# The Precession of Mercury's Perihelion

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# 1 Introduction

In this paper, we will attempt to give a demonstration that General Relativity predicts a rate of perihelion precession equal to that of Mercury's orbit around the Sun (when the influences due to other planets have already all been accounted for). First, we will use classical physics to serve a two-fold purpose: to demonstrate that classical orbits are (closed) ellipses, and also to illustrate the methods involved in the relativistic solution. Second, we will apply these methods to a general relativistic treatment of geodesics in the Schwarzschild metric, and show that an "orbit" matching Mercury's specifications can be expected to shift by approximately 43 arcseconds per century.

# 2 The Classical Solution

We will begin with three-dimensional polar coordinates, where the metric is

$$ds^2 = dr^2 + r^2 d\Omega^2$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. In these coordinates, we can express the Lagrangian as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x}) \\ &= \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right] + \frac{GMm}{r} \end{aligned}$$

where we have substituted the gravitational potential $V(\mathbf{x}) = -\frac{GMm}{|\mathbf{x}|}$. The equations of motion for a particle are then given by the Euler-Lagrange equations $\frac{\partial\mathcal{L}}{\partial x^i} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}^i}$. These become, with $x^i = r, \theta, \phi$ respectively:

$$\begin{aligned} mr\left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right] - \frac{GMm}{r^2} &= \frac{d}{dt}(m\dot{r}) = m\ddot{r} \\ mr^2\sin\theta\cos\theta\dot{\phi}^2 &= \frac{d}{dt}\left(mr^2\dot{\theta}\right) \\ 0 &= \frac{d}{dt}\left(mr^2\sin^2\theta\dot{\phi}\right) \end{aligned}$$

Notice that these equations are invariant under $\theta \mapsto \pi - \theta$, under which $\sin\theta \mapsto \sin\theta$, $\cos\theta \mapsto -\cos\theta$, and $\dot{\theta} \mapsto -\dot{\theta}$. Then for an initial value problem with $\theta(0) = \frac{\pi}{2}$, $\dot{\theta}(0) = 0$ (that is, the motion of the particle begins in the equatorial plane), any solution with $(r(t), \theta(t), \phi(t))$ immediately gives us another solution $(r(t), \pi - \theta(t), \phi(t))$, which contradicts local uniqueness of the solution to the initial value problem unless $\theta(t) \equiv \frac{\pi}{2}$. Since any initial value problem can be rotated into one of this form, we will now assume that $\theta \equiv \frac{\pi}{2}$, reducing the Euler Lagrange equations (where we have also canceled $m$) to:

$$\begin{aligned} r\dot{\phi}^2 - \frac{GM}{r^2} &= \ddot{r} \\ 0 &= \frac{d}{dt}\left(r^2\dot{\phi}\right) \end{aligned}$$

The second equation says that $L = r^2\dot{\phi}$ is a constant of the motion; if it is zero we find that $-\frac{GM}{r^2} = \ddot{r} < 0$ and hence $r$ is concave down. Since concave down functions are unbounded below, we would find that for some $t$, $r = 0$, which describes the uninteresting event in which the object crashes into the sun. Hence, we will restrict our attention to $L \neq 0$. In that case, $\dot{\phi}$ is either always negative or always positive, in which case $\phi(t)$ is monotone and we can write $t = t(\phi)$. (Here we are letting $\phi$ range through $\mathbb{R}$, and considering the fact that $\phi$ and $\phi + 2\pi$ describe the same point only as a curiosity.) Then

$$\frac{\partial}{\partial t} = \dot{\phi}\frac{\partial}{\partial \phi} = \frac{L}{r^2}\frac{\partial}{\partial \phi}.$$

Hence we may rewrite the other equation of motion as

$$\begin{aligned} & r\dot{\phi}^2 - \frac{GM}{r^2} = \ddot{r} \\ \Rightarrow \quad & \frac{L^2}{r^3} - \frac{GM}{r^2} = \frac{L}{r^2}\frac{\partial}{\partial \phi}\left(\frac{L}{r^2}\frac{\partial r}{\partial \phi}\right). \end{aligned}$$

To make this equation more readily solvable, we make a change of variables to $u = 1/r$. Then denoting differentiation with respect to $\phi$ by a prime, we have $u' = -r'/r^2$, and so the differential equation becomes

$$\begin{aligned} & L^2u^3 - GMu^2 = -L^2u^2u'' \\ \Rightarrow \quad & u'' + u = \frac{GM}{L^2} \end{aligned}$$

We can easily solve this equation as $u(\phi) = A\cos(\phi - \phi_0) + \frac{GM}{L^2}$. By suitably translating $\phi$, we can choose $\phi_0 = 0$ and $A \leq 0$, in which case we can rewrite this as

$$u(\phi) = \frac{GM}{L^2}(1 - e\cos(\phi)) \qquad (1)$$

with $e = -\frac{AL^2}{GM} \geq 0$. It is well-known [1] that Equation 1 describes an ellipse of eccentricity $e$.

