Connectivity of Mandelbrot's Percolation Process

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June 6, 2013

Contents

1 Abstract

The purpose of this paper is to review the results of Chayes, Chayes, and Durett (CC&D) in their paper *Connectivity Properties of Mandelbrot's Percolation Process* [1]. The Mandelbrot percolation process involves subdividing a square and discarding each sub-square independently with probability p, forming a set. The process is then applied recursively for every remaining sub-square to form successive sets. The most striking result from CC&D's paper is that if the process is iterated indefinitely, and if the plane is tiled with independently generated copies of the set, then there exists a critical probability $p_c \in (0, 1)$ such that if p is greater than or equal to p_c, then the union of the sets contains an unbounded, connected component; however if p is less than p_c, the largest connected component is a point; that is, any point still remaining in the set is disconnected from every other point in the set.

2 Introduction and Definitions

2.1 Notation

Throughout this paper, we will be using notation related to probability theory. We shall let $\{event\}$ denote a probabilistic event; that is, a set of possible outcomes in a sample space. The notation $P(E)$, where E is an event, shall denote the probability measure of that event taking place. Unfortunately, a discourse on probability measure is outside the scope of this paper, so we shall rely on the reader's intuition for the construction of the necessary measure.

2.2 Ordinary Percolation

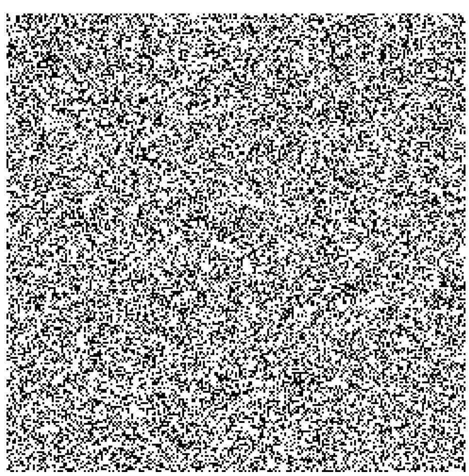

Fig. 2.1: Site percolation, open with $p = 0.644$

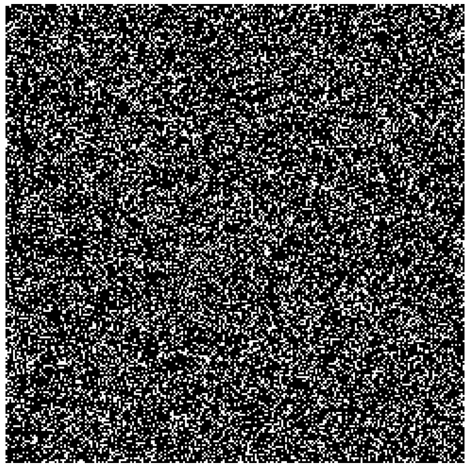

Fig. 2.2: Site percolation, open with $p = 0.257$

We start with a definition of "ordinary" percolation, which is in some sense a simpler process than Mandelbrot percolation. Specifically, will be looking at site percolation, as it is more closely related to Mandelbrot percolation.

We shall start with the finite case. Fix a probability p, such that $0 < p < 1$. Let $R = [0, L] \times [0, L]$ be a square of side length L. Let N be a positive integer, and subdivide R into N^2 subsquares of equal size. For each subsquare $[iL/N, (i+1)L/N] \times [jL/N, (j+1)L/N]$, independently fill in the subsquare with probability p. We shall refer to a filled in subsquare as *closed* and an unfilled subsquare as *open*. Though the resulting set has many interesting properties, the one we are chiefly concerned with is the probability of finding a path through open points.

In the infinite case, we typically take the subsquares above to be the squares of the form $[i, i+1] \times [j, j+1]$ where $i, j \in \mathbb{Z}$, and apply the above procedure to every such square. The main focus of interest in the infinite case is, given a point in the plane, whether or not it can be connected to the origin through paths that only go through open subsquares.

2.3 Mandelbrot Percolation

Here we give a somewhat more rigorous definition of the Mandelbrot percolation process. Just as in site percolation, we fix a probability parameter p. We also set N to be the degree of subdivision such that at each step, we subdivide into N^2 smaller squares. Letting $A_0 = [0, 1]^2$, we then define A_n as follows: with $1 \leq i, j \leq N$, let

$$B^n_{i,j} = \left[\frac{i-1}{N^n}, \frac{i}{N^n}\right] \times \left[\frac{j-1}{N^n}, \frac{j}{N^n}\right].$$

