Newton's Method

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This note will explain Newton's method and quadratic convergence.

Newton's method for the solution of a non-linear equation $f(x) = 0$ is the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This iteration looks for a fixed point of the function $x - \frac{f(x)}{f'(x)}$. Let s be a fixed point of g and assume that $f'(s) =\neq 0$. Then $s - \frac{f(s)}{f'(s)} = s$ implies that $f(s) = 0$. Conversely if $f(s) = 0$ and $f'(s) \neq 0$ then $g(s) = s - \frac{f(s)}{f'(s)} = s$. Computing $g'(s)$ we find that $g'(s) = \frac{f(s)f''(s)}{f'(s)^2} = 0$. Taylor's theorem at the point s is

$$g(x) = g(s) + g'(s)(x - s) + \frac{g"(\xi)}{2}(x - s)^2 = s + \frac{g"(\xi)}{2}(x - s)^2, \tag{1}$$

where ξ is between x and s. Suppose that x_0 is an initial guess and the errors are denoted by $e_n = x_n - s$. Suppose that $|\frac{g"(\xi)}{2}| \leq K$. Then

$$|e_{n+1}| \leq K|e_n|^2. \tag{2}$$

This follows easily from (1). Now we have a good general result:

Theorem 1. *The errors in Newton's method satisfy*

$$|e_n| \leq \frac{1}{K}(K|e_0|)^{2^n}. \tag{3}$$

Proof. The proof is by induction. It is true for $n = 0$. Assume (3). Then by (2).

$$|e_{n+1}| \leq K\left(\frac{1}{K}(K|e_0|)^{2^n}\right)^2 \tag{4}$$

$$= \frac{1}{K}(K|e_0|)^{2^{n+1}} \tag{5}$$

□

We will apply this to Newton's method for finding square roots. Let $c > 1$. Newton's method for solving $x^2 - c = 0$ is to find a fixed point of $g(x) = \frac{x}{2} - \frac{c}{2x}$. Then $g''(x) = \frac{c}{x^3}$. Suppose $x \geq \sqrt{c}$. Then $|\frac{g''(x)}{2}| \leq K = \frac{\sqrt{c}}{2}$. Now suppose we choose x_0 so that $x_0 \geq \sqrt{c}$ and $K|e_0| \leq .1$. Then by (3)

$$|e_n| \leq \frac{2}{\sqrt{c}}10^{-2^n} \leq 2(10)^{-2^n}.$$

So after 4 steps we have better than double precision (16 decimal digits) of precision.