The mth Ratio Convergence Test and Other Unconventional Convergence Tests

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May 14, 2012

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1 Introduction

The simplest ratio test for the convergence of infinite series is d'Alembert's ratio test, which is taught in most undergraduate calculus courses. However, this test fails quite often, sometimes for complicated series such as the hypergeometric series and sometimes for the simplest series such as $\sum \frac{1}{n^p}$.

In these cases, mathematicians have created several more ratio convergence tests to utilize when this simplest test is inconclusive. These further ratio convergence tests form the De Morgan Hierarchy of Ratio Tests which, in order from lowest to highest, are: d'Alembert's ratio test, Raabe's test, Bertrand's test, Gauss's test, and finally, Kummer's test. These tests use the condition under which d'Alembert's test fails ($\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$) to rewrite the series $\sum a_n$ as $\sum(1 + b_n)$ for some sequence b_n that converges to 0.

However, in this paper we will discuss a different kind of ratio test which is more akin to d'Alembert's test than the other tests in De Morgan's hierarchy, *the second ratio convergence test*. This test can be used to successfully check for convergence when d'Alembert's test fails, without the possibly cumbersome application of the higher ratio convergence tests. A specific troublesome sequence is the hypergeometric sequence, defined below. We will show that this sequence converges for the argument $x = 1$. We will also use the second ratio convergence test to prove the convergence portion of Raabe's test. We will then conclude with the statement of *the* m*th ratio convergence test*, the generalization of the second ratio convergence test.

2 Definitions, Lemmas, and Theorems

2.1 Defintions

Definition The hypergeometric series $F(a, b; c; x)$ is defined as

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)n!} x^n.$$

2.2 Lemmas

Lemma 2.1 *For fixed m,*

$$\lim_{n\to\infty} \left[\frac{tn+a}{tn}\right]^{n-m} = e^{a/t}$$

Lemma 2.2 *For fixed m,*

$$\lim_{n\to\infty}\left[\frac{tn+b}{tn+c}\right]^{n-m} = e^{(b-c)/t}$$

2.3 De Morgan's Hierarchy

Theorem 2.3 (d'Alembert's Ratio Test) [3] *Suppose $\{a_n\}$ is a sequence of positive numbers.*

a. If $\frac{a_{n+1}}{a_n} < r$ for all sufficiently large n, where $r < 1$, then the series $\sum_0^\infty a_n$ converges. On the other hand, if $\frac{a_{n+1}}{a_n} \geq 1$ for all sufficiently large n, then the series $\sum_0^\infty a_n$ diverges.

b. Suppose that $l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists. Then the series $\sum_0^\infty a_n$ converges if $l < 1$ and diverges if $l > 1$. No conclusion can be drawn if $l = 1$.

Theorem 2.4 (Raabe's Test) [2] *If $a_n > 0, \epsilon_n \to 0$, and*

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\beta}{n} + \frac{\epsilon_n}{n},$$

where β is independent of n, then $\sum_{n=1}^\infty a_n$ converges if $\beta > 1$ and diverges if $\beta < 1$.

Theorem 2.5 (Bertrand's Test) [4] *Let $\{a_n\}$ be a sequence of positive numbers. Suppose that*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\rho_n}{n \ln n}.$$

Then the series $\sum a_n$ converges if $\liminf \rho_n > 1$ and diverges if $\limsup \rho_n < 1$.

Theorem 2.6 (Gauss's Test) [5] *Let $\{a_n\}$ be a sequence of positive numbers and $B(n)$ a bounded function of n as $n \to \infty$, and*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{h}{n} + \frac{B(n)}{n^r}, \qquad r > 1.$$

Then the series $\sum a_n$ converges for $h > 1$ and diverges for $h \leq 1$.

Theorem 2.7 (Kummer's Test) [2] *If $a_n > 0, d_n > 0, \sum_{n=1}^\infty d_n$ diverges, and*

$$\lim_{n\to\infty}\left(\frac{1}{d_n} - \frac{a_{n+1}}{a_n}\cdot\frac{1}{d_{n+1}}\right) = h,$$

then the series $\sum_{n=1}^\infty a_n$ converges if $h > 0$ and diverges if $h < 0$.

