An Introduction of Basic Lie theory

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1 Introduction

Lie theory, initially developed by Sophus Lie, is a fundamental part of mathematics. The areas it touches contain classical, differential, and algebraic geometry, topology, ordinary and partial differential equations, complex analysis and ect. And it is also an essential chapter of contemporary mathematics. A development of it is the Uniformization Theorem for Riemann surface. The final proof of such theorem is the invention from Einstein to the special theory of relativity and the Lorentz transformation. The application of Lie thoery is astonishing. In this paper, we are going to follow the work by Roger Howe to show the essential phenomenon of the theory that Lie groups G may be associated naturally by it Lie algebra g. And we will go through the proof of the fact that Lie group G is determined by g and its Lie bracket. And g is a vector space which is endowed with a bilinear nonassociative product called the Lie bracket. G is a complicated nonlinear object. Thus for many purposes we can replace G with g.

2 Background

2.1 Homeomorphism Groups

In this section, we will introduce some basic background of hemeomorphism group. Let X be a set, A group is the collection $Bi(X)$ of bijections from X to itself. Then the set $Hm(X)$ of homeomorphisms from X to itself is a subgroup of $Bi(X)$. Then let G be a group, if the topology of it satisfies,

(i) The multiplicaiton map $(g_1, g_2) \to g_1 g_2$ from $G \times G$ should be continuous;

(ii) the inverse map $g \to g^{-1}$ from G to G should be continuous, then the topology on G is called a *group topology.* And a group endowed with a group topology is called a *topological group.*

Define λ_g is the left-translation by g, and ρ_g is the right-translation by g.

LEMMA 1. Let G be a topological group, and $g \in G$.

(a) the map $\lambda_g: G \to G$ is a homeomorphism. Similarly, $\rho_g: G \to G$ is a homeomorphism.

(b) If $U \subseteq G$ is a neighborhood of the identity 1_G of G, then gU and gU are heighborhood of g. Similarly if $V \subseteq G$ is a neighborhood of g, then $g^{-1}V$ and Vg^{-1} are neighborhoods of 1_G.

COROLLARY 2. A group topology is determined by its system of neighborhood of the identity.

LEMMA 3. A homogeneous topology on a group G is a group topology if and only if the system of neighborhoods of 1_G satisfies conditions (a) and (b) below.

(a) If U is a neighborhood of 1_G, there is another neighborhood V of 1_G such that $V \subseteq U^{-1}$, where

$$U^{-1} = \{g^{-1}: g \in U\}$$

(b) If U is a neighborhood of 1_G, there are other neighborhoods V, W of 1_G such that $VW \subseteq U$, where

$$VW = \{gh: g \in V\, h \in W\}$$

Now we would like to make $Hm(X)$ into a topological group. By Corollary 2, we can only define neighborhoods of the identity map 1_X on X for the definition of the topology on $Hm(X)$. And by Lemma 3, we know that what we must check to know our definition yields a group topology.

Then, let X be a locally compact Hausdorff space. Let $C \subseteq X$ be compact, and let $O \subseteq C$ be open. Define

$$U(C, O) = \{h \in Hm(X): h(C) \subseteq O, h^{-1}(C) \in O\}$$

If $\{C_i\}$, $1 \leq i \leq n$, are compact subsets of X, and $\{O_i\}$ are open subsets of X such that $C_i \subseteq O_i$, set

(2.1)

$$U(\{C_i\}\{O_i\}) = \bigcap_{i=1}^{n} U(C_i, O_i)$$

DEFINITION. Let X be a locally compact Hausdorff space. The compact-open topology on $Hm(X)$ is the homogeneous topology such that a base for the neighborhoods of 1_X consists of the sets $U(\{C_i\}, \{O_i\})$ of equation (2.1).

PROPOSITION 4. The compact-open topology on $Hm(X)$ is a Hausdorff group topology.

2.2 One-Parameter Groups Flow and Differential Equations

The basic object mediating between Lie groups and Lie algebras is the one-parameter group. A Lie group is a very coherent system of one-parameter groups. In this section, we will define one-parameter group in a general context.