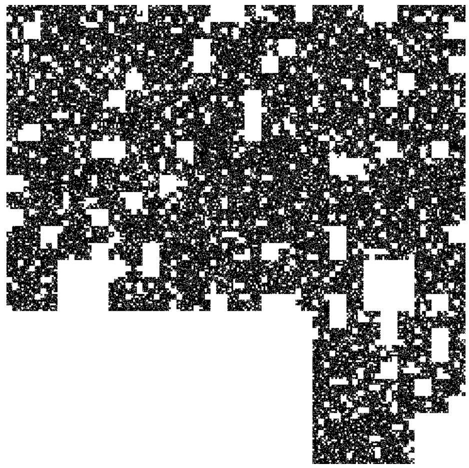

Fig. 2.3: Mandelbrot percolation, kept with $p = 0.8888$

Let $\lambda^n_{i,j} \in \{0, 1\}$ be independent random "coin flips" such that $P(\lambda^n_{i,j} = 1) = p$. We then let

$$A_n = A_{n-1} \cap \left(\bigcup_{\lambda^n_{i,j}=1} B^n_{i,j} \right).$$

We shall define A_∞ as the set resulting when n is taken to infinity.

The set A_∞ has some interesting properties that, while not the focus of this paper, are still worthwhile to briefly discuss. A slightly more thorough discussion can be found in CC&D [1].

(1) $A_\infty \neq \emptyset$ with positive probability if and only if $p > 1/N^2$.

If the probability parameter p falls too low, then not only will A_∞ be sparse but it will cease to contain any points at all.

(2) If $p \leq 1/N$ and x is not of the form m/N^n for some integers m and n, then $P(A_\infty \cap (\{x\} \times [0, 1]) = \emptyset) = 1$.

If p is once again too low, then with probability 1 and except for certain values of x, one can draw a horizontal line through $[0, 1]^2$ at height x without hitting a single point of A_∞. From this, we can conclude that the largest connected component of A_∞ must be a point.

(3) If $p \leq 1/\sqrt{N}$, then the largest connected component of A_∞ is a point.

This lemma simply strengthens the previous statement by increasing the lower bound for which A_∞ ceases to be disconnected.

These statements imply that we can find an actual value p for which A_∞ stops being completely disconnected; we shall call this value p_d.

Definition. Let $p_d = \sup\{p : P(A_\infty \text{ is completely disconnected}) = 1\}$.

Just as in site percolation, we wish to examine the probabilities of a left-right crossing of A_∞, and at what point a left-right crossing becomes assured. This motivates another definition:

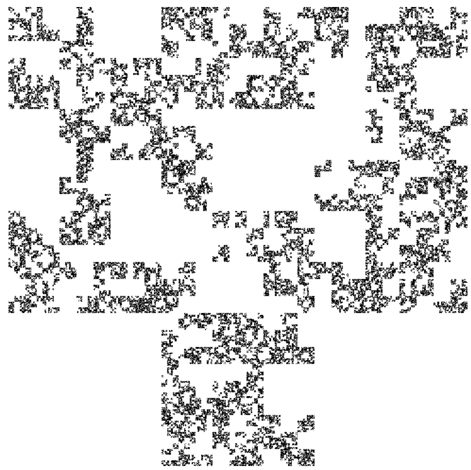

Fig. 2.4: Mandelbrot percolation, kept with $p = 0.7071$

Definition. Let $p_c = \inf\{p : P(A_\infty \text{ has a left-right crossing}) = 1\}$.

It is intuitive that p_d must be less than p_c; after all, we must cease being disconnected before we can start crossing the square. However, as we will prove, the values p_c and p_d are in fact exactly equal to each other.

3 Analysis of Connectivity

We wish to define some notion of "connectedness" in A_∞; for the purposes of this paper, the existence of a left-right crossing of $[0,1]^2$ within A_∞ is considered sufficient.

Definition. Let B_n be the set of all points x in A_n such that x can be connected to the left and right sides of the square (that is, $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$) by paths in A_n.

Definition. Let B_∞ be the intersection $\bigcap_{n=1}^{\infty} B_n$.

Definition. Let Ω_1^n be the event $\{B_n \neq \emptyset\}$, and Ω_1^∞ be the event $\{B_\infty \neq \emptyset\}$.

Definition. A *left-right crossing* of $[0,1]^2$ exists if and only if the event Ω_1^∞ occurs.

Theorem 1. *$p_c(N) < 1$ for all $N \geq 2$.*

This theorem states that the minimum required parameter p for having a positive probability of a left-right path in A_∞ does in fact exist with a non-trivial value. The proof follows the case where $N = 3$, but can be easily generalized to $N \geq 3$. The case for $N = 2$ can be compared to the case for $N = 4$. We start with $N = 3$ because this gives us the nice property that if at least 8 of the 9 sub-squares of a square are occupied, then any two adjacent sub-squares have adjacent occupied boundary sub-squares.

