A Discourse on Mathematical Physics

Green's Functions and their Applications to the Electrodynamics of Swift Ions

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Abstract

This work aims to explore several areas of mathematical physics, focusing on the Dirac Delta Function, Fourier Transforms between conjugate spaces, the Green's Function method of solving partial differential equations, and time-dependent electrodynamics. These topics are explored in the context of the Wave, Helmholtz, and Poisson equations which are shown to be very closely related by Fourier Transform and have very similar Green's Functions. This paper culminates with the application of such explorations to computing the electric field generated by a high energy, relativistic charged particle. Such a calculation is immediately applicable to experiemtns such as the Electron Energy Loss Spectroscopy Experiment, a relatively new experiment which studies plasmonic excitations generated by a swift electron interacting with the electron cloud of a nanometal. I am currently studying this experiment from a theoretical chemistry perspective and am using the results of this work to computationally model said excitations.

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1 Introduction

1.1 The Dirac Delta Function

The Dirac delta function, denoted $\delta(x - x')$, is a commonly-used tool in Physics. The first thing to note about the that Dirac delta function is that it is not a function at all. A function is a rule that assigns another number to each number in a set. The delta function, as used in physics, is instead a shorthand for a rather compicated limiting process whose use greatly simplifies calcuations.

Defintion: The delta 'function' $\delta(x - x')$ is such that for any other function $f : \mathbb{R} \to \mathbb{R}$

$$\int_{-\infty}^{\infty} f(x')\delta(x' - x)\, dx' = f(x)$$

A special case of this is when f is identically 1, giving

$$\int_{-\infty}^{\infty} \delta(x' - x)\, dx = 1$$

The delta function is best thought of as a **functional**, something that takes a function to a value. In the above examples, the delta 'plucks out' the value of f at x from under an integral.

> **Example:** $\frac{d}{dx}$ is an **operator**, as it takes a function to another function (sometimes to a constant function, but a fuction nontheless). $\frac{d}{dx}\big|_{x=0}$ is a **functional**, as it evaluates a function at a point. Similarly, the delta 'function' maps a function f to a single value.
>
> $$\int_{-\infty}^{\infty} (x^3 - x^2 - 1)\delta(2 - x)\, dx = 2^3 - 2^2 - 1 = 3$$

The above discussion may lead you to wonder how such a 'function' can possibly be defined. Indeed, $\delta(x' - x)$ is better thought of as shorthand for the limit of a sequence of normalized functions $\delta_n(x)$ that satisfy

$$\lim_{n\to 0} \delta_n(x' - x) = 0 \qquad \text{for all } x \neq x'$$

$$\lim_{n\to 0} \int_{-\infty}^{\infty} f(x')\delta_n(x' - x)\, dx' = f(x') \qquad \text{for all } x \neq x'$$

As an example, consider a sequence of functions defined by

$$\delta_n(x' - x) = \begin{cases} \frac{1}{n} & \text{for } |x' - x| \leq \frac{n}{2} \\ 0 & \text{for } |x' - x| > \frac{n}{2} \end{cases}$$

Notice that each δ_n satisfies $\int_{-\infty}^{\infty} \delta_n(x' - x) = 1$, therefore $\lim_{n\to 0} \int_{-\infty}^{\infty} \delta_n(x' - x)$ exists and equals 1. Also notice that this sequence of functions satisfies the above properties of the delta function:

$$\lim_{n\to 0} \int_{\infty}^{\infty} f(x')\delta_n(x' - x)\, dx' = \lim_{n\to 0} \int_{x'-\frac{n}{2}}^{x'+\frac{n}{2}} f(x')\delta_n(x' - x)\, dx' = \lim_{n\to 0} \frac{1}{n} \int_{x'-\frac{n}{2}}^{x'+\frac{n}{2}} f(x')\, dx'$$

By the mean value theorem of integrals, there exists a $c \in \left[-\frac{n}{2}, \frac{n}{2}\right]$ such that $\int_{x'-n/2}^{x'+n/2} f(x')\, dx' = f(x' + c)\left(x' + \frac{n}{2} - x' + \frac{n}{2}\right) = nf(c)$. Therefore we have:

$$\lim_{n\to 0} \frac{1}{n} \int_{x'-\frac{n}{2}}^{x'+\frac{n}{2}} f(x')\, dx' = \lim_{n\to 0} f(x' + c)$$