DEFINITION. A one-parameter group of homeomorphisms of (the loccaly compact Hausdorff space) X is a continuous homomorphism

$$\varphi: \mathbf{R} \to Hm(X)$$

Denotes the image under φ of t by φ, then $\{\varphi_t\}$ is a family of homeomorphisms of X satisfying the rule

$$\varphi_t \circ \varphi_s = \varphi_{t+s} \quad t, s \in \mathbf{R}$$

Let ϕ_t be a one-parameter group of homeomorphisms of X, we can define a map

(2.2)

$$\phi: \mathbf{R} \times X \to X,$$
$$\phi(t, x) = \phi_t(x).$$

$t \to \phi_t$ is a homomorphism is captured by the identities

(2.3)

$$(i) \quad \phi(0, x) = x,$$
$$(ii) \quad \phi(s, \phi(t, x)) = \phi(s + t, x).$$

LEMMA 6. Let $\phi: \mathbf{R} \times X \to X$ be a map. For $t \in \mathbf{R}$, define $\varphi_t: X \to X$ by (2.2) (ii). Then $\{\varphi_t\}$ is a one-parameter group of homeomorphisms if and only if

$$(a) \quad \phi \quad satisfies \quad identities (2.3) \quad and$$

$$(b)\ \ \phi\ \ is\ \ continuous.$$

LEMMA 7. Let $\varphi: G \to H$ be a homomorphism between topological groups.Then φ is continuous if and only if φ is continuous at 1_G.

By the map we define in (2.2), if we make x fixed and let t vary, then the map $t \to \phi(t, x) = \phi_t(x)$ defines a continuous curve in X. That is, as t varies, each point of x moves continuously inside X, and various points move in a coherent fashion. Thus one-parameter group of homeomorphisms of X is also be called a flow of X. And the notion of flow is closely related to the theory of differential equations. I am going to skip the details here. Conclusively, the notion of one-parameter group gives us a geometric way of looking at the solutions of a system of ordinary differential equations. And it also provides a link between ordinary differential equations and Lie groups and Lie algebras.

3 One-Parameter Groups of Linear Transformation

Lie algebras are vector spaces. In order to show the relationship between Lie group and Lie algebra, we are going to show how one-parameter groups of linear transformations of a vector space can be described by using the exponential map on matrices first.

Let V be a finite dimensional real vector space, $End(V)$ be the algebra of linear maps from V to itself, and $GL(V)$ be the group of invertible linear maps from V to itself. We usually call $GL(V)$ the *general linear group* of V.

DEFINITION. A one-parameter group of linear transformations of V is a continuous homomorphism

$$M: \mathbf{R} \to GL(V).$$

$M(t)$ is a collection of linear maps such that

$$(i)\ \ M(0) = 1_V, the\ \ identity\ \ of\ \ V,$$

$$(ii)\ \ M(s)M(t) = M(s+t)\ \ \ s, t \in \mathbf{R},$$

$$(iii)\ \ M(t)\ \ depends\ \ continuously\ \ on\ \ t.$$

For $A \in End(V)$, define

$$exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

PROPOSITION 8. If A and B in $End(V)$ commute with each other, then

$$exp(A+B) = expAexpB.$$

COROLLORY 9. For any $A \in End(V)$, the map $t \to exp(tA)$ is a one-parameter group of linear transformations on V. In particular $exp(A) \in GL(V) and (exp(A))^{-1} = exp(-A)$.

THEOREM 10. Every one-parameter group M of linear transformations of V has the form

$$M(t) = exp(tA)$$

for some $A \in End(V)$.

4 Properties of the Exponential Map

As we shown in last section, the map exp is the basic link between $End(V)$ and $GL(V)$. Now we are going to describe some properties of this link.

For $A, B \in End(V)$, write

$$[A, B] = AB - BA.$$

The quantity $[A, B]$ is called the *commutator* or *Lie bracket* of A and B,

PROPOSITION 13. Suppose A, B, C have norm at most 1/2 and satisfy equation(4.1). Then we have

$$C = A + B + \frac{1}{2}[A, B] + S,$$

where the reminder term S satisfies

$$\|S\| \leq 65(\|A\| + \|B\|)^3.$$

PROPOSITION 14. (Trotter Product Formula). For $A, B \in End(V)$, one has

$$exp(A+B) = \lim_{n\to\infty} (exp(A/n)exp(B/n))^n.$$

PROPOSITION 15. (Commutator formula) For $A, B \in End(V)$, one has

$$exp[A,B] = \lim_{n\to\infty}(exp(A/n)exp(B/n)exp(-A/n)exp(-B/n))^{n^2}$$

$$= \lim_{n\to\infty}(exp(A/n){:}\, exp(B/n))^{n^2}.$$

There is one further concept involving the exponential map. And it involves conjugation.