Theorem 2. *There is an $\varepsilon_0 > 0$ so that if $P(\Omega_1^n) \leq \varepsilon_0$ then $P(\Omega_1^\infty) = 0$. Furthermore, the largest connected component is a point.*

This theorem shows that if the probability of a Ω_1^n occurring drops below some threshold, then there will be no left-right path for A_∞ and the set will "disintegrate": any given point that still remains in A_∞ will not be connected to any other points in A_∞. It also follows from this theorem that $P(\Omega_1^\infty)$ holds a positive value at p_c (and is zero if $p < p_c$). The proof of this theorem is based off of a similar proof from "ordinary" percolation, after some details are generalized to work with our setting.

Definition. Let $\Omega_{1,K}^n$ be the event that there is a left-right crossing of $[0,1] \times [0,K]$, where for $1 \leq k \leq K$ each square $[0,1] \times [k-1,k]$ is filled with an independent copy of A_n.

Lemma 2.1. *if $P(\Omega_{1,1}^n) \leq \varepsilon$ then $P(\Omega_{1,K}^n) \leq f_K(\varepsilon)$ where $f_K(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Lemma 2.2. *if $P(\Omega_{1,2}^n) \leq 0.01$ then $P(\Omega_{1,2}^{n+k}) \leq \frac{1}{25}\exp(-N^{k-1})$.*

With these two lemmas, it is fairly simple to prove Theorem 2. We first pick ε_0 such that $f_2(\varepsilon) \leq 0.01$ for all $\varepsilon \leq \varepsilon_0$. If we have n such that $P(\Omega_{1,1}^n) < \varepsilon_0$, then the above two lemmas imply $P(\Omega_{1,1}^n) \leq P(\Omega_{1,2}^n)$ and that $P(\Omega_{1,2}^n)$ heads to zero as n heads to infinity. Applying Lemma 2.1 again shows $P(\Omega_{1,K}^n) \to 0$ for all $K < \infty$. Note that the event "there does not exist a left-right crossing through elements in the set" is equivalent to the event "there exists a top-bottom crossing through elements removed from the set". We shall call this latter event a *crack*. Due to the self-similarity properties of A_n, we can then derive that there exists, with probability 1, a crack in $[a,b] \times [0,1]$ for all $a < b$ of the form $a = j/N^m, b = k/N^m$. Since this will also be true for rectangles of the form $[0,1] \times [a,b]$, it follows that as $n \to \infty$, every point will be separated from every other point by these cracks.

With Theorem 2, we can now determine that the probability of a left-right crossing is greater than zero when $p = p_c$. To prove this, note that since $P_p(\Omega_1^n)$, as a function of p, is continuous and decreases to $P_p(\Omega_1^\infty)$ as $n \to \infty$, $P_p(\Omega_1^\infty)$ is upper semi-continuous. Since $P_p(\Omega_1^\infty)$ is non-decreasing, it must be right continuous on $[0,1]$, and hence must be greater than zero at $p = p_c$.

The last theorem requires the definition of an additional construct: we take A'_∞ to be the tiling across the plane of independently formed copies of A_∞. That is, in each square $[z_1, z_1+1] \times [z_2, z_2+1]$ for $z \in \mathbb{Z}^2$, we place in independent copy of A_∞.

Definition. Let Ω_∞ be the event that there exists in A'_∞ an unbounded connected component.

Definition. Let $p_b = \inf\{p : P(\Omega_\infty) > 0\}$. It is easy to see that $p_b \geq p_c$. To prove our final results, we wish to show that p_b is in fact equal to p_c (which is equal to p_d by the theorem above).

Theorem 3. *If $p \geq p_c$, then with probability 1, A'_∞ has a unique unbounded connected component.*

The consequences of this theorem are the most striking. We know that if p is less than p_c, then the largest connected component in A'_∞ is a point, and yet the moment p equals p_c, we witness the appearance of a unique unbounded connected component.

The proof proceeds as follows: We can rescale A'_∞ by dividing by N. We then observe that if we flip new coins to see which squares of the form $[(i/N), (i+1)/N] \times [j/N, (j+1)/N]$ we keep, the result has the same distribution as A'_∞. If we rescale and keep all of our existing squares, then this is the equivalent of having all coins land "heads". Thus,

$$A'_\infty/N \overset{d}{=} (A'_\infty|\text{all } \lambda^1_{ij}).$$

where the symbol $\overset{d}{=}$ is used to denote the equivalence of the distributions. Generalizing this result, we have

$$A'_\infty/N^n \overset{d}{=} (A'_\infty|\text{all } \lambda^m_{ij}, m \leq n).$$