3 The Second Ratio Convergence Test

The second ratio test is the main result of this paper. The generalized result, the mth ratio test, serves the same purpose but is more cumbersome to apply in practice. All results proven in this paper will follow from the second ratio test; the mth ratio test is presented to show that the second ratio test can be generalized to the third, fourth, and so on.

3.1 Statement and Proof

Theorem 3.1 (The Second Ratio Test) [1] *Let $\{a_n\}$ be a sequence of positive numbers. Let*

$$L = \max\left\{\limsup_{n\to\infty} \frac{a_{2n}}{a_n},\ \ \limsup_{n\to\infty} \frac{a_{2n+1}}{a_n}\right\}$$

and

$$l = \max\left\{\liminf_{n\to\infty} \frac{a_{2n}}{a_n},\ \ \liminf_{n\to\infty} \frac{a_{2n+1}}{a_n}\right\}$$

Then $\sum_{n=1}^{\infty} a_n$:

1. *converges if $L < \frac{1}{2}$;*
2. *diverges if $l > \frac{1}{2}$;*
3. *may either converge or diverge if $l \leq \frac{1}{2} \leq L$.*

Proof 1. Suppose $L < \frac{1}{2}$. Let r be such that $L < r < \frac{1}{2}$. Then there exists an integer N such that

$$\frac{a_{2n}}{a_n} \leq r \quad \text{and} \quad \frac{a_{2n+1}}{a_n} \leq r$$

for all $n \geq N$. Now,

$$\begin{aligned}\sum_{n=N}^{\infty} a_n &= (a_N + a_{N+1} + \cdots + a_{2N-1}) + (a_{2N} + a_{2N+1} + \cdots + a_{4N-1}) \\ &+ (a_{4N} + a_{4N+1} + \cdots + a_{8N-1}) + \cdots \\ &+ (a_{2^k N} + a_{2^k N+1} + \cdots + a_{2^{k+1}N-1}) + \cdots\end{aligned}$$

$$= \sum_{k=0}^{\infty} (a_{2^k N} + a_{2^k N+1} + \cdots + a_{2^{k+1}N-1}).$$

Let $S_k = a_{2^k N} + a_{2^k N+1} + \cdots + a_{2^{k+1}N-1}$ for $k = 0, 1, 2, 3, \ldots$. Then, for $k \geq 1$,

$$S_k = (a_{2^k N} + a_{2^k N+1}) + (a_{2^k N+2} + a_{2^k N+3}) + \cdots + (a_{2^{k+1}N-2} + a_{2^{k+1}N-1}).$$

Since $\frac{a_{2n}}{a_n} \leq r$ and $\frac{a_{2n+1}}{a_n} \leq r$, we then obtain

$$\begin{aligned} S_k &= (a_{2^k N} + a_{2^k N+1}) + (a_{2^k N+2} + a_{2^k N+3}) + \cdots + (a_{2^{k+1}N-2} + a_{2^{k+1}N-1}) \\ &\leq 2(a_{2^{k-1}N})r + 2(a_{2^{k-1}N+1})r + \cdots + 2(a_{2^k N-1})r \\ &= 2r(a_{2^{k-1}N} + a_{2^{k-1}N+1} + \cdots + a_{2^k N-1}) = 2rS_{k-1}. \end{aligned}$$

So, by induction on k it follows that

$$S_k \leq 2^k r^k (a_N + a_{N+1} + \cdots + a_{2N-1}) = s^k r^k S_0$$

for $k \geq 1$. Thus,

$$\sum_{n=N}^{\infty} a_n = \sum_{k=0}^{\infty} S_k \leq \sum_{k=0}^{\infty} S_0 (2r)^k < \infty$$

since $r < \frac{1}{2}$. Therefore, $\sum_{n=1}^{\infty} a_n$ converges if $L < \frac{1}{2}$.

