Divided Differences

This document will discuss general divided differences. A formula for Herite interpolation will be derived.

Let values of a function f be given: $f(x_0), f(x_1), \dots, f(x_n)$, or generally pairs (x_j, y_j). First assume that $x_i \neq x_j$ if $i \neq j$. Define by induction,

$$\begin{aligned} f[x_j] &= f(x_j) \\ [y_j] &= y_j \\ f[x_j, \dots, x_{j+k+1}] &= \frac{f[x_{j+1}, \dots, x_{j+k+1}] - f[x_j, \dots, x_{j+k}]}{x_{j+k+1} - x_j} \\ [y_j, \dots, y_{j+k+1}] &= \frac{[y_{j+1}, \dots, y_{j+k+1}] - [y_j, \dots, y_{j+k}]}{x_{j+k+1} - x_j}. \end{aligned}$$

Proposition 1. *Let*

$$p_n(x) = [y_0] + [y_0, y_1](x - x_0) + \cdots + [y_0, \dots, y_n](x - x_0)(x - x_1)\dots(x - x_{n-1}).$$

Then p_n is the unique polynomial of degree n such that $p(x_j) = y_j, j = 0, \dots, n$.

Proof. The proof is by induction. It is true when $n = 0$. It is clear, by induction, that

$$p_{n-1}(x_j) = y_j, j = 0, \dots, n - 1.$$

Let P be the interpolating polynomial. Then $P(x_j) - p_{n-1}(x_j) = 0, j = 0, \dots, n - 1$, so $P - p_{n-1} = c(x - x_0)\dots)(x - x_{n-1})$. Hence c is the coefficient of x^n in P. P is unique and can be represented in another way. Let u, v be the polynomials of degree $n - 1$ that interpolate at $x_0, \dots, x_{n-1}$ and $x_1, \dots, x_n$ respectively. By uniqueness

$$P = \frac{(x - x_0)v + (x_n - x)u}{x_n - x_0},$$

hence the coefficient of x^n in P is

$$\frac{[y_1, \dots, y_n] - [y_0, \dots, y_{n-1}]}{x_n - x_0}.$$

□

Remark 1. *If σ is a permutation,*

$$[x_0, x_1, \dots, x_n] = [x_{\sigma(0)}, x_{\sigma(1)}, \dots, x_{\sigma(n)}].$$

Proof. The coefficients in the interpolating polynomial are unique. No matter how we order the points, the leading term $(x - x_0)(x - x_1)\dots(x - x_n) = (x - x_{\sigma(0)})\dots(x - x_n)$. The result follows. □

Remark 2.

$$[x_0, x_1, \ldots, x_n] = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} + \ldots + \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}.$$

Proof. The proposition implies that the coefficient of x^n in the interpolating polynomial is $[y_0, y_1, \ldots, y_n]$. The result follows by computing the coefficient of x^n in the Lagrange form of the interpolating polynomial

$$\frac{y_0(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} + \cdots + \frac{y_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}.$$

□

Now we consider the problem of finding the polynomial that interpolates derivative data. To simplify notation consider the special case of finding the polynomial p of degree 7 that satisfies

$$p(x_0) = y_{00}, p^{(k)}(x_1) = y_{1k}, k = 0\ldots, 3, p^j(x_2) = y_{2j}, j = 0\ldots2.$$

By using linear independence of the terms in this expression we see that every polynomial of degree 8 is uniquely expressible in the form

$$\begin{aligned} p(x) =& a_0 + b_0(x - x_0) + b_1(x - x_0)(x - x_1) + b_2(x - x_0)(x - x_1)^2 \\ &+ b_3(x - x_0)(x - x_1)^3 + b_4(x - x_0)(x - x_1)^4 + c_0(x - x_0)(x - x_1)^4(x - x_2) \\ &+ c_1(x - x_0)(x - x_1)^4(x - x_2)^2. \end{aligned}$$

This can be proved, starting with a_0, by differentiating and evaluating. We want to find a formula for the coefficients. We consider the following matrix

$$\begin{matrix} x_0 & x_1 & x_1 & x_1 & x_1 & x_2 & x_2 & x_2 \\ y_{00} & y_{10} & y_{10} & y_{10} & y_{10} & y_{20} & y_{20} & y_{20} \end{matrix}$$

We define

$$[p(x_0)] = y_{10} = p(x_0) = y_0, [p(x_1)] = y_{10} = p(x_1) = y_1, [p(x_2)] = y_{20} = p(x_2) = y_2.$$

Next

$$\begin{aligned} [y_0, y_1] = [p(x_0), p(x_1)] =& \frac{[p(x_1)] - [p(x_0)]}{x_1 - x_0}, [y_1, y_1] = [p(x_1), p(x_1)] = y_{11} \\ [y_1, y_2] = [p(x_2), p(x_1)] =& \frac{[p(x_2)] - [p(x_1)]}{x_2 - x_1}, [y_2, y_2] = [p(x_2), p(x_2] = y_{21} \end{aligned}$$

Continuing,

$$\begin{aligned} [y_0, y_1, y_1] =& \frac{[y_1, y_1] - [y_0, y_1]}{x_1 - x_0}, [y_1, y_1, y_1] = y_{12} \\ [y_1, y_1, y_2] =& \frac{[y_1, y_2] - [y_1, y_1]}{x_2 - x_1}, [y_1, y_2, y_2] = \frac{[y_2, y_2] - [y_1, y_2]}{x_2 - x_2} \\ [y_2, y_2, y_2] =& y_{22} \end{aligned}$$

This pattern continues, in particular

$$[y_1, y_1, y_1, y_1] = y_{13}$$

Theorem 1. *The coefficients of terms of the form*

$$(x - x_0)^j (x - x_1)^k (x - x_2)^\ell$$

are

$$[y_0, y_1, y_1, \ldots, y_2, y_2, \ldots, y_2].$$

Explicitly

$$\begin{aligned}
a_0 =& [y_0], b_0 = [y_0, y_1], b_1 = [y_0, y_1, y_1], b_2 = [y_0, y_1, y_1, y_1] \\
b_3 =& [y_0, y_1, y_1, y_1, y_1], c_0 = [y_0, y_1, y_1, y_1, y_1, y_2], c_1 = [y_0, y_1, y_1, y_1, y_1, y_2, y_2], \\
c_2 =& [y_0, y_1, y_1, y_1, y_1, y_2, y_2, y_2].
\end{aligned}$$