Math 466 Numerical Analysis

Time and place:	T, 8:30-9:20, C-401 Padelford		
Instructor:	Jim Morrow	office hours:	MW 8:30-9:20 C439 Padelford
Prerequisites:	Math 465 and Math 307		
Textbooks:	*Numerical Analysis* by Johnson and Riess		

This course will introduce some numerical techniques for solving differential equations. It will be expected that you spend some time reading the textbook. You can tell from the homework assignments which sections you should be reading.

The homework assignments are as follows:

due date	assignment
April 6	§7.1: 8,11; §7.2.2: 4,10,13,14
April 13	§7.2.3: 5,6,8,9;
April 20	§7.2.5: 2,4,6,7
April 27	§7.3: 1,3,7
May 4	§7.4: 2,7,8,9; 7.5.1: 5,13
May 11	§7.5.3: 1c,2c,4,10; 7.5.4: 3,4,5
May 18	§7.6.3: 6,7,8
May 25	§8.1: 5,8
June 1	§8.2.1: 3,4

Project # 1 (due April 20)

Write a program RK4 which uses the fourth order Runge-Kutta method (*Johnson & Riess*, p369-371) to calculate a numerical solution to an initial value problem:

$$y' = f(x,y), y(a) = ya$$

RK4 should take as inputs the following arguments:
initial x value a
final x value b
initial y value ya
positive integer m, the number of subdivisions

The step-size is then $h = (b-a)/m$. There is also a parameter VALMAX, which may be taken to be 1.e4. RK4 should call on a subroutine for the function evaluations. This subroutine $func(x,y,z)$ takes the values x, y and places the computed value in z.

RK4 should return an error message if some computed y or y' has an absolute value that exceeds VALMAX. The program should return the computed values (x_i, y_i) at $m+1$ equally spaced points in $[a,b]$, or as many points as are computed before VALMAX is exceeded. Have the results sent to an external file.

1. Run the program with $f(x,y) = x - 100y$, and $a = 0, b = 1, ya = 1$. Let $m = 2^k$, where $k = 2, 3, \ldots, 8$. Compare the computed solution values with the values obtained by solving

the differential equation exactly. Plot the computed solutions and the true solution on a single graph.
2. Let $f(x, y) = y(2 - y)$. Run the program with $a = 0, b = 4, k = 8$, for the three initial y values $ya = .1, 1, 3$, and let $m = 2^8$. Plot the results on a single graph.
3. Let $f(x, y) = x^2 + y^2$. Let $a = 0, b = 1, ya = 1$, and let $m = 2^k$, where $k = 2, \ldots, 8$. Plot the solutions and explain the results.

Project # 2 (due May 11)

Write a program that uses the Heun method to solve a system of ODE's:

$$\mathbf{u}' = \mathbf{F}(x, \mathbf{u})$$

where $\mathbf{u}$ and $\mathbf{F}$ are n-vectors. Take the suggestion of *Johnson & Riess*, p396, and write a subroutine $func(x, \mathbf{u}, \mathbf{y})$, which puts the value of $\mathbf{F}(x, \mathbf{u})$ in $\mathbf{y}$. The program should take as inputs the following arguments:

integer $n \leq 10$, where n is the dimension of the system
initial x value
final x value
initial values $u_1, u_2, \ldots, u_n$
positive integer m, where the number of subdivisions is m

The step-size is then $h = (b - a)/m$. There is also a parameter VALMAX, which may be taken to be 1.e6.

The program should place the values $\mathbf{u}(x_i) = (u_1(x_i), u_2(x_i), \ldots, u_n(x_i))$, calculated at the intermediate points x_i, in an external file or print an error message if some calculated value of $\|\mathbf{u}(x_i)\|$ or $\|\mathbf{y}(x_i)\|$ exceeds VALMAX.

1. Run your program on the system:

$$\begin{aligned} u_1' &= -30u_1 + 28u_2 \\ u_2' &= \qquad -2u_2 \end{aligned}$$

with $0 \leq x \leq 4$, $u_1(0) = 2, u_2(0) = 1$ and $k = 3, \ldots, 8$. Plot the result for $m = 2^8$ in the (u_1, u_2)-plane. Do the same thing for:

$$\begin{aligned} u_1' &= -30u_1 + 28u_2 \\ u_2' &= \qquad -2u_2 \end{aligned}$$

which values of m give reasonable answers? 2. Convert the second order ODE

$$y'' + xy' + x^2 y = 0$$

to a first order system, and then run your program on the system to find the approximate value of $y(4)$, given the initial conditions $y(0) = 1, y'(0) = 0$. Let $m = 2^k$ where $k = 3, \ldots, 8$. Plot the result for $k = 8$ in the (x, y)-plane.

Project # 3 (due May 18)

Use three-step Adams-Bashforth as a predictor and two-step Adams-Moulton (formulas 7.47a and 7.47b of *Johnson & Riess*) to write a program to solve $y' = f(x, y)$. Use your RK4 routine to compute the starting values. The program should take as inputs:

initial x value xa

final x value xb

initial y value ya

number of subdivisions m

There should be a parameter VALMAX set equal to 1.e5.