As $n \to 0$, c is squeezed to 0, giving the final result

$$\lim_{n\to 0} \int_{\infty}^{\infty} f(x')\delta_n(x'-x)\, dx' = f(x')$$

In this paper, the symbol $\delta(x-x')$ will refer to the above limit of a sequence of functions: $\delta(x'-x) = \lim_{n\to 0} \delta_n(x'-x)$. There are many other, equivalent ways to define the Dirac delta, some of which are pictured below:

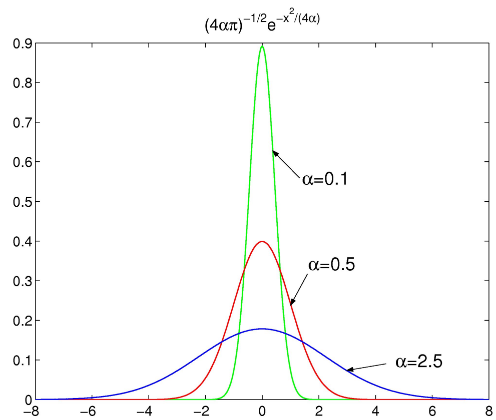

Figure 1: Delta function as the limit of the Gaussian $(4\pi\alpha)^{-1/2}e^{-x^2/4\alpha}$ as $\alpha \to 0$

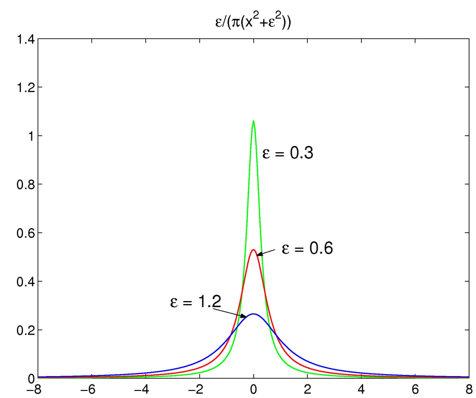

Figure 2: Delta function as the limit of $\frac{\epsilon}{\pi(x^2+\epsilon^2)}$ as $\epsilon \to 0$

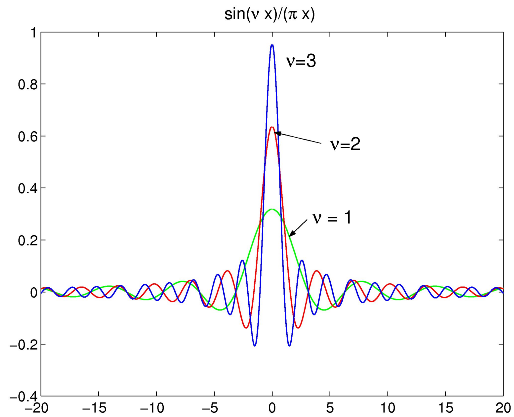

Figure 3: Delta function as the limit of the sinc function $\frac{\sin \nu x}{\pi x}$ as $\nu \to \infty$

The results of this discussion are generalizable to multiple dimensions, where the multi-dimensional Dirac delta $\delta(\mathbf{x}' - \mathbf{x}) = \lim_{n\to 0} \delta_n(\mathbf{x}' - \mathbf{x})$ has the properties

$$\lim_{n\to 0} \delta_n(\mathbf{x}' - \mathbf{x}) = 0 \qquad \text{for all } \mathbf{x} \neq \mathbf{x}'$$

$$\lim_{n\to 0} \int_S f(\mathbf{x}')\delta_n(\mathbf{x}' - \mathbf{x})\, d\mathbf{x}' = f(\mathbf{x}') \qquad \text{for all } \mathbf{x} \neq \mathbf{x}'$$

1.2 Green's Functions

Green's Functions are an ingenious approach to solving ordinary and partial differential equations. Consider a linear differential operator L, and two bounded function $f(x)$ and $y(x)$ such that

$$Ly(x) = f(x)$$

Definition: The **Green's Function** of the operator L is defined as the function $G(x, x')$ such that

$$LG(x, x') = \delta(x' - x)$$

The Green's function of an operator is unique given that the problem is well posed (given sufficent boundary conditions). A more complete statement and proof of the uniqueness of Green's Functions can be found in Barton, pages 201-202 [**?**].