2. Suppose $l > \frac{1}{2}$. Let r be such that $\frac{1}{2} < r < l$. Then there is an integer N such that

$$\frac{a_{2n}}{a_n} > r \text{ and } \frac{a_{2n+1}}{a_n} > r$$

for all $n \geq N$. Thus, $a_{2n} > ra_n$ and $a_{2n+1} > ra_n$ for all $n \geq N$. Let S_k be as above. We follow the same induction process but use the previous two inequalities to obtain $S_k \geq S_0(2r)^k$ for $k \geq 1$. Therefore, since $r > \frac{1}{2}$, we have

$$\sum_{n=N}^{\infty} a_n = \sum_{k=0}^{\infty} S_k \geq \sum_{k=0}^{\infty} S_0 (2r)^k = \infty.$$

Thus, $\sum_{n=1}^{\infty} a_n$ diverges if $l > \frac{1}{2}$.

3. The series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n(\ln n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$. But

$$\lim_{n\to\infty} \frac{a_{2n}}{a_n} = \lim_{n\to\infty} \frac{n(\ln n)^p}{2n[\ln(2n)]^p} = \frac{1}{2}.$$

This completes the proof.

3.2 Corollaries

Some corollaries are immediately apparent. These serve to make the second ratio test easier to apply.

Corollary 3.2 *Let* $\sum_{n=1}^{\infty} a_n$ *be a series with* $a_n > 0$ *for* $n \geq 1$. *Suppose*

$$\lim_{n\to\infty} \frac{a_{2n}}{a_n} \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{a_{2n+1}}{a_n}$$

both exist. Let $L_1 = \lim_{n\to\infty} \frac{a_{2n}}{a_n}, L_2 = \lim_{n\to\infty} \frac{a_{2n+1}}{a_n}, L = \max\{L_1, L_2\}$, *and* $l = \min\{L_1, L_2\}$. *Then* $\sum_{n=1}^{\infty} a_n$*:*

1. *converges if* $L < \frac{1}{2}$*;*
2. *diverges if* $l > \frac{1}{2}$*;*
3. *may either converge or diverge if* $l \leq \frac{1}{2} \leq L$.

This first corollary is just the second ratio test when the limits of the two ratios $\frac{a_{2n}}{a_n}$ and $\frac{a_{2n+1}}{a_n}$ both exist. The only difference is this presumption, which allows us to get rid of both instances of $\limsup$ found in the original statement.

Corollary 3.3 *If* $\{a_n\}$ *is a positive decreasing sequence of numbers, then* $\sum_{n=1}^{\infty} a_n$ *converges if* $\lim_{n\to\infty} \frac{a_{2n}}{a_n} < \frac{1}{2}$ *and diverges if* $\lim_{n\to\infty} \frac{a_{2n+1}}{a_n} > \frac{1}{2}$.

The second corollary follows from the first one when given that the sequence $\{a_n\}$ is decreasing.

3.3 Strength

To examine the second ratio test and what makes it so strong when compared to d'Alembert's ratio test, we look at the relations

$$\begin{aligned} \frac{a_{2n}}{a_n} &= \frac{a_{n+1}}{a_n}\frac{a_{n+2}}{a_{n+1}}\cdots\frac{a_{2n}}{a_{2n-1}} \\ \frac{a_{2n+1}}{a_n} &= \frac{a_{n+1}}{a_n}\frac{a_{n+2}}{a_{n+1}}\cdots\frac{a_{2n+1}}{a_{2n}}. \end{aligned}$$

The first thing to note is that the first term in this product is $\frac{a_{n+1}}{a_n}$. This means that if a series converges by d'Alembert's ratio test, then it converges by the second ratio test. Further, if a series converges by d'Alembert's test, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, so then $\lim_{n\to\infty} \frac{a_{2n}}{a_n} = \lim_{n\to\infty} \frac{a_{2n+1}}{a_n} = 0$. In the same way, divergence by d'Alembert's test implies divergence by the second ratio test; if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, then $\lim_{n\to\infty} \frac{a_{2n}}{a_n} = \lim_{n\to\infty} \frac{a_{2n+1}}{a_n} = \infty$. So the second ratio test is at least as powerful as the ordinary ratio test.