## 3 Classical Calculation of the Period

As an alternate demonstration that $r(\phi)$ is periodic with period $2\pi$, consider that the binding energy (per unit mass) of the system is another constant of the motion:

$$-E = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{GM}{r},$$

where we have used $-E$ instead of $E$ so that $E > 0$. Then we can solve this for $\dot{r}$:

$$\begin{aligned}
\dot{r}^2 &= -2E + \frac{2GM}{r} - \frac{L^2}{r^2} \\
\left(\frac{L}{r^2}r'\right)^2 &= -2E + \frac{2GM}{r} - \frac{L^2}{r^2} \\
(r')^2 &= -\frac{2E}{L^2}r^4 + \frac{2GM}{L^2}r^3 - r^2 \qquad (2) \\
&= r^2\left(1 - \frac{r}{R_+}\right)\left(\frac{r}{R_-} - 1\right) \\
r' &= \pm r\sqrt{\left(1 - \frac{r}{R_+}\right)\left(\frac{r}{R_-} - 1\right)}
\end{aligned}$$

where we have introduced the notation $R_\pm$ for the nonzero roots of the quartic polynomial in (2); since these are the only places where $r' = 0$ and $r \neq 0$, we may identify them as the aphelion and the perihelion of a closed orbit. Since $r' = \frac{\partial r}{\partial \phi} = \frac{1}{\partial\phi/\partial r}$, we can find the amount of $\phi$ required to pass from $R_-$ to $R_+$ by integrating:

$$\begin{aligned}
\phi_+ - \phi_- &= \int_{R_-}^{R_+} \frac{dr}{r\sqrt{\left(1 - \frac{r}{R_+}\right)\left(\frac{r}{R_-} - 1\right)}} \\
&= \arctan\left[\frac{(R_+ - r)(r - R_-) + r^2 - R_+R_-}{2\sqrt{(R_+ - r)(r - R_-)R_-R_+}}\right]_{R_-}^{R_+} \\
&\to \arctan[+\infty] - \arctan[-\infty] = \frac{\pi}{2} + \frac{\pi}{2} = \pi
\end{aligned}$$

Hence the particle will travel from $R_-$ to $R_+$ and back every time $\phi \to \phi + 2\pi$, so the orbit $r(\phi)$ is periodic with period $2\pi$, and so closed.

## 4 The Relativistic Solution

In the general relativistic case, we assume that the particle is a test particle traveling along a geodesic through spacetime. Geodesics can also be described as stationary points of the integral

$$I = \int \langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle \, d\tau,$$

which is the formulation of the geodesics we will use. Assume that the metric for the solar system is spherically symmetric, static, and asymptotically flat, so that it can be represented as follows:

$$ds^2 = -e^{2\alpha(R)} dT^2 + e^{2\beta(R)} dR^2 + e^{2\gamma(R)} d\Omega^2, \tag{3}$$

where the $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ term comes from spherical symmetry and $T$ is the coordinate produced by the timelike Killing vector field, of which the metric components are all independent. We would like to change coordinates from $R$ to $r$, where $r$ corresponds to physical measurements of radius. If we define the radius $r$ of a sphere as the square root of its area divided by $4\pi$, then the coefficient of $d\Omega^2$ is fixed as $r^2$, and so we can reexpress (3) as

$$ds^2 = -e^{2A(r)} dT^2 + e^{2B(r)} dr^2 + r^2 d\Omega^2, \tag{4}$$

where we define $A(r) = \alpha(R) = \alpha \circ \gamma^{-1}(\ln r)$ and similarly for $B(r)$. If we assume that the orbit of Mercury is a geodesic in a vacuum, this further constrains $ds^2$ to satisfy the vanishing of the Ricci Tensor: $R_{\mu\nu} = 0$. We can compute the nonvanishing Christoffel symbols $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left( \frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right)$ for (4) as [2]:

$$\begin{aligned}
\Gamma^r_{rr} &= B'(r) & \Gamma^r_{\theta\theta} &= -re^{-2B(r)} \\
\Gamma^r_{\phi\phi} &= -r\sin^2\theta e^{-2B(r)} & \Gamma^r_{tt} &= A'(r)e^{2A(r)-2B(r)} \\
\Gamma^\theta_{r\theta} &= \Gamma^\theta_{\theta r} = \frac{1}{r} & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^\phi_{r\phi} &= \Gamma^\phi_{\phi r} = \frac{1}{r} & \Gamma^\phi_{\phi\theta} &= \Gamma^\phi_{\theta\phi} = -\sin\theta\cos\theta \\
\Gamma^t_{tr} &= \Gamma^t_{rt} = A'(r)
\end{aligned}$$

Then we can compute the Ricci tensor components [2]:

$$R_{\mu\nu} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\lambda\eta} \text{ as}$$

$$
\begin{aligned}
R_{rr} &= A'' + 2(A')^2 - A'(A'+B') - \frac{2}{r}B' \\
R_{\theta\theta} &= -1 + re^{-2B}(A'-B') + e^{-2B} \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta} \\
R_{tt} &= -[A'' + 2(A')^2]e^{2A-2B} + A'e^{2A-2B}(A'+B') - \frac{2}{r}A'e^{2A-2B} \\
R_{\mu\nu} &= 0 \quad \mu \neq \nu
\end{aligned}
$$

Note that if we take the combination $R_{rr}+R_{tt}e^{2B-2A}$, we obtain $-\frac{2}{r}(A'+B')$. Then the vacuum requirement $R_{\mu\nu} = 0$ implies that $A'+B' = 0$, i.e. $A+B =$ const. Since $A, B \to 0$ as $r \to \infty$ by asymptotic flatness, we must have $A = -B$. Then the vacuum conditions become:

$$
\begin{aligned}
R_{\theta\theta} &= -1 + 2rA'e^{2A} + e^{2A} = 0 \\
R_{rr} &= A'' + 2(A')^2 + \frac{2}{r}A' = \frac{1}{2re^{2A}}R'_{\theta\theta} = 0
\end{aligned}
$$

Since the second condition follows from the first, we need only choose $A$ so that $1 = 2rA'e^{2A} + e^{2A} = (re^{2A})'$. The general solution to this is $re^{2A} - r =$ const., i.e. $e^{2A} = 1 - \frac{\text{const.}}{r}$. It is known that in a gravitational field that resembles Newtonian gravity, we must have $g_{rr} \approx 1-2\Phi$, where $\Phi = -\frac{GM}{r}$ is the Newtonian potential. Then the observation that our gravitational field approximates Newtonian gravity gives us the Schwarzschild metric:

$$
ds^2 = -\left[1 - \frac{R_S}{r}\right]dT^2 + \left[1 - \frac{R_S}{r}\right]^{-1}dr^2 + r^2 d\Omega^2, \tag{5}
$$

where $R_S = 2GM$ is the Schwarzschild radius of the sun.

Now if we parameterize a curve $\mathbf{x}(\tau) = (T(\tau), r(\tau), \theta(\tau), \phi(\tau))$ by proper time, then we find that letting $\mathcal{L} = \langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle$ (where the dot refers to differentiation with respect to proper time), $\mathcal{L}$ is both a constant of the motion ($-1$, in fact) and also satisfies the Euler-Lagrange equations so that $I = \int \mathcal{L}\, d\tau$ is stationary. By exactly the same reasoning as in the classical case, we may restrict our attention to motion in the equatorial plane and assume that $\theta(\tau) \equiv \pi/2$, so that the "Lagrangian" becomes

$$
\mathcal{L} = -\left[1 - \frac{R_S}{r}\right]\dot{T}^2 + \left[1 - \frac{R_S}{r}\right]^{-1}\dot{r}^2 + r^2\dot{\phi}^2 \tag{6}
$$

Then the Euler-Lagrange equations for $\phi$ and $T$ read:

$$0 = \frac{d}{d\tau}\left(2r^2\dot{\phi}\right)$$
$$0 = \frac{d}{d\tau}\left(-2\left(1-\frac{R_S}{r}\right)\dot{T}\right)$$