For g, h are elements of a group, then the group commutator of g and h, written $(g{:}\, h)$, is the expression

$$(g{:}\, h) = ghg^{-1}h^{-1}.$$

For $g \in GL(V)$ and $T \in EndV$, we can form the conjugate

$$Adg(A) = gAg^{-1}.$$

PROPOSITION 16. $(i) Adg(aA+bB) = aAdg(A)+bAdg(B) for A, B \in End(V)$; $a, b \in \mathbf{R}$; and $g \in GL(V)$.

$$(ii) \quad Adg(AB) = Adg(A)Adg(B).$$

$$(iii) \quad Adg_1g_2(A) = Adg_1(Adg_2(A)).$$

for (iii): $Adg1g2(A) = Adg1(Adg2(A))$, implies in particular that if $\exp(tA)$ is one parameter subgroup of $GL(V)$, then $Adexp(tA)$ is a one-parameter groups of linear transformations on $EndV$. Hence $Adexp(tA)$ has infinitesimal generator $\lambda \in End(EndV)$.

Then if we define

$$adA{:}\, EndV \in EndV,$$

by

$$adA(B) = [A, B].$$

we have,

PROPOSITION 17. For $A \in End(V)$

$$Ad(expA) = exp(adA).$$

5 The Lie Algebra of a Matrix Group

A *matrix group* is a closed subgroup of $GL(V)$ for some vector space V. In this section, we will show a matrix group is a Lie group. And also shows that every matrix group can be associated to a Lie algebra which is related to its group in a close and precise way.

First, we need to define Lie algebra.

DEFINITION. A real Lie algebra g is a real vector space equipped with a product

$$[,]: g \times g \to g.$$

satisfying the identities

$$(i) \quad (Bilinearity).For \quad a, b \in \mathbf{R} \quad and \quad x, y, z \in g,$$

$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[z, ax + by] = a[z, x] + b[z, y].$$
$$(ii) \quad (skew \quad symmetry).For \quad x, y \in g,$$
$$[x, y] = -[y, x].$$
$$(iii) \quad (Jacobi \quad Identity).For \quad x, y, z \in g,$$
$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

Then, we come to our main statement.

THEOREM 17. (a) The the Lie algebra g of a matrix group G is a Lie algebra. (b) The map $exp: g \to G$ maps a neighborhood of 0 in g bijectively onto a neighborhood of 1_v in G.

It provides a geometric picture of the relation between Lie algebra and Lie group. If a one-parameter group $exp(tA)$ is regarded as a curve inside the vector space $End(V)$, then this curve passes through the identity 1_V at time $t = 0$. By differentiating the formula for $exp(tA)$, we see the tangent vector at the point 1_V to this curve is just A. Thus g consists of all tangent vectors to the curves defined by one-parameter groups in G.

But it also asserts that these tangent vectors actually fill out some linear subspace g of $End(V)$. Then, G is shown to be a smooth multidimensional surface inside $End(V)$, and g is its tangent space at the point 1_V.

In order to prove Theorem 17, we need another technical result.

LEMMA 18. Suppose A_n is a sequence in $exp^{-1}(G)$, and $\|A_n\| \to \mathbf{0}$. Let s_n be a sequence of real numbers. Then any cluster point of $s_n A_n$ is in g.

Now we can do the proof.

Proof of Theorem 17. First, we need to show g is a subspace of $End(V)$. By definition, g is closed under scalar multiplication, then we need to prove that g is also closed under addition. Take $A, B \in g$, by Proposition 14, for n is large enough,

$$exp(A/n)exp(B/n) = exp(C_n),$$

where $\|C_n\| \to 0$, and $nC_n \to A + B$. Then $A + B \in g$ by Lemma 18. Secondly, we need to show $[A, B] \in g$ if $A, B \in g$. By Proposition 15, for n is large enough,

$$(exp(A/n)\colon exp(B/n)) = expE_n$$

with $E_n \to 0$ and $n^2 E_n \to [A, B]$. Then $[A, B] \in g$ by Lemma 18. This proves the part (a) of Theorem 17.