In the next section we will demonstrate this strength by proving the convergence of a few series on which the ordinary ratio test fails. We begin with the simplest series and then ramp up to proving the convergence portion of Raabe's test.

3.4 Applications

3.4.1 The p-series

An immediate, easy application of the second ratio test and its corollaries is the convergence of the series $\sum_{n=1}^{\infty} n^{-p}$.

Example *(The p-series)* Let $a_n = n^{-p}$. Then

$$\frac{a_{2n}}{a_n} = \frac{(2n)^{-p}}{n^{-p}} = \frac{1}{2^p} \text{ and } \frac{a_{2n+1}}{a_n} = \frac{(2n+1)^{-p}}{n^{-p}} = \frac{1}{(2+\frac{1}{n})^p}$$

and thus

$$\lim_{n\to\infty} \frac{a_{2n}}{a_n} = \lim_{n\to\infty} \frac{a_{2n+1}}{a_n} = \frac{1}{2^p}.$$

This limit is less than $\frac{1}{2}$ if $p > 1$ and greater than $\frac{1}{2}$ if $p < 1$. So by the second ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges in the first case and diverges in the second. Thus this well-known result is proven without the need to use the integral comparison test.

3.4.2 The Hypergeometric Series

Gauss's test stated in (2.6) was originally created by him in order to test the convergence of the hypergeometric series (2.1) with the unit argument $x = 1, F(a, b; c; 1)$. However, this test can be done in greater simplicity with the second ratio test. In this example, we will use the results (2.2.1) and (2.2.2).

Example *(The Hypergeometric Series)* Let a, b, and c be positive numbers. Let

$$a_n = \frac{a(a+1)(a+2)\cdots(a+n-1)b(b+1)(b+2)\cdots(b+n-1)}{c(c+1)(c+2)\cdots(c+n-1)n!}.$$

Then

$$\begin{aligned}
\frac{a_{2n}}{a_n} &= \frac{(a+n)(a+n+1)\cdots(a+2n-1)(b+n)(b+n+1)\cdots(b+2n-1)}{(n+1)(n+2)\cdots(2n)(c+n)(c+n+1)\cdots(c+2n-1)} \\
&= \frac{(a+n)(b+n)}{2n(c+n)}\left[\frac{(a+n+1)(b+n+1)}{(n+1)(c+n+1)}\cdot\frac{(a+n+2)(b+n+2)}{(n+2)(c+n+2)}\right. \\
&\qquad \left.\cdots\frac{(a+2n-1)(b+2n-1)}{(2n-1)(c+2n-1)}\right].
\end{aligned}$$

We note that each of the expressions inside of the brackets has the form $\frac{(a+x)(b+x)}{x(c+x)}$. So we let $f(x) = \frac{(a+x)(b+x)}{x(c+x)}$. Then

$$f(x) = \frac{(a+x)(b+x)}{x(c+x)} = 1 + \frac{(a+b-c)x+ab}{x(c+x)}.$$

So if $a+b<c$, then there is some $N>0$ such that $f(x)$ is increasing for all $x>N$. Therefore, if $a+b<c$,

$$\frac{a_{2n}}{a_n} \le \frac{(a+n)(b+n)}{2n(c+n)}\left[\frac{(a+2n-1)(b+2n-1)}{(2n-1)(c+2n-1)}\right]^{n-2}$$

for all $n>N$. Similarly, with this same process as all of the above, we can write, for $\frac{a_{2n+1}}{a_n}$:

$$\frac{a_{2n+1}}{a_n} \le \frac{(a+n)(b+n)}{(2n+1)(c+n)}\left[\frac{(a+2n)(b+2n)}{2n(c+2n)}\right]^{n-1}$$

for all $n>N$. Then using the lemmas (2.2.1) and (2.2.2),

$$\lim_{n\to\infty}\left[\frac{(a+2n-1)(b+2n-1)}{(2n-1)(c+2n-1)}\right]^{n-2} = e^{(a-1+1)/2}e^{(b-1-c+1)/2} = e^{\frac{a+b-c}{2}} \text{ and}$$

$$\lim_{n\to\infty}\left[\frac{(a+2n)(b+2n)}{2n(c+2n)}\right]^{n-1} = e^{a/2}e^{(b-c)/2} = e^{\frac{a+b-c}{2}}.$$

So then

$$\limsup_{n\to\infty}\frac{a_{2n}}{a_n} = \limsup_{n\to\infty}\frac{a_{2n+1}}{a_n} \le \frac{1}{2}e^{\frac{a+b-c}{2}} < \frac{1}{2}$$

when $a+b<c$. Thus, by the second ratio test, the hypergeometric series converges under the condition $a+b<c$.

