t - \left(\frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial S_t}\, dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\, dS_t^2 \right) \\
&= \Delta \cdot dS_t - \left(\frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial S_t}\, dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\, \sigma^2 S_t^2\, dt \right) \\
&= \left(\Delta - \frac{\partial f}{\partial S_t} \right) dS_t - \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2 \right)\, dt.
\end{aligned}
$$

The only Brownian randomness comes from dS_t terms. Take $\Delta = \dfrac{\partial f}{\partial S_t}$. 'Continuously update' number of stocks Δ to form a 'Delta Hedge'.

With $\Delta = \dfrac{\partial f}{\partial S_t}$ stocks in the portfolio, Π_t grows deterministically.

Because there is no arbitrage, Π_t must grow at the risk free rate:

$$d\Pi_t = r\Pi_t \, dt.$$

Substitute previous expressions and obtain the Black-Scholes PDE:

$$rf = \frac{\partial f}{\partial t} + r\frac{\partial f}{\partial S_t}S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2.$$

Notice the lack of μ in this PDE!

With $\Delta = \dfrac{\partial f}{\partial S_t}$ stocks in the portfolio, Π_t grows deterministically. Because there is no arbitrage, Π_t must grow at the risk free rate:

$$d\Pi_t = r\Pi_t \, dt.$$

Substitute previous expressions and obtain the Black-Scholes PDE:

$$rf = \frac{\partial f}{\partial t} + r\frac{\partial f}{\partial S_t}S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2.$$

Notice the lack of μ in this PDE!

With $\Delta = \frac{\partial f}{\partial S_t}$ stocks in the portfolio, Π_t grows deterministically.
Because there is no arbitrage, Π_t must grow at the risk free rate:

$$d\Pi_t = r\Pi_t \, dt.$$

Substitute previous expressions and obtain the Black-Scholes PDE:

$$rf = \frac{\partial f}{\partial t} + r\frac{\partial f}{\partial S_t} S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2.$$

Notice the lack of μ in this PDE!

With $\Delta = \dfrac{\partial f}{\partial S_t}$ stocks in the portfolio, Π_t grows deterministically.
Because there is no arbitrage, Π_t must grow at the risk free rate:

$$d\Pi_t = r\Pi_t \, dt.$$

Substitute previous expressions and obtain the Black-Scholes PDE:

$$rf = \frac{\partial f}{\partial t} + r\frac{\partial f}{\partial S_t}S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2.$$

Notice the lack of μ in this PDE!

Boundary conditions (for call option):

- $f(t, 0) = 0$
- $f(T, S_T) = (S_T - K)^+$
- $f(t, S_t) \to \infty$ as $S_t \to \infty$

How can we solve this?

$$rf = \frac{\partial f}{\partial t} + r\frac{\partial f}{\partial S_t}S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2.$$

Change variables....

$$S = Ke^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad f(T, S) = K \cdot v(x, \tau).$$

Change variables again and set $c = 2r/\sigma^2$ ...

$$v(x, \tau) = \exp\left(-\frac{1}{2}(c-1)x - \frac{1}{4}(c+1)^2\tau\right) \cdot u(x, \tau).$$

Miraculously this reduces to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

Change variables....

$$S = Ke^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad f(T, S) = K \cdot v(x, \tau).$$

Change variables again and set $c = 2r/\sigma^2$ ...

$$v(x, \tau) = \exp\left(-\frac{1}{2}(c-1)x - \frac{1}{4}(c+1)^2\tau\right) \cdot u(x, \tau).$$

Miraculously this reduces to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

Change variables....

$$S = Ke^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad f(T,S) = K \cdot v(x,\tau).$$

Change variables again and set $c = 2r/\sigma^2$ ...

$$v(x,\tau) = \exp\left(-\frac{1}{2}(c-1)x - \frac{1}{4}(c+1)^2\tau\right) \cdot u(x,\tau).$$

Miraculously this reduces to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

Can use the Fourier Transform to solve this!

$$f(t, S_t) = S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-) \qquad \text{(call)},$$

$$f(t, S_t) = Ke^{-r(T-t)}\Phi(-d_-) - S\Phi(-d_+) \qquad \text{(put)},$$

where $\Phi(x)$ is the normal CDF,

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx,$$

$$d_\pm = \frac{\log(S/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Risk-Neutral Measure

Suppose Z is a RV with $E(Z) = 1$ and $Z > 0$ a.s. on $(\Omega, \mathcal{F}, P)$. Can define

$$\widetilde{P}(A) = \int_A Z(\omega) dP(\omega), \qquad \text{for } A \in \mathcal{F}.$$

We call Z the Radon-Nikodym derivative of $\widetilde{P}$ w.r.t P.

Definition

If Z is the R-N derivative of $\widetilde{P}$ w.r.t P, then

$$Z_t = E[Z|\mathcal{F}_t], \qquad 0 \leq t \leq T$$

is the R-N derivative process w.r.t. $\mathcal{F}_t$.