From this, we can intuitively see that the probability of a left-right crossing of $[0, N^n]^2$ in A'_∞ heads to 1 as N^n increases to infinity. We can prove this formally by first observing that $P(\Omega^n_1)$ approaches $P(\Omega^\infty_1)$ as $n \to \infty$, so with $\varepsilon > 0$ and n large, we have

$$P(\Omega^n_1 - \Omega^\infty_1) \leq \varepsilon P(\Omega^\infty_1) \leq \varepsilon P(\Omega^n_1)$$

where $\Omega^n_1 - \Omega^\infty_1$ represents the event of a left-right crossing in A_m for $m \leq n$, but not for $m > n$. Through some simple manipulation, we see that this is equivalent to the statement

$$P(\Omega^\infty_1|\Omega^n_1) \geq 1 - \varepsilon,$$

and thus

$$P(\Omega^\infty_1|\lambda^m_{ij} = 1, m \leq n) \geq 1 - \varepsilon.$$

With this, we have shown that the crossing probabilities of large squares is close to 1. The remaining proof follows from two additional lemmas, similar to those from Theorem 2. To make the proof slightly easier, we will consider the situation after n subdivisions and prove results which are independent of n.

Definition. Let A'_n be the set which results when we place independent copies of A_n in each square $[z_1, z_1 + 1] \times [z_2, z_2 + 1]$ for $z \in \mathbb{Z}^2$.

Definition. Let $\Omega^n_{J,K}$ be the event that there is a left-right crossing of $[0, J] \times [0, K]$ in A'_n.

Lemma 3.1. *if* $P(\Omega^n_{L,L}) \geq 1-\varepsilon$ *then* $P(\Omega^n_{kL,L}) \geq 1 - g_k(\varepsilon)$ *where* $g_k(\varepsilon)$ *is independent of* n *and* $g_k \to 0$ *as* $\varepsilon \to 0$.

Lemma 3.2. *if* $P(\Omega^n_{2L,L}) \geq 0.99$ *then* $P(\Omega^n_{2^kL,2^{k-1}L}) \geq 1 - \frac{1}{25}\exp(-2^{k-1})$.

We invite the reader to notice the similarity between these two lemmas and the two we required for Theorem 2.

Taking L such that $P(\Omega^n_{2L,L}) \geq 0.99$ for all n when $p = p_c$, let $R_1 = [0,2L]\times[0,L]$, $R_2 = [0,2L]\times[0,4L]$, and for $j \geq 2$ let

$$R_{2j-1} = [0,2^jL]\times[0,2^{j-1}L]$$
$$R_{2j} = [0,2^jL]\times[0,2^{j+1}L].$$

Furthermore, we define the following set of events:

$$X^n_{2j-1} = \text{there is a left-right crossing of } R_{2j-1} \text{ in } A'_n$$
$$X^n_{2j} = \text{there is a top-bottom crossing of } R_{2j} \text{ in } A'_n.$$

These events are chosen so that if all crossing occur, then we are guaranteed an infinite path in A'_n starting in $\{0\}\times[0,L]$. Using the estimate from Lemma 3.2, we have

$$P(X^{n,c}_{2j-1}) \leq \tfrac{1}{25}\exp(-2^{j-1})$$
$$P(X^{n,c}_{2j}) \leq \tfrac{1}{25}\exp(-2^{j})$$

where the superscript c represents the complement. Using the facts $2(j-1) \leq 2^{j-1}$ and $e \geq 2$, we have

$$\sum_{k=1}^{\infty} P(X^{n,c}_k) \leq \sum_{j=1}^{\infty} 2\cdot\tfrac{1}{25}(e^{-2})^{j-1} = \tfrac{2}{25}(1-e^{-2})^{-1} \leq \tfrac{2}{25}\cdot\tfrac{4}{3} < \tfrac{1}{9}.$$

From this it immediately follows that $P\left(\bigcap_{k=1}^{\infty} X^n_k\right) > 8/9$, and thus with a probability greater than 8/9 there is an infinite path in A'_n starting in $\{0\}\times[0,L]$. Letting $n\to\infty$ it is easy to see that this result still holds for $n=\infty$.

We now have the existence of an unbounded component when $p \geq p_c$. The proof that it is unique is the same as in the ordinary case, and is properly developed in Harris [3].

It is interesting to note that in ordinary site percolation, the transition in the existence of the component is continuous, while in Mandelbrot percolation there is a violent discontinuity. As is hopefully clear to the reader, this is largely due to the fact that while ordinary percolation is "symmetric" in some sense with respect to the open/closedness of its subsquares, Mandelbrot percolation contains a fundamental asymmetry: while vacant crossings persist to all further iterations, occupied crossings may be lost in any subsequent steps.

3.1 Proof of Theorem 1

Recall that we will only be proving the case $N = 3$, as cases for larger N will proceed in the same manner.

Definition. Let $A^1_{i,j}$ be a subsquare of A_1, and let $A^n_{i,j}$ be a subsquare of $A^{n-1}_{i,j}$.