Now let's turn to a general methodology for determining Green's functions. Although rarely used in practice, this method is an excellent way to better understand Green's Functions. Suppose that L possess a complete, (possibly infinite) orthonormal set of eigenfunctions $\{\phi_n(x)\}$ such that

$$L\phi_n(x) = \lambda_n \phi_n(x)$$

This means that we can expand $y(x)$ and $f(x)$ onto this orthonormal set:

$$y(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x)$$

$$f(x) = \sum_{n=1}^{\infty} \beta_n \phi_n(x)$$

Substituting back into the above equation yields

$$Ly = L\sum_{n=1}^{\infty}\alpha_n\phi_n(x) = \sum_{n=1}^{\infty}\alpha_n L\phi_n(x) = \sum_{n=1}^{\infty}\alpha_n\lambda_n\phi_n(x)$$

This must be equal to $f(x)$, therefore

$$\sum_{n=1}^{\infty}\alpha_n\lambda_n\phi_n(x) = \sum_{n=1}^{\infty}\beta_n\phi_n(x)$$

Since the ψ_n's are linearly independent, we have that

$$\alpha_n\lambda_n - \beta_n = 0$$

$$\alpha_n = \frac{\beta_n}{\lambda_n}$$

Notice that this manipulation is only allowed if $\lambda_n \neq 0$. If indeed for some n $\lambda_n = 0$ then any solution we obtain for $y(x)$ will not be unique, as we can add any multiple of ϕ_n to it. For the sake of simplicity we will exclude this case.

We can think of the β's as the projection of f onto the ϕ basis, ie $\beta_n = (\psi_n, f)$. This lets us write

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\phi_n(x)(\phi_n, f) \\ &= \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\phi_n(x)\int \phi_n^*(x')f(x')\,dx' \\ &= \int\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n^*(x')}{\lambda_n}f(x')\,dx' \\ &= \int G(x,x')f(x')\,dx' \end{aligned}$$

Where we define

$$G(x,x') \equiv \sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n^*(x')}{\lambda_n}$$

to be the Green's function of the operator L. Notice that it is appropriate to associate G with L (rather than with y or f) because G is a combination of the eigenfunctions and eigenvalues of L.

Example: Consider the operator

$$L = \frac{d^2}{dx^2}$$

for $x \in [0,1]$. The normalized eigenfunctions of L that vanish at the endpoints are $\sqrt{2}\sin(n\pi x)$ ($n \in \mathbb{N}$) with eigenvalues $-n^2\pi^2$. Thus the Green's Function in this case is

$$G(x,x') = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)\sin(n\pi x')}{n^2}$$

By the M-Test (comparison to $\frac{1}{n^2}$) the above series converges uniformly and absolutely to a continuous function.

The applications of Green's Functions can be seen by oberving how the operator L transforms G:

$$L(x)G(x,x') = \sum_{n=1}^{\infty} \frac{L\phi_n(x)\phi_n^*(x')}{\lambda_n} = \sum_{n=1}^{\infty} \phi_n(x)\phi_n^*(x) \equiv I(x,x')$$

For any function $f(x)$ we can write

$$\begin{aligned}\int I(x,x')f(x')\,dx' &= \sum_{n=1}^{\infty} \phi_n(x) \int \phi_n(x')^* f(x')\,dx' \\ &= \sum_{n=1}^{\infty} \phi_n(x)(\psi_n, f)\end{aligned}$$