This implies that $L = r^2\dot{\phi}$ and $E = \dot{T}(R_S/r - 1)$ are two constants of the motion. Then the relation $\mathcal{L} = -1$ gives us:

$$1 = \left[1-\frac{R_S}{r}\right]\dot{T}^2 - \left[1-\frac{R_S}{r}\right]^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$
$$= \frac{E^2}{1-R_S/r} - \frac{\dot{r}^2}{1-R_S/r} - \frac{L^2}{r^2}, \quad \text{i.e.}$$
$$(\dot{r})^2 = (E^2-1) + \frac{R_S}{r} - \frac{L^2}{r^2} + \frac{R_S L^2}{r^3}$$

Once again, assuming $L \neq 0$ allows us to invert $\phi = \phi(\tau)$, so we may obtain $r$ as a function of $\phi$ with $\dot{r} = \frac{L}{r^2}r'$, and hence we have

$$(r')^2 = \frac{E^2-1}{L^2}r^4 + \frac{R_S}{L^2}r^3 - r^2 + R_S r$$

Now the requirement that of a closed orbit with $(r')^2 \geq 0$ imposes some constraints on $L, E$, and $R_S$; we need a connected component of $\{r : r' \geq 0\}$ to be a compact subset of $\mathbb{R}^+$. This means there exist at least two values $R_+$ and $R_-$ where $r' = 0$, i.e. aphelion and perihelion. Then the angle shift from $R_-$ to $R_+$ is given, as in the classical case, by

$$\phi_+ - \phi_- = \int_{R_-}^{R_+} \frac{dr}{\sqrt{\frac{E^2-1}{L^2}r^4 + \frac{R_S}{L^2}r^3 - r^2 + R_S r}}. \tag{7}$$

Given that $(r-R_+)$ and $(r-R_-)$ are factors of $\frac{E^2-1}{L^2}r^4 + \frac{R_S}{L^2}r^3 - r^2 + R_S r$, we can solve for $E^2-1$ and $L^2$ in terms of $R_\pm$ and $R_S$:

$$(E^2-1)R_+^4 + (L^2)(-R_+^2 + R_S R_+) = R_S R_+^3$$
$$(E^2-1)R_-^4 + (L^2)(-R_-^2 + R_S R_-) = R_S R_-^3$$

which give

$$E^2 - 1 = \frac{-R_+R_-R_S + (R_+ + R_-)R_S^2}{R_+R_-(R_+ + R_- + R_S) - (R_+ + R_-)^2 R_S}$$
$$L^2 = \frac{R_+^2 R_-^2 R_S}{R_+R_-(R_+ + R_- + R_S) - (R_+ + R_-)^2 R_S}$$

It is convenient to introduce the combination

$$D = \frac{R_+ R_-}{R_+ + R_-},$$

which has units of distance. Then the above expressions for $E^2 - 1$ and $L^2$ become:

$$\begin{aligned} E^2 - 1 &= \frac{(-R_S/R_+R_-) + (R_S^2/DR_+R_-)}{1/D + (R_S/R_+R_-) - (R_S/D^2)} \\ L^2 &= \frac{R_S}{1/D + (R_S/R_+R_-) - (R_S/D^2)} \end{aligned}$$

We would like an expression for $\varepsilon$, the third nonzero root of $\frac{E^2-1}{L^2}r^4 + \frac{R_S}{L^2}r^3 - r^2 + R_S r = 0$. We know that the sum of the three nonzero roots is $\frac{R_S}{E^2-1}$ (the coefficient of $r^3$ with the polynomial in standard form); using the above expressions we can swiftly obtain:

$$\varepsilon = \frac{R_S}{1 - R_S/D}$$

Now we can approximate (7), by writing

$$\frac{E^2-1}{L^2}r^4 + \frac{R_S}{L^2}r^3 - r^2 + R_S r = \frac{1-E^2}{L^2}(R_+ - r)(r - R_-)(r - \varepsilon)r.$$

We obtain:

$$\begin{aligned} \phi_+ - \phi_- &= \sqrt{\frac{L^2}{1-E^2}} \int_{R_-}^{R_+} \frac{1}{\sqrt{r(R_+ - r)(r - R_-)(r - \varepsilon)}} dr \\ &= \sqrt{\frac{L^2}{1-E^2}} \int_{R_-}^{R_+} \frac{1}{r\sqrt{(R_+ - r)(r - R_-)}} \left(1 - \frac{\varepsilon}{r}\right)^{-1/2} dr \end{aligned}$$