Theorem 1. *$p_c(N) < 1$ for all $N \geq 2$.*

Proof. We will call an outcome *good* if A_1 contains at least 8 squares $A^1_{i,j}$. Additionally, we will call an outcome *very good* if each $A^1_{i,j}$ of a good A_1 is itself good. Thus, we can define the concept of (very)^m goodness of A_1 as containing at least 8 occupied sub-squares $A^1_{i,j}$ which are each (very)$^{m-1}$ good. Let θ_m be the probability that the outcome is (very)^m good. We can see from the definitions above that

$$\theta_0 = p^9 + 9p^8(1-p)$$

and

$$\theta_m = p^9(\theta^9_{m-1} + 9\theta^8_{m-1}(1-\theta_{m-1})) + 9p^8(1-p)\theta^8_{m-1} \tag{3.1}$$

for $m \geq 1$. If we assign θ_{-1} the value 1, then we can in fact calculate θ_0 from (3.1). We can thus think of θ_m as the value of $\varphi^{m+1}(1)$, where

$$\varphi(x) = p^9(9x^8 - 8x^9) + 9p^8(1-p)x^8$$

and $\varphi^{m+1}(x) = \varphi(\varphi^m(x))$. It follows that as n increases to infinity, $\varphi^n(1)$ decreases to a value ρ, the largest fixed point of φ in the interval $[0, 1]$.

Letting $\alpha = p^9$ and $\beta = 9p^8(1-p)$, we can rewrite (3.1) as

$$\varphi(x) = (9\alpha + \beta)x^8 - 8\alpha x^9.$$

Let $x = 1 - \varepsilon$. We can expand the expression $(1-\varepsilon)^k$ as

$$(1-\varepsilon)^k = 1 - k\varepsilon + \frac{k(k-1)}{1 \cdot 2}\varepsilon^2 - \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3}\varepsilon^3 + \cdots$$

From this, we can easily see that when $(9-3)\varepsilon/4 < 1$,

$$(1-\varepsilon)^9 \leq 1 - 9\varepsilon + 36\varepsilon^2. \tag{3.2}$$

Then for $\varepsilon < 2/3$,

$$\begin{aligned}\varphi(1-\varepsilon) &\geq (9\alpha + \beta)(1 - 8\varepsilon) - 8\alpha(1 - 9\varepsilon + 36\varepsilon^2) \\ &= (\alpha + \beta) - 8\beta\varepsilon - 288\alpha\varepsilon^2.\end{aligned}$$

We know that both α and β are greater than zero, and also that their sum is less than or equal to one. Thus, if we take $\varepsilon < 1/8 (< 2/3)$, we can see that $\varphi(1-\varepsilon) \geq \alpha - 288\varepsilon^2$. With this result, we now let $\varepsilon = 0.001$ and $\alpha = 1 - 0.5\varepsilon$ to calculate $\varphi(1-\varepsilon) \geq 1 - 0.788\varepsilon$. Thus, φ has a fixed point in the interval $[0.999, 1]$. Since $\alpha = p^9$, and from (3.2) we have $(1-\delta)^9 \geq 1 - 9\delta$ when $(9-2)\delta/3 < 1$, it follows that if $p > 0.9999$ then $P(\Omega_1) > 0.999$. $\square$

3.2 Connectedness for Ordinary Percolation

In order to prove the next few results about Mandelbrot percolation, we wish to generalize from similar results from "ordinary" percolation. To begin, we work up from some basic lemmas.

Lemma 4.1 (Harris' Inequality)**.** *If E and F are both increasing or both decreasing events, then*

$$P(E \cap F) \geq P(E)P(F).$$

A detailed proof of this lemma is given in Kesten (1982) [5], though it is available in many of the other references we consulted. If the reader is so inclined, we invite them to read through the proof, as it is interesting in its own right.

Lemma 4.2 (The Square Root Trick)**.** *Let A_1 and A_2 be increasing events. If $A = A_1 \cup A_2$ and $P(A_1) = P(A_2)$, then*

$$P(A_1) \geq 1 - (1 - P(A))^{1/2}.$$

Proof. Utilizing Harris' Inequality and some basic set theory, we get:

$$\begin{aligned}(1 - P(A_1))^2 &= P(A_1^c)^2 = P(A_1^c)P(A_2^c) \\ &\leq P(A_1^c \cap A_2^c) = 1 - P(A)\end{aligned}$$

and so $P(A_1) \geq 1 - (1 - P(A))^{1/2}$. □

Definition. Let $\rho_{J,K}$ be the probability there is a left-right crossing of $[0, J] \times [0, K]$ by open sites when sites are independently open with probability p (and thus independently closed with probability $1 - p$).