But since the ψ_n's are orthonormal, we can also write

$$\int I(x,x')f(x')\,dx' = f(x)$$

It is now clear that $I(x,x')$ satisfies the properties of the Dirac delta $\delta(x-x')$, therefore we may write

$$LG = \delta(x-x')$$

Multiplying both sides by $f(x')\,dx'$ and integrating yields

$$\begin{aligned}\int f(x')L(x)G(x',x)\,dx' &= \int f(x')\delta(x-x')\,dx' \\ L(x)\int f(x')G(x',x)\,dx' &= f(x) \\ \int f(x')G(x',x)\,dx' &= L^{-1}(x)f(x)\end{aligned}$$

Returning to our original problem, $Ly = f$, we see that finding the Green's function is tantamount to *inverting* L:

$$y = L^{-1}f = \int G(x,x')f(x')\,dx'$$

Specific applications of Green's Functions as a tool for solving partial differential equations can be found in Part 2.

1.3 Fourier Transforms

From now on we will restrict our discussion to three-dimentional functions, as these are most relevant to Classical Mathematical Physics. The three-dimensional Fourier Transform $\tilde{f}$ is defined as

$$\tilde{f}(\mathbf{k}) = \int_V e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x})\,d\mathbf{x}$$

And its inverse mapping is given by

$$f(\mathbf{x}) = \int_V \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{f}(\mathbf{k})$$

Notice that the Fourier transform maps a spatially-dependent function $f(\mathbf{x})$ to a momentum-dependent function $\tilde{f}(\mathbf{k})$ (where k is the wave vector and is directly related to momentum and the propagation direction of the wave). We can write a similar transform to take a time-dependent function into a frequency (energy) - dependent function:

$$\tilde{f}(\omega) = \int_\tau e^{-i\omega t} f(t)\, dt$$

$$f(t) = \int_\tau \frac{d\omega}{2\pi} e^{i\omega t} \tilde{f}(\omega)$$

For a Fourier Transform to be possible, we say that f and $\tilde{f}$ must satisfy the **Dirichlet Conditions**:

1. f and $\tilde{f}$ must be square-integrable.
2. f and $\tilde{f}$ must be single-valued.
3. f and $\tilde{f}$ must be piecewise continuous.
4. $|f|$ and $|\tilde{f}|$ must be bounded. This last condition is considered "sufficent but not necessary". The dirac delta function, for instance, does not satisfy this property and yet is considered to be transformable.

Since the Fourier functions are a complete set of expansion functions we demand that a fourier transform from time to frequecy followed by a transfrom from frequency to time must return the original function:

$$f(t) = \int_\infty^\infty \frac{d\omega}{2\pi} e^{i\omega t} \tilde{f}(\omega) = \int_\infty^\infty \frac{d\omega}{2\pi} e^{i\omega t} \int_\infty^\infty dt'\, e^{-i\omega t'} f(t') = \int_\infty^\infty dt' f(t') \int_\infty^\infty \frac{d\omega}{2\pi} e^{i\omega (t-t')}$$

This can only be true if

$$\int_\infty^\infty \frac{d\omega}{2\pi} e^{i\omega(t-t')} = 2\pi\delta(t'-t)$$

Therefore we see that the Fourier transform of a plane wave from frequency space to time space yields a delta function (Bracewell 1999, pp. 74-75). In retrospect, this result is very predictable. Since the Fourier transform can be thought of as a projection of a function onto the space of plane waves, projecting a plane wave would yield a single, infintesimally small peak in the fourier spectrum. For consistency, let's convince ourselves that the Fourier transform of a delta function gives back the original plane wave:

> **Example:** Fourier transforming the delta function ($f(t) = \delta(t-t')$) from time space to frequency space yields
>
> $$\tilde{f}(\omega) = \int_\infty^\infty \frac{dt'}{2\pi} f(t) = \int_\infty^\infty dt'\, e^{-i\omega t} \delta(t-t') = e^{-i\omega t'}$$

One of the greatest uses of Fourier Transforms is in solve Partial and Ordinary Differetial Equations. This applciation is made clear by the following theorem.