g is a linear subspace of $End(V)$. Let $Y \subseteq End(V)$ be a complementary subspace of g, then $End(V) = g \bigoplus Y$. Then let p_1 and p_2 be the projections of $End(V)$ on g and Y, define a map $E\colon End(V) \to GL(V)$ by

$$E(A) = exp(p_1(A))exp(p_2(A)).$$

then, we have

$$\frac{d}{dt}(exp(p_1(tA))exp(p_2(tA)))\bigg|_{t=0} = p_1(A) + p_2(A) = A,$$

by Proposition 13. This computation says that the differential of E at 0 is the identity map on $End(V)$, thus, by the Inverse Function Theorem, E takes small neighborhoods of 0 to neighborhoods of 1_V bijectively. Select a small ball $B_r(0) \subseteq End(V)$, and suppose $exp(B_r(0) \bigcap g)$ does not cover a neighborhood of 1_V in G. Then we can find a sequence $B_n \in exp^{-1}(G)$ such $B_n \to 0$, but $B_n \notin g$. Then when B_n is close enough to 0, we may write

$$expB_n = E(A_n)$$

for some A_n. Then, we will have $A_n \to 0$ as $B_n \to 0$. Then

$$exp(p_2(A_n)) = exp(p_1(A_n))^{-1}expB_n$$

is also in G and is not zero. Since $A_n \to 0$, then $p_2(A_n) \to 0$. The sequence $\|p_2(A_n)\|^{-1}p_2(A_n)$ will have cluster points, by Lemma 18 there must be in g.

Since $p_2(An) \in Y$, so all cluster points must be in Y. This contradicts that Y was supposed to be the complementary to g. Thus this concludes the part (b) of Theorem 17.

The second essentail feature of g is its natural.

Let g and h be real lie algebra. A *homomorphism* from g to h is a linear map

$$L: g \to h$$

satifying

$$L([x, y]) = [Lx, Ly] \quad x, y \in g$$

.

Let V, U be real vector spaces.

THEOREM 19. Let $G \subseteq GL(V)$ be a matrix group with Lie algebra g. let $\phi: G \to GL(U)$ be a continuous homomorphism. Then there is a homomorphism of Lie algebras

$$d\phi: g \to EndU$$

such (3.1)

$$exp(d\phi(A) = \phi(exp(A)))$$

.

Proof. If $A \in g$, the $exp(tA)$ is a one-parameter subgroup of G, thus $\phi(exp(tA))$ is a one-parameter subgroup of $\phi(G) \subseteq GL(U)$. Then by Theorem 10, we may write

$$\phi(exp(tA)) = exp(tB)$$

for some $B \in End(U)$. Then if we define

$$d\phi(A) = B,$$

the equation (3.1) will be satisfied. To prove this theorem, it suffices to show that $d\phi$ is a homomorphism of Lie algebras, and this follows from Propositions 14 and 15 which show that the Lie algebra operations in g are determined by operations in G.

Then an consequence of Theorem 19 leads to the next corollary,

COROLLARY 20. If $G_1 \subseteq GL(V)$ and $G_2 \subseteq GL(U)$ are isomorphic matrix groups, then their Lie algebras g_1 and g_2 are isomorphic as Lie algebras.

6 Conclusion

We have shown that for each matrix group G, which can be realized by Lie group, there is a Lie algebra g can be associated to it in a close and natural way. And the connection between these two can be held via one parameter groups and the exponentail map. These results we have proved consist an essentail part of the foundation of Lie theory. But we have omitted from the standard accout. First of all, we have dealt with Lie groups as subgroups of a standard alnguage of differential manifold instead of treaing them as abstract things in themselves. For the next, we haven't shown the relationship between Lie groups and their Lie algebras completely. For instance, we haven't got a chance to attach a group to every Lie subalgebra of $End(V)$.

As Roger mentions at the begining of the article, Lie theory is poorly known in comprison to its importance. And Lie theory hasn't been penetrated into undergraduate curriculum and graduate programs. So at the end Roger gives some examples of the relations between the Lie theory and the standard curriculum. For example, the theory of Fourier series and Fourier transform is best understood group-theoretically.

REFERENCES

[1]Roger Howe: Very Basic Lie Theory. The American Mathematical, Vol.90, No.9(Nov., 1983), 600-623