Lemma

The R-N derivative process is a martingale w.r.t $\mathcal{F}_t$.

Proof.

$$E[Z_t|\mathcal{F}_s] = E[E[Z|\mathcal{F}_t]|\mathcal{F}_s] = E[Z|\mathcal{F}_s] = Z_s.$$

□

Lemma

Let Y be an $\mathcal{F}_t$ measurable RV. Then for $0 \leq s \leq t$,

$$\widetilde{E}[Y|\mathcal{F}_s] = \frac{1}{Z_s} E[YZ_t|\mathcal{F}_s].$$

Lemma

The R-N derivative process is a martingale w.r.t $\mathcal{F}_t$.

Proof.

$$E[Z_t|\mathcal{F}_s] = E[E[Z|\mathcal{F}_t]|\mathcal{F}_s] = E[Z|\mathcal{F}_s] = Z_s.$$

□

Lemma

Let Y be an $\mathcal{F}_t$ measurable RV. Then for $0 \leq s \leq t$,

$$\widetilde{E}[Y|\mathcal{F}_s] = \frac{1}{Z_s} E[YZ_t|\mathcal{F}_s].$$

Theorem (Girsanov)

On $(\Omega, \mathcal{F}, P)$ with W_t and $\mathcal{F}_t$, let Θ_t be an adapted process for $0 \leq t \leq T$. Define

$$\widetilde{W}_t = W_t + \int_0^t \Theta_s \, ds, \quad \text{i.e.} \quad d\widetilde{W}_t = dW_t + \Theta_t \, dt.$$

Then there exists an explicit Z_t that defines $\widetilde{P}$ for which $E(Z_T) = 1$ and under $\widetilde{P}$ the process $\widetilde{W}_t$ is a BM.

Recall our stock process

$$dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dW_t.$$

The risk free rate can be given as a stochastic process r_t, so we have discount factor

$$D_t = \exp\left(-\int_0^t r_s \, ds\right).$$

Discount process has zero quadratic variation, so

$$D_t S_t = S_0 \exp\left(\int_0^t \sigma_s \, dW_s + \int_0^t \mu_s - r_s - \frac{1}{2}\sigma_s^2 \, ds\right).$$

In differential form, the discounted stock process is

$$d(D_t S_t) = \sigma_t D_t S_t \left(\Theta_t \, dt + dW_t\right), \quad \Theta_t = \frac{\mu_t - r_t}{\sigma_t}.$$

Use Girsanov with Θ_t to change to $\widetilde{P}$, then

$$d(D_t S_t) = \sigma_t D_t S_t \, d\widetilde{W}_t.$$

Discounted stock process is a martingale! From this we get discounted payoff $D_t V_t$ also a martingale.

The solution to this SDE is

$$S_t = S_0 \exp\left(\int_0^t \sigma_s \, d\widetilde{W}_s + \int_0^t r_s - \frac{1}{2}\sigma_s^2 \, ds\right).$$

Evaluate call option price from martingale statement:

$$f(0, S_0) = \widetilde{E}\left[e^{-rT}(S_T - K)^+ | \mathcal{F}_0\right].$$

Straightforward integral to recover Black-Scholes.

Theorem (Martingale Representation)

On $(\Omega, \mathcal{F}, P)$ with W_t and $\mathcal{F}_t$ for $0 \leq t \leq T$, let M_t be a martingale w.r.t. $\mathcal{F}_t$. Then there exists a unique $\mathcal{F}_t$ adapted process Γ_t such that

$$M_t = M_0 + \int_0^t \Gamma_s \, dW_s, \quad 0 \leq t \leq T.$$

We really want the existence of $\mathcal{F}_t$ adapted $\widetilde{\Gamma}$ for martingale $\widetilde{M}$ under $\widetilde{P}$, such that

$$\widetilde{M}_t = \widetilde{M}_0 + \int_0^t \widetilde{\Gamma}_s \, d\widetilde{W}_s.$$

General hedging problem for European style option:

- Let V_T be $\mathcal{F}_T$ measurable - derivative payoff
- Start with initial capital X_0 and form portfolio process X_t
- Want $X_T = V_T$
- Call $V_0 = X_0$ the price of the option at time zero

Change to risk-neutral measure, get $D_t V_t$ to be a martingale:

$$D_t V_t = V_0 + \int_0^t \widetilde{\Gamma}_s d\widetilde{W}_s, \qquad 0 \leq t \leq T.$$

On the other hand, we want the portfolio process X_t to be 'self-financing':

$$dX_t = \Delta_t dS_t + r_t \left(X_t - \Delta_t S_t \right) dt,$$

where we hold Δ_t shares of the stock, and invest/borrow at r_t to finance with X_0 initial capital.

