Mertens Conjecture (is not true)

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1 Introduction

Mertens conjecture, which concerns the growth rate of the difference between the number of squarefree integers with an even number of prime divisors and those with odd prime divisors, was thought of as one avenue to prove the Riemann hypothesis. Not only did Mertens conjecture imply the Riemann hypothesis, but it also implied that all zeros of the Riemann zeta function were simple, a fact that while not proven, is generally thought to be true. In addition to this, Mertens conjecture was verified numerically up to very $7.8x10^9$ [4].

However, all of this evidence in favor of Mertens conjecture did not amount to a proof. Andrew Odlyzko and Herman te Riele published a paper [4] in which they disprove Mertens conjecture. After a short summary of previous results about the subject, they move on to various ways of approximating values of $M(x)$, which they use to prove that $|M(x)| > 1$ for some value of x. Unfortunately, that value of x is speculated to be on the order of 10^{30}, meaning they are unable to produce an explicit counterexample to Mertens conjecture. They also claim, though they do not prove, that $|M(x)x^{-1/2}|$ is undbounded as $x \to \infty$.

2 Definitions

- The **Mobius-μ** function is defined as:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is not squarefree} \\ (-1)^k & \text{if } n \text{ has } k \text{ distinct prime factors} \end{cases}$$

- **Mertens function**:

$$M(x) = \sum_{j=1}^{\lfloor x \rfloor} \mu(j).$$

- **Mertens conjecture** states that

$$M(x) < x^{1/2}, \quad x > 1$$

- The **Riemann zeta function** is defined as:

$$\zeta = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re} s > 1$$

  and is extended to be meromorphic on the entire complex plane by the functional equation (from [5]):

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{1}{2} s\pi\right) \Gamma(1-s)\zeta(1-s)$$

- The **Riemann Hypothesis** states that the non-trivial zeroes of the zeta function (which has trivial zeroes at every negative even integer) all lie on the line $\operatorname{Re} \rho = \frac{1}{2}$. It is important to note that the functional equation guarentees that for any s in the strip $0 < \operatorname{Re} s < 1$, if $\zeta(s) \neq 0$, then $\zeta(1-s) \neq 0$ as well.

3 Mertens Conjecture and the Riemann Hypothesis

One of the reasons Mertens Conjecture was considered important is that Mertens Conjecture implies the Riemann Hypothesis. To see that, we start with the following lemma [3]:

Lemma 1.

$$\frac{1}{\zeta(s)} = \sum_{1}^{\infty} \frac{\mu(n)}{n^s}, \quad \operatorname{Re} s > 1$$

Proof. We begin with the well known Euler product formula for the Zeta function:

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \quad \operatorname{Re} s > 1$$

Taking reciprocals, we get:

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} (1 - p^{-s}) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots$$

When you expand this product, you take a finite number of terms of the form $1/p^s$ for some prime p, and an infinite number of ones. Writing this as a sum, we have:

$$\begin{aligned}
\frac{1}{\zeta(s)} &= 1 - \sum_{p \text{ prime}} \frac{1}{p^s} + \sum_{p,q \text{ prime}} \frac{1}{p^s q^s} - \cdots \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{p_1^s \cdots p_j^s} \\
&= \sum_{n=0}^{\infty} \frac{\mu(n)}{n^s}
\end{aligned}$$

where $n = p_1 p_2 \cdots p_j$ and $p_1, p_2, ...p_j$ represents all possible combinations of distinct primes. $\square$

Theorem 1. *If Mertens Conjecture is true, then* $1/\zeta(s)$ *is analytic on the half plane* $\operatorname{Re} s > \frac{1}{2}$.