Theorem on Fourier Transforms *The Fourier transform of the function* $\frac{d^r f}{dt^r}$ *is* $(-i\alpha)^r$ *times the Fourier transform of the function* $f(t)$ *provided that the first* $(r-1)$ *derivatives of* $f(t)$ *vanish as* $t \to \pm\infty$ *(this is true for all reasonable physical applications).*

Proof: Consider the Fourier transform of $\frac{d^r f}{dt^r}$, where $F(\omega)$ is the Fourier transform of $f(t)$:

$$\int_{-\infty}^\infty \frac{d^r f}{dt^r} e^{-\omega t} dt = F^{(r)}(\omega)$$

Integrating the left hand side by parts ($u = e^{i\omega t}$ and $dv = \frac{d^r f}{dt^r} dt$) yields

$$\frac{d^{r-1}f}{dt^{r-1}} e^{i\omega t}\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d^{r-1}f}{dt^{r-1}} (i\omega) e^{i\omega t}$$

Using our assumption that the derivative does to 0 as $t \to \infty$, the above simplifies to

$$-i\omega \int_{-\infty}^{\infty} \frac{d^{r-1}f}{dt^{r-1}} e^{i\omega t} = -i\omega F^{(r-1)}(\omega)$$

Therefore we have

$$F^{(r)}(\omega) = -i\omega F^{(r-1)}(\omega)$$

Applying the same procedure another $r-1$ times (each time using the assumption that the derivative goes to 0 at ∞) we find that

$$F^{(r)}(\omega) = (-i\omega)^r F(\omega)$$

□

This theorem will be used repeatedly in solving linear differential equations in Section 3.

1.4 Electrodynamics

It is impossible to give the extensive field of electrodynamics justice in the span of a few pages, but in this section my goal is to introduce some of the terminology and key formulas that arise as a result of electrodynamics. For a more detailed consider the relatively simple text *Introduction to Electrodynamics* by David Griffiths [**?**] or the more advanced *Classical Electrodynamics* by John David Jackson [**?**].

An electric field $\mathbf{E}$ is a vector field depciting the force of attraction due to a charge distributions ρ. The magnetic field $\mathbf{B}$ is a vector field that represents the torque acted on a charge by a moving charge distribution, known as a **current**. In many models we consider our charge distribution ρ to be distributed in a vacuum, where the **dielectric constant** ϵ is 1 (in the natural unit of measurement). In general the dielectric constant is the relative electric permeability of material - how well an electric field 'travels' through the material, and is measured relative to the permeability of vacuum.

To deal with the different behavior of electric fields in a **dielectric**, a material with polarizable molecules (basically anything but vacuum), we introduce the displacement $\mathbf{D} = \epsilon\mathbf{E}$, where ϵ is the relative permeability of the dielectric. Analogously to electric permeability there is also **magnetic permativity** μ, measured relative to the permativity of vacuum, such that in vacuum $\mu = 1$. Similarly to permeability, in non-vaccum spaces we use $\mathbf{B} = \mu\mathbf{H}$ (notice that this seems backwards to the $\mathbf{D} = \epsilon\mathbf{E}$) definition.

Now that we have defined some terminology, we can look at the famous Maxwell Equations. Maxwell's four equations each have their own names, and I will refer to these names in Section 3.

Gauss's Law:

$$\nabla \cdot \mathbf{E} = \rho \qquad \nabla \cdot \mathbf{D} = \rho_f$$

Where ρ_f is the free (non-induced charge). Only the equation on the left will be used in this paper - so the relation of ρ_f to ρ is not crucial to ones understanding of this text.

Gauss's Law for Magnetism:

$$\nabla \cdot \mathbf{B} = 0$$

Faraday's Law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Ampere's Law:

$$\nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Where $\mathbf{J}$ is the current density (similar to electric flux) and $\mathbf{J}_f$ is the free electron density. Here, again, we are only interested in the left equation. Maxwell's Equations are also commonly written in integral form, and although this paper will only use the differential forms above, integral forms are included below for completness.

$$\oiint_{\partial\Omega} \mathbf{E} \cdot d\mathbf{S} = Q(\Omega) \qquad \oiint_{\partial\Omega} \mathbf{D} \cdot d\mathbf{S} = Q_f(\Omega)$$

$$\oiint \mathbf{B} \cdot \mathbf{S} d\Omega = 0$$

$$\oint_{\partial\Sigma} \mathbf{E} \cdot d\mathbf{l} = -\oiint_{\Sigma} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\oint_{\partial\Sigma} \mathbf{B} \cdot d\mathbf{l} = I + \oiint_{\Sigma} \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S} \qquad \oint_{\partial\Sigma} \mathbf{H} \cdot d\mathbf{l} = I_f + \oiint_{\Sigma} \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

$Q(V)$ represents the total charge enclosed inside the volume Ω, $Q_f(\Omega)$ is the total free charge in volumne Ω, and I is the electric current. Most important to our discussion will be the differential forms of Faradays and Ampere's Laws.