Now use the Taylor series expansion $(1 - \varepsilon/r)^{-1/2} \approx 1 + \varepsilon/2r$, with an error $\mathcal{E}$ bounded by $|\mathcal{E}| \le \frac{3}{8}(1-\varepsilon/r)^{-5/2}(\varepsilon/r)^2 \le \frac{3}{8}(1-\varepsilon/R_+)^{-5/2}(\varepsilon/R_-)^2$, which produces:

$$= \sqrt{\frac{L^2}{1-E^2}} \int_{R_-}^{R_+} \frac{1+\mathcal{E}}{r\sqrt{(R_+ - r)(r - R_-)}} + \frac{\varepsilon/2}{r^2\sqrt{(R_+ - r)(r - R_-)}} dr$$

We already evaluated the integral of the first term in the classical case; it is just $\pi(1+\mathcal{E})/\sqrt{R_+R_-}$. The second integral is trickier, but can be evaluated in closed form:

$$\int_{R_-}^{R_+} \frac{\varepsilon/2}{r^2\sqrt{(R_+ - r)(r - R_-)}} dr = \frac{\pi\varepsilon/2}{2\sqrt{R_+R_-}} \frac{R_+ + R_-}{R_+R_-} = \frac{1}{\sqrt{R_+R_-}} \frac{\pi\varepsilon}{4D}.$$

Then if we recognize that $\frac{L^2/R_+R_-}{1-E^2} = \frac{1}{1-R_S/D}$, we find that

$$\begin{aligned}\phi_+ - \phi_- &= \pi(1+\mathcal{E})\sqrt{\frac{L^2/R_+R_-}{1-E^2}} + \sqrt{\frac{L^2/R_+R_-}{1-E^2}}\frac{\pi\varepsilon}{4D} \\ &= \frac{\pi}{\sqrt{1-R_S/D}}\left(1+\frac{1}{4}\frac{R_S/D}{1-R_S/D}\right) + \frac{\pi}{\sqrt{1-R_S/D}}\mathcal{E}.\end{aligned}$$

Using the observed values $R_+ = 69.8 \cdot 10^6$km, $R_- = 46.0 \cdot 10^6$km (from which we obtain $D = 27.7 \cdot 10^6$km), and $R_S = 2GM/c^2 = 2.95$km, we find that the second term is bounded above by $\pi\frac{3}{8}(1-\varepsilon/R_+)^{-5/2}(\varepsilon/R_-)^2/\sqrt{1-R_S/D} \approx 4.88 \cdot 10^{-15}$, making the first term $\frac{\pi}{\sqrt{1-R_S/D}}\left(1+\frac{1}{4}\frac{R_S/D}{1-R_S/D}\right) \approx \pi + 2.515 \cdot 10^-7$ a trustworthy estimate of $\phi_+ - \phi_-$ (half a revolution, in radians). Since Mercury completes 415.2 revolutions each century, and there are $360 \cdot 60 \cdot 60/2\pi$ arcseconds per radian, we find that Mercury's perihelion advances by

$$(2.515 \cdot 10^{-7})\left(\frac{360 \cdot 60 \cdot 60}{\pi}\right) \cdot 415.2 = 43.084 \text{ arcseconds per century.}$$

# 5 Remarks

In the general relativity solution, we opted to estimate a single integral, rather than attempt a sort of "first-order" approximation to a differential equation. The reason for this is that such approximations are typically not well justified, and neglect certain terms as small without providing estimates for the neglected error.

On the other hand, we made some assumptions in our treatment as well. Apart from the standard assumptions that the solar system is spherically symmetric (which it is not), Mercury is a test particle and travels along a geodesic of this background spacetime (which it is not and does not), and that spacetime is asymptotically flat (who knows?), etc., we also assumed that there even existed a geodesic corresponding to the specifications we gave for Mercury's orbit. In addition, in our derivation of the Schwarzschild metric, we used a few arguments that stand on somewhat shaky ground (such as the use of $\sqrt{A(S_R)/4\pi}$, with $A(S_R)$ the area of the foliating sphere of "radius" $R$ in our original coordinates, as a smooth coordinate), though the use of the Schwarzschild metric is also standard. In any case, we need to match our physical observations to theory at some point, and we have demonstrated that the assumption of Mercury traveling along a geodesic of the Schwarzschild metric models our observations well.

## References

[1] Etgen, Hille, Salas. *Calculus: One and Several Variables.* Wiley, 2002.

[2] Weinberg, Steven. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* John Wiley and Sons, 1972.