Proof.

$$\begin{aligned}
\frac{1}{\zeta(s)} &= \sum_{1}^{\infty} \frac{M(n)-M(n-1)}{n^s} \\
&= \sum_{1}^{\infty} M(n)\left[\frac{1}{n^s} - \frac{1}{(n+1)^s}\right] = \sum_{1}^{\infty} M(n)\int_n^{n+1} \frac{s}{x^{s+1}}\,dx \\
&= s\sum_{1}^{\infty} \frac{M(n)\,dx}{n^{s+1}} = s\int_1^{\infty} \frac{M(x)\,dx}{x^{s+1}}
\end{aligned}$$

and if Mertens conjecture is true,

$$\int_1^{\infty} \left|\frac{M(x)}{x^{s+1}}\right| dx < \int_1^{\infty} \left|\frac{x^{1/2}}{x^{s+1}}\right| dx = \int_1^{\infty} \frac{dx}{x^{\sigma+1/2}}$$

where $\sigma = \operatorname{Re} s$. That last integral converges for $\sigma > \frac{1}{2}$, so

$$s\int_1^{\infty} \frac{M(x)\,dx}{x^{s+1}}$$

defines an analytic continuation of $1/\zeta(s)$ to the half plane $\operatorname{Re} s > \frac{1}{2}$. $\square$

If Mertens conjecture were true, the Riemann hypothesis would immediately follow, becase $1/\zeta(s)$ is analytic where $\zeta(z) \neq 0$ so $\zeta(s)$ has no zeros in $\sigma > \frac{1}{2}$, "which is exactly the statement of the Riemann hypothesis" [4]. Notice also that this proof is valid if $|M(x)| < Ax^{1/2}$ for any fixed A, not just 1.

4 Disproof of Mertens Conjecture

4.1 Heuristics

As is typical with step functions, define

$$M_0(x) = \begin{cases} M(x) - \mu(x)/2 & \text{if } x \in \mathbb{Z}^+ \\ M(x) & \text{if } x \notin \mathbb{Z}^+ \end{cases}$$

Titchmarsh [5] proved that assuming the Riemann hypothesis and that all zeros of the zeta function are simple (both of which follow from Mertens conjecture),

$$M_0(x) = \lim_{k\to\infty} \sum_{\substack{\rho \\ |\gamma|<k}} \frac{x^\rho}{\rho\zeta'(\rho)} - 2 + \sum_{1}^{\infty} \frac{(-1)^{n-1}(2m/x)^{2n}}{(2n)!n\zeta(2n+1)} \tag{4.1}$$

where $\rho = \frac{1}{2} + \gamma i$ represents the nontrivial zeroes of the zeta function. Let

$$m(x) = M(x)x^{-1/2}.$$

Make the variable substitution $x = e^y$ and let

$$h(x) = \lim_{k\to\infty} \sum_{\substack{\rho \\ |\gamma|<k}} \frac{x^\rho}{\rho\zeta'(\rho)} = e^{y/2} \lim_{k\to\infty} \sum_{\substack{\rho \\ |\gamma|<k}} \frac{e^{i\gamma y}}{\rho\zeta'(\rho)}.$$

Because M_0 is discontinuous at the square-free integers and the series on the right side of (4.1) represents a rapidly diminishing continuous function,

$$m(y) = h(y) + O(\min(1, e^{-y/2})) \tag{4.2}$$

and thus proving that $|M(x)| > x^{1/2}$ for some value of x is equivalent to proving $\limsup h(y) > 1$ as $y \to \infty$. Sadly, we do not know enough about the zeros of the zeta function to make statements about the behavior of $h(y)$. To circumvent this problem, we use a technique from [1] to approximate $h(y)$ with a finite sum. Let $K(y)$ be a suitably behaved (this is defined more precisely later) function with bounded support, and let

$$k(t) = \int_{-\infty}^{\infty} K(y)e^{-ity}dy$$

and let

$$h_K(y) = \int_{-\infty}^{\infty} h(y-t)K(t)\,dt = \sum_\rho \left[\frac{e^{i\gamma y}}{\rho\zeta'(\rho)} \int_{-\infty}^{\infty} K(t)e^{-i\gamma t}\,dt\right] = \sum_\rho k(\gamma)\frac{e^{i\gamma y}}{\rho\zeta'(\rho)}$$