We now wish to prove a crucial building block:

$$\rho_{3L/2,L} \geq (1 - (1 - \rho_{L,L})^{1/2})^3. \tag{3.3}$$

The following procedure is based off of Russo (1981) [7], as well as Harris [3]. It appears both were developed from a version by Seymour and Welsh [8]. The reader is referred to any of these papers for alternative statements and explanations if the following seems confusing. We now direct the reader's eyes to Fig. 3.1 and set up a series of definitions:

Let s be a left-right crossing of $[0, L] \times [0, L]$.

Let E_s be the event that s is the lowest such left-right crossing.

Let s_r be the portion of s from the time it last hits $\{L/2\} \times [0, L]$ until it reaches $\{L\} \times [0, L]$ (represented by the thick line in Fig. 3.1).

Let s_{rr} be the reflection of s_r through $\{L\} \times [0, L]$ (represented by the dotted line in Fig. 3.1).

Let $\mathscr{A}(s_r \cup s_{rr})$ be the points in $[L/2, 3L/2] \times [0, L]$ strictly above $s_r \cup s_{rr}$.

Let F_s be the event that there is a path starting from $[L/2, 3L/2] \times \{L\}$ and connected to s_r in $\mathscr{A}(s_r \cup s_{rr})$.

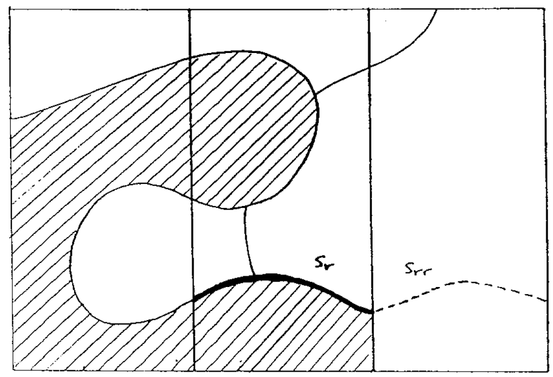

Fig. 3.1

Let S_l be the set of all paths s for which the first point of s_r has a y-coordinate less than $L/2$.

Let G be the union of $E_s \cap F_s$ over all paths $s \in S_l$.

Let H be the event that there is a left-right crossing of $[L/2, 3L/2] \times [0, L]$ which starts at a point with y-coordinate greater than $L/2$.

If we have both G and H, then it is easy to see that there exists a left-right crossing of $[0, 3L/2] \times [0, L]$. Thus, to prove (3.3) it suffices to show

$$P(G \cap H) \geq (1 - (1 - \rho_{L,L})^{1/2})^3.$$

Proof. By Harris' Inequality, we have

$$P(G \cap H) \geq P(G)P(H).$$

We then apply the square root trick using the events H and H', where H' is the event H reflected over the line $y = 1/2$, to get:

$$P(H) \geq (1 - (1 - \rho_{L,L})^{1/2}).$$

We can estimate $P(G)$ as:

$$P(G) = \sum_{s \in S_l} P(E_s \cap F_s) = \sum_{s \in S_l} P(E_s)P(F_s|E_s).$$

Another application of the square root trick, again using a reflection over $y = 1/2$, yields

$$\sum_{s \in S_l} P(E_s) \geq 1 - (1 - \rho_{L,L})^{1/2}.$$

Note that, since F_s is independent of E_s,

$$P(F_s|E_s) = P(F_s). \qquad (*)$$

Since a top-bottom crossing of the square $L/2, 3L/2]$ would necessarily imply either F_s or F'_s (the reflection over $x = L$; note that s_{rr} does not have to be an open path), we can apply the square root trick one last time to yield

$$P(F_s) \geq 1 - (1 - \rho_{L,L})^{1/2}.$$

Putting everything together we have (3.3). □

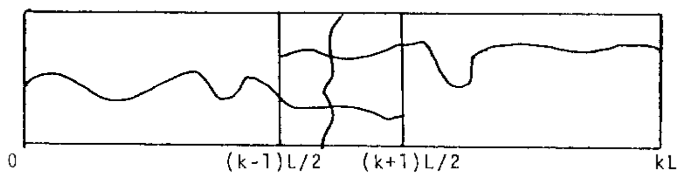

Fig. 3.2

We now wish to show that, with $k \geq 1$:

$$1 - \rho_{kL,L} \leq 3(1 - \rho_{(k+1)L/2,L}). \tag{3.4}$$

Proof. Observe in Fig. 3.2 that if all three paths exist, then there is a left-right crossing of the rectangle. Let X_1 be the event of a left-right crossing of $[0, (k-1)L/2] \times [0, L]$, and X_2 and X_3 be similar events for $[0, L] \times [0, L]$ and $[0, (k+1)L/2] \times [0, L]$. From basic probability theory, we know that

$$P\left(\bigcup_{i=1}^{3} X_i^c\right) \leq \sum_{i=1}^{3} P(X_i^c).$$

Since it is easy to see that $\rho_{L,L} \geq \rho_{(k+1)L/2,L}$ for $k \geq 1$, our proof is complete. □