The program should return the computed values (x_i, y_i) or an error message if $|y_i|$ or $|f(x_i, y_i)|$ exceeds VALMAX. Use this program as follows:

1. Let $f(x, y) = y/2, \quad xa = 0, xb = 2, ya = 1, m = 2^k, k = 3, \ldots, 8$. Plot the results.

2. Let $f(x, y) = 80(\sin(x) - y), \quad xa = 0, xb = 3, ya = 0, m = 2^k, k = 3, 5, 7, 9$. Plot the results. Explain the results. Find experimentally the smallest m that can be used to give a reasonable approximation to the true solution. Solve this equation using the RK4 method of project # 1.

3. The global error is usually not the sum of the local errors. Take the initial value problem of part 1 and $k = 8$, and compute the difference between the sum of the local errors and the global error.

4. The relative global error should be closer to the sum of the relative local errors. Use part 1 again and compute the difference between the relative global error and the sum of the relative local errors.

Project # 4 (Extra Credit, due June 1)

1. A disposal plant has an ejector which satisfies the ODE:

$$80y'' + 20(y')^2 + y = 60, \quad y(0) = 0, y'(0) = 0$$

where $y(t)$ is the height of the fluid in the ejector chamber at time t. Convert this equation to a first order system of ODE's. Using stepsize $h = 2^{-6}$, compute a numerical approximation, for $0 \leq t \leq 4$. Plot the results.

2. (Lorenz's example) Consider the system

$$\begin{aligned} X' &= -10X + 10Y \\ Y' &= -XZ + rX - Y \\ Z' &= XY - 8Z/3 \end{aligned}$$

with initial conditions $X(0) = 0, Y(0) = 1, Z(0) = 0$. Solve this system on $0 \leq t \leq 30$, using the method of your choice with a stepsize of .01, with r assuming the values $5, 20, 28$. In each case make the following plots: $(t, Y), (X, Y), (Y, Z)$.

3. Solve the system

$$\begin{aligned} s' &= cd \\ c' &= -ds \\ d' &= -sc/3 \end{aligned}$$

with initial conditions $s(0) = 0, c(0) = d(0) = 1$. This is the system satisfied by the angular momentum of a rigid body with moments of inertia $(1, 2, 3)$ (an asymmetric top). The solution is periodic in t. Find the period P by finding the first $P > 0$ for which $s(P) = 0, c(P) = d(P) = 1$. This period is also given by an integral

$$4 \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-x^2/3)}}$$

Use a Gaussian quadrature routine to compute this integral. There is yet another way to compute P (also due to Gauss) using the arithmetic- geometric mean. Let $a_0 = 1, b_0 = \sqrt{2/3}$. For $n \geq 0$, let $a_{n+1} = (a_n + b_n)/2$, $b_{n+1} = \sqrt{a_n b_n}$. The sequences $\{a_n\}, \{b_n\}$ converge *very rapidly* to a common limit L. Gauss showed that $P = 2\pi/L$.

Project # 5 (Extra Credit, due June 1)

1. Use finite differences to approximate the solution of Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$, with the boundary conditions $u(x, 0) = 0, u(0, y) = 0, u(x, 1) = 10 \sin \pi x, u(1, y) = 0$.

(a) Use the Gauss-Seidel method to solve the matrix equation that arises. Continue the iteration until two successive approximations differ by less than TOL or until MAX iterations have occurred. Take $TOL = .01, MAX = 200$. Let the step size $h = \pi/N$, where N takes on the values $10, 20, 50$. For this problem find the number of iterations required for each step size. *Do not print out the solution.*

(b) Do the this problem (with $N = 50$) using SOR with the acceleration parameter ω assuming the values $1.0, 1.1, \ldots, 1.9$ as well as the optimal value $\omega_{opt} = 2/(1 + \sin(\pi/50))$. In each case find the number of iterations until TOL is obtained.

2. Consider the heat equation $u_t = u_{xx}, 0 \leq x \leq 1, t \geq 0$, with boundary conditions $u(x, 0) = f(x), u(0, t) = 0 = u(1, t)$.

(a) Solve the equation by the direct method. Let $h = 1/M$ be the step size in the x direction. Let the t step size be $k = rh^2$. Let the final t value be c. The program should return the values $u(x_j, c), j = 0, \ldots, M$. Run the program with the data: $f(x) = \sin \pi x + 4 \sin 4\pi x, M = 20$, and $c = .0625, .125, .25$, and let $k = 1/400$ $(r = 1)$, and $k = 1/1200$ $(r = 1/3)$. Plot the results.

(b) Solve the equation with the implicit method where $k = 1/40, 1/100, 1/400$ and $c = .0625, .125, .25$. Plot the results for $k = 1/40$.

(c) Solve the equation with the Crank-Nicholson method, taking $k = 1/40, 1/100$ and $c = .0625, .125, .25$. Plot the results for $k = 1/40$.