Write this under $\widetilde{P}$ and get

$$D_t X_t = X_0 + \int_0^t \Delta_s \sigma_s D_s S_s \, d\widetilde{W}_s, \qquad 0 \leq t \leq T.$$

Equating this with our previous expression

$$D_t V_t = V_0 + \int_0^t \widetilde{\Gamma}_s d\widetilde{W}_s, \qquad 0 \leq t \leq T,$$

we should pick

$$\Delta_t = \frac{\widetilde{\Gamma}_t}{\sigma_t D_t S_t}, \qquad 0 \leq t \leq T.$$

Theorem (Fundamental Theorem of Asset Pricing)

A market has a risk-neutral measure if and only if it does not admit arbitrage.

In an arbitrage free market, every derivative can be hedged if and only if the risk neutral measure is unique.

American Options and Duality

An American style option can be exercised at any time up to maturity T.

- Closed form solutions rarely available
- Approximate with discrete exercise times $G = \{0, 1, 2, \dots, T\}$
- Discounted payoff $h_t(S_t)$ if exercised at time t
- Discounted value $V_t(S_t)$, inherent value/price at time t

Notice that

$$V_t(S_t) \geq h_t(S_t), \qquad V_0 = \sup_{\tau} \widetilde{E}(h_\tau(S_\tau)).$$

An American style option can be exercised at any time up to maturity T.

- Closed form solutions rarely available
- Approximate with discrete exercise times $G = \{0, 1, 2, \dots, T\}$
- Discounted payoff $h_t(S_t)$ if exercised at time t
- Discounted value $V_t(S_t)$, inherent value/price at time t

Notice that

$$V_t(S_t) \geq h_t(S_t), \qquad V_0 = \sup_{\tau} \widetilde{E}(h_\tau(S_\tau)).$$

The value of an American option follows

- $V_T(S_T) = h_T(S_T)$
- $V_i(S_i) = \max\{h_i(S_i), \widetilde{E}[V_{i+1}(S_{i+1})|\mathcal{F}_i]\}$

Use some sort of dynamic programming algorithm to find V_0.

How do you compute this continuation value?

- Lattice
- Least squares
- Whatever works...

The value of an American option follows

- $V_T(S_T) = h_T(S_T)$
- $V_i(S_i) = \max\{h_i(S_i), \widetilde{E}[V_{i+1}(S_{i+1})|\mathcal{F}_i]\}$

Use some sort of dynamic programming algorithm to find V_0.

How do you compute this continuation value?

- Lattice
- Least squares
- Whatever works...

Consider martingale M_t with $M_0 = 0$ and τ attaining values in G. Then by Optional Sampling,

$$\widetilde{E}(h_\tau(S_\tau)) = \widetilde{E}(h_\tau(S_\tau) - M_\tau) \leq \widetilde{E}(\max_{k \in G}(h_k(S_k) - M_k)).$$

Take inf over M and sup over τ to arrive at

$$V_0 = \sup_\tau \widetilde{E}(h_\tau(S_\tau)) \leq \inf_M \widetilde{E}(\max_{k \in G}(h_k(S_k) - M_k)).$$

Claim: This is actually an equality!

Theorem

There exists a martingale M_t with $M_0 = 0$ for which

$$V_0 = \widetilde{E}(\max_{k \in G}(h_k(S_k) - M_k)).$$

Proof.

For $i = 1, \ldots, T$ define $N_i = V_i(S_i) - \widetilde{E}[V_i(S_i)|\mathcal{F}_{i-1}]$. Then take

$$M_0 = 0, \qquad M_i = \sum_{j=1}^{i} N_j, \qquad i = 1, \ldots, T.$$

Notice that $\widetilde{E}[N_i|\mathcal{F}_{i-1}] = 0$. Thus M_t is a martingale. □

continued...

With induction: $V_i(S_i) = \max\{h_i(S_i), h_{i+1}(S_{i+1}) - N_{i+1}, \ldots, h_T(S_T) - N_T - \cdots - N_{i+1}\}$.
True at T since $V_T(S_T) = h_T(S_T)$. Assuming true at i,

$$\begin{aligned} V_{i-1}(S_{i-1}) &= \max\{h_{i-1}(S_{i-1}), \widetilde{E}[V_i(S_i)|\mathcal{F}_{i-1}] \\ &= \max\{h_{i-1}(S_{i-1}), V_i(S_i) - N_i\} \end{aligned}$$

Therefore

$$V_0 \geq \widetilde{E}[V_1(S_1)|\mathcal{F}_0] = V_1(S_1) - N_1 = \max_{k=1,\ldots,T}(h_k(S_k) - M_k).$$

□

Duality gives a way to form lower/upper bounds:

$$\sup_{\tau} \widetilde{E}(h_\tau(S_\tau)) = V_0 = \inf_{M} \widetilde{E}(\max_{k \in G}(h_k(S_k) - M_k)).$$

- An approximate stopping time gives a lower bound
- An approximate martingale gives an upper bound
- What constitutes a good approximation?