Combining (3.3) and (3.4), we can derive:

$$\begin{aligned} \rho_{3L/2,L} &\geq (1 - (1 - \rho_{L,L})^{1/2})^3 \\ 1 - \rho_{2L,L} &\leq 3(1 - \rho_{3L/2,L}) \\ 1 - \rho_{3L,L} &\leq 3(1 - \rho_{2L,L}) \end{aligned}$$

and so on, which bounds $\rho_{kL,L}$ in terms of $\rho_{L,L}$. Thus, we can state the following:

Lemma 4.3. *if $\rho_{L,L} \geq 1 - \varepsilon$ then $\rho_{kL,L} \geq 1 - h_k(\varepsilon)$ where $h_k(\varepsilon)$ is independent of L and $h_k \to 0$ as $\varepsilon \to 0$.*

We now need just one other lemma:

Lemma 4.4. *if $\rho_{2L,L} \geq 0.99$ then*

$$\rho_{2^k L, 2^{K-1} L} \geq 1 - \tfrac{1}{25} \exp(-2^{k-1}).$$

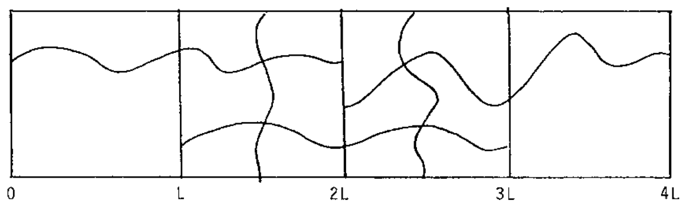

Fig. 3.3

Proof of this lemma requires the use of the following two inequalities:

$$1 - \rho_{4L,L} \leq 5(1 - \rho_{2L,L}). \tag{3.5}$$

$$\rho_{4L,2L} \geq 1 - (1 - \rho_{4L,L})^2. \tag{3.6}$$

Proof. To prove (3.5), refer to Fig. 3.3 and observe that if all five paths exist, then there is a left-right crossing of the entire rectangle. The proof can be argued as in the proof of (3.4).

To prove (3.6), observe that the existence of a crossing in $[0, 4L] \times [0, L]$ and $[0, 4L] \times [L, 2L]$ are independent events. Furthermore, a crossing of $[0, 4L] \times [0, 2L]$ must occur if at least one of the above crossing occurs. $\square$

Combining (3.5) and (3.6), we have

$$\rho_{4L,2L} \geq 1 - 25(1 - \rho_{2L,L})^2 \tag{3.7}$$

Now if $\rho_{2L,L} = 1 - \delta/25$, where $\delta < 1$, then from (3.7) we have:

$$\rho_{4L,2L} \geq 1 - \delta^2/25$$
$$\rho_{8L,4L} \geq 1 - \delta^4/25$$

and so on. That is to say,

$$\rho_{2^k L, 2^{k-1} L} \geq 1 - \tfrac{1}{25} \exp(2^{k-1} \log \delta).$$

Letting $\delta = 1/4$, and using the fact that $\log(1/4) < -1$, we have 4.4.

While these two lemmas apply only to "ordinary" percolation, we only need to make a few minor adjustments to adapt them to our discussion of Mandelbrot percolation.

3.3 Proofs of Theorems 2 and 3

The reader should be informed that Harris' Inequality generalizes to our discussion easily, as the variables that indicate whether the squares $[j/N^n, (j + 1)/N^n] \times [k/N^n, (k + 1)/N^n]$ are occupied or not are increasing functions of independent random variables. Thus, any step using or deriving from Harris' Inequality can remain unchanged, and we only need to watch out for the two places we made assumptions about independence. We will first prove Lemmas 3.1 and 3.2, as these follow more easily from the results in the previous section. We will then finish up by proving Lemmas 2.1 and 2.2.

Lemma 3.1. *if* $P(\Omega^n_{L,L}) \geq 1 - \varepsilon$ *then* $P(\Omega^n_{kL,L}) \geq 1 - g_k(\varepsilon)$ *where* $g_k(\varepsilon)$ *is independent of* n *and* $g_k \to 0$ *as* $\varepsilon \to 0$.