and since K has bounded support, each of those integrals and sums are finite, so there are no convergence issues to deal with. While we no longer have the relationship (4.2), we do know that if $h_K(y_0)$ is large for some value of y_0, so is $h(y)$. To be exact:

$$\sup_y |h(y)| \int_{-\infty}^{\infty} |K(t)|\,dt \geq |h_K(y_0)|.$$

If we restrict K to functions such that $K(t) \geq 0$ and $k(-\gamma) = \gamma$ is real, we no longer need to take moduli, and we have:

$$\sup_y h(y) \int_{-\infty}^{\infty} K(t)\,dt \geq h_K(y_0)$$

and

$$\inf_y h(y) \int_{-\infty}^{\infty} K(t)\,dt \leq h_K(y_0)$$

All of this leads us to the following theorem from [1]:

Theorem 2. *Suppose that $K(y) \in C^2(-\infty,\infty)$. Assume $K(y) \geq 0$ and $K(-y) = K(y)$. Assume $K(y) = O((1+y^2)^{-1})$ as $y \to \infty$ and $k(t) = \int_{-\infty}^{\infty} K(t)e^{-ity}\,dy$ satisfies $k(t) = 0$ for $|t| \geq T$ for some $T > 0$ and $k(0) = 1$. If the zeroes $\rho = \beta + i\gamma$ of the zeta functions with $0 < \beta < 1$ and $|\gamma| < T$ satisfy $\beta = \frac{1}{2}$ and are simple, then for any y_0,*

$$\limsup_{y\to\infty} m(y) \geq h_K(y_0),$$

$$\liminf_{y\to\infty} m(y) \leq h_K(y_0)$$

where

$$h_K(y) = \sum_\rho k(\gamma)\frac{e^{i\gamma y}}{\rho\zeta'(\rho)}$$

4.2 Calculation

While proving that $|h_K(y_0)| > 1$ is easier than finding values for which $|M(x)| > 0$, it is still not a trivial task. The method used by Odlyzko and Riele is long and technical, and the details do not provide any particular insight. They use the function $k(t) = g(t/T)$, where $T \approx 2515$ is the imaginary part of the 2000th zero of the zeta function and

$$g(t) = \begin{cases} (1 - |t|)\cos(\pi t) + \pi^-1 \sin(\pi|t|) & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| \geq 1 \end{cases}$$

as their kernel for defining $h_K(y)$. After calculating the first 2000 zeros of the zeta function to a very high degree of accuracy (100 digits) they calculate that:

$$h_K(-14045289680592998046790361630399781127400591999789738039965960762.521505) = 1.061545$$

and

$$h_K(3209702577292265586974000018621130709979714454034906268280532165 1.697419) = -1.009749$$

and by Theorem 2, we now know that

$$\limsup_{y\to\infty} m(y) \geq 1.061545 > 1$$

and

$$\liminf_{y\to\infty} m(y) \leq -1.009749 < -1$$

so Mertens conjecture is not true.

5 Further Results

Odlyzko and te Riele believe that their method could be used to prove that counterexamples exist to any inequality of the form $|M(x)| < Ax^{1/2}$ where A is a positive number, but finding values of h_K that would prove that $m(y) > A$ becomes prohibitively difficult very quickly. Their proof is not constructive, and an explicit counterexample is not given, and has not been found. In the twenty years since the disproof was originally published, the strongest result that has been found is that $|M(x)| > 1.218429$ for some x ([2]).

Other work has been done on the growth rate of $M(x)$. "$M(x)x^{-1/2}$ is expected to be unbounded, and the Riemann hypothesis is known to be equivalent to $M(x) = O(x^{1/2+\epsilon})$" [4], but as it currently stands, Titchmarsh [5] gives the best growth rate as

$$M(x) = O\left(x^{1/2} \exp\left(A \frac{\log(x)}{\log\log(x)}\right)\right)$$

References

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[3] T. Nagell, *Introduction to Number Theory,* Chelsea Pub. Co., 1964.

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[5] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function,* Oxford Univ. Press, 1951.