3.4.3 Raabe's Test

Raabe's test (2.4) is the second test in the De Morgan Hierarchy and so its delicacy is outdone by the tests further up the hierarchy. We will now present a proof of the convergence portion Raabe's test based on the second ratio test. The divergence portion is excluded because of irrelevancies.

Proof *(Raabe's Test)* Suppose $\frac{a_{n+1}}{a_n} = 1 - \frac{\beta}{n} + \frac{\epsilon_n}{n}$, where $a_n > 0$ and $\epsilon_n \to 0$. Assume $\beta > 1$, and choose r such that $\beta > r > 1$. Then there is some N such that

$$\frac{a_{n+1}}{a_n} < 1 - \frac{r}{n}$$

for $n \geq N$. Then

$$\frac{a_{2n}}{a_n} = \frac{a_{n+1}}{a_n}\frac{a_{n+2}}{a_{n+1}}\cdots\frac{a_{2n}}{a_{2n-1}} < \left(a - \frac{r}{n}\right)\cdots\left(1 - \frac{r}{2n-1}\right)$$

for $n \geq N$. Since $1 - x \leq e^{-x}$ for $0 < x < 1$,

$$\left(a - \frac{r}{n}\right)\cdots\left(1 - \frac{r}{2n-1}\right) \leq e^{-\left(\frac{r}{n}+\frac{r}{n+1}+\cdots+\frac{r}{2n-1}\right)},$$

and since $\frac{r}{n} + \frac{r}{n+1} + \cdots + \frac{r}{2n-1} > r\ln\left(\frac{2n}{n}\right) = r\ln 2$, we have

$$\frac{a_{2n}}{a_n} < \left(a - \frac{r}{n}\right)\cdots\left(1 - \frac{r}{2n-1}\right) < e^{-r\ln 2} = \frac{1}{2^r}.$$

Thus we have

$$\limsup_{n\to\infty}\frac{a_{2n}}{a_n} \leq \frac{1}{2^r} < \frac{1}{2}.$$

Similarly,

$$\frac{a_{2n+1}}{a_n} < \left(a - \frac{r}{n}\right)\cdots\left(1 - \frac{r}{2n}\right) < e^{-r\ln\frac{2n+1}{n}},$$

so

$$\limsup_{n\to\infty}\frac{a_{2n+1}}{a_n} \leq \limsup_{n\to\infty} e^{-r\ln\frac{2n+1}{n}} < e^{-r\ln 2} = \frac{1}{2^r} < \frac{1}{2}.$$

Thus by the second ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges.

4 The mth Ratio Test

Theorem 4.1 (The mth Ratio Test) *Let $\{a_n\}$ be a positive sequence and let $m > 1$ be a fixed positive integer. Let L_k and l_k be such that for $1 \leq k \leq m$,*

$$L_k = \limsup_{n\to\infty} \frac{a_{mn+k-1}}{a_n},$$

$$l_k = \liminf_{n\to\infty} \frac{a_{mn+k-1}}{a_n}.$$

Let $L = \max\{L_k\}$ and $l = \min\{l_k\}$. Then the series $\sum_{n=1}^{\infty} a_n$

1. *converges if $L < \frac{1}{m}$;*
2. *diverges if $l > \frac{1}{m}$;*
3. *may either converge or diverge if $l \leq \frac{1}{m} \leq L$.*

This test is interesting for theoretical purposes but serves very little practical purpose, especially when m is large. It is for this reason that we have focused primarily on the second ratio test instead of this more general mth ratio test.

5 Bibliography

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