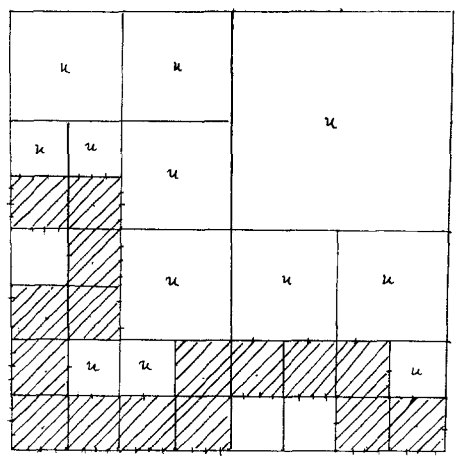

Fig. 3.4

Proof. Although F_s and E_s are no longer independent, if we can still show that $P(F_s|E_s) \geq P(F_s)$, we can follow the same proof as Lemma 4.3. To this end, we introduce some new notation; referring to Fig. 3.4 will be helpful to the reader (shaded squares are occupied, blank squares are vacant, and squares marked with u are unconditioned; that is, their occupancy or vacancy does not affect our discussion).

Let s be the lowest left-right crossing of $[0,1]^n$ in A_n. In this case, s is a union of adjacent squares. We consider squares to be adjacent if they share a common side. Let $\mathscr{A}(s)$ be the region above s. A square of the form $[(i-1)/N^m, i/N^m] \times [(j-1)/N^m, j/N^m], m \leq n$ is said to be unconditioned if it lies in $\mathscr{A}(s)$, since its coin flip $\lambda^m_{i,j}$ is independent of the event $\{s$ is the lowest left-right crossing of $A_n\}$.

However, for any square that intersects s, we know that its coin flip λ^m_{ij} must have resulted in a 1; otherwise it would have been removed and the path s would not exist. From this, we can conclude that the inequality holds, and thus Lemma 3.1 follows exactly as Lemma 4.3. $\square$

The second lemma is even easier to adapt.

Lemma 3.2. *if* $P(\Omega^n_{2L,L}) \geq 0.99$ *then* $P(\Omega^n_{2^k L, 2^{k-1} L}) \geq 1 - \frac{1}{25}\exp(-2^{k-1})$.

Proof. The existence of left-right crossings in $[0,4L]\times[0,L]$ and $[0,4L]\times[L,2L]$ are independent events by the nature of Mandelbrot percolation, and so (3.5) is satisfied. Thus, Lemma 4.4 generalizes nicely to our situation. $\square$

We now move on to the lemmas required for Theorem 2.

Lemma 2.1. *if* $P(\Omega^n_{1,1})\le\varepsilon$ *then* $P(\Omega^n_{1,K})\le f_K(\varepsilon)$ *where* $f_K(\varepsilon)\to 0$ *as* $\varepsilon\to 0$.

Proof. Observe that by turning the lemma into a statement about paths through vacant squares, as opposed to occupied squares, we get something that looks very similar to previous results. It is can be seen that, if we allow vacant crossings to cross through diagonal neighbors as well as regular ones, for $[0,J]\times[0,K]$ there is always either an occupied left-right crossing or a vacant top-bottom crossing, but not both. Letting $\tilde{\Omega}^n_{J,K}$ be the probability of a top-bottom crossing of $[0,J]\times[0,K]$ through vacant squares, it is sufficient to prove the following:

$$\text{If } P(\tilde{\Omega}^n_{1,1})\ge 1-\varepsilon \text{ then } P(\tilde{\Omega}^n_{1,K})\ge 1-f_K(\varepsilon) \text{ where } f_K(\varepsilon)\to 0 \text{ as } \varepsilon\to 0. \quad (3.8)$$

As can be seen, we can prove this statement just as we proved Lemma 3.1 once we show that $P(\tilde{F}_s|\tilde{E}_s)\ge P(\tilde{F}_s)$, where we use the symbol $\sim$ to indicate the corresponding events defined for vacant crossings. However, unlike Lemma 3.1, we cannot conclude that the corresponding coin flip for a square is 0 simply because it intersects s. Fortunately we are able to apply Harris' inequality in this case, and thus (3.8) follows nicely. $\square$

We are now left with one final proof.

Lemma 2.2. *if* $P(\Omega^n_{1,2})\le 0.01$ *then* $P(\Omega^{n+k}_{1,2})\le\frac{1}{25}\exp(-N^{k-1})$.

Proof. Just as in the previous proof, we will convert the statement into one regarding vacant crossings, and so it is sufficient to prove:

$$\text{If } P(\tilde{\Omega}^n_{1,2})\ge 0.99 \text{ then } P(\tilde{\Omega}^{n+k}_{1,2})\ge 1-\tfrac{1}{25}\exp(-N^{k-1}). \quad (3.9)$$

Since

$$P(\tilde{\Omega}^n_{N^m,2N^m})=P(\tilde{\Omega}^{n+m}_{1,2}|\text{all } \varepsilon^k_{ij}=1 \text{ when } k\le m),$$

it suffices to show $P(\tilde{\Omega}^n_{N^m,2N^m})$ heads to 1 exponentially fast. However, due to our perspective change to vacant crossings, we now have the necessary independence just as in Lemma 3.2, and the rest of the proof follows in the same way. $\square$

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