2 Some Important Green's Functions

2.1 The Wave Equation

The wave equation occurs frequently throughout physics (with the proper choice of gauge, Maxwell's Equations all *become* a single wave equation):

$$-\Box\psi(\mathbf{x}, t) = -\left\{\nabla^2 + \frac{1}{v^2}\frac{d^2}{dt^2}\right\}\psi(\mathbf{x}, t) = \rho(\mathbf{x}, t)$$

Our goal is to find the Green's Function of the d'Alembertian operator $\Box$: we want to find the G such that

$$\Box G(\mathbf{x} - \mathbf{x}', t - t') = \delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$$

Rather than working with the rather complicated d'Alembertian, consider taking the Fourier transform from time to frequency:

$$\int_{\infty}^{\infty} d(t' - t)e^{i\omega(t'-t)}(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2})G(\mathbf{x} - \mathbf{x}', t - t') = \int_{\infty}^{\infty} d(t' - t)e^{i\omega(t'-t)}\delta(\mathbf{x}' - \mathbf{x})\delta(t' - t)$$

By our theorem on Fourier Transforms (Section 1.3) the above simplifies to

$$\left(\nabla^2 - \frac{1}{v^2}(i\omega^2)\right)\tilde{G}(\mathbf{x} - \mathbf{x}', \omega) = \delta(\mathbf{x}' - \mathbf{x})$$

Letting $k = \frac{\omega}{v}$ we have

$$\left(\nabla^2 + \frac{\omega^2}{v^2}\right)\tilde{G}(\mathbf{x} - \mathbf{x}', \omega) = \left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{x} - \mathbf{x}', t) = \delta(\mathbf{x}' - \mathbf{x})$$

The above equation is now in the form of the Helmholtz equation, which we shall study next.

2.2 The Helmholtz Equation

We have simplified our original problem of solving the Wave Equation (a two variable operator) to solving the Helmholtz Equation (a one variable operator). We can further simplify by Fourier transforming from space ($\mathbf{x}$) to momentum ($\mathbf{q}$)

$$\int d(\mathbf{x}-\mathbf{x}')e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\left\{\nabla^2+k^2\right\}\tilde{G}(\mathbf{x}-\mathbf{x}') = \int d(\mathbf{x}-\mathbf{x}')e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\delta(\mathbf{x}'-\mathbf{x}) = 1$$

Applying the theorem on Fourier Transforms again yields

$$\int d(\mathbf{x}-\mathbf{x}')e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\left\{\nabla^2+k^2\right\}\tilde{G}(\mathbf{x}-\mathbf{x}') = \left\{(-i\mathbf{q})^2+k^2\right\}\tilde{\tilde{G}}(\mathbf{q},\omega) = \{-q^2+k^2\}\tilde{\tilde{G}}(\mathbf{q},\omega)$$

Where $\tilde{\tilde{G}}$ is the double Fourier Transform of G, ie the transform of $G(\mathbf{x}-\mathbf{x'}, t'-t)$ from time to frequency (energy) space and from configuration space $\mathbf{x}$ to momentum space $\mathbf{q}$. Thus we have

$$\{-q^2+k^2\}\tilde{\tilde{G}}(\mathbf{q},\omega) = 1$$

As a result of two Fourier transforms we have diagonalized the wave operator, making it easy to invert:

$$\tilde{\tilde{G}}(\mathbf{q},\omega) = -\frac{1}{q^2+k^2}$$

In the context of electrodyanmics, since $\epsilon\mu = \left(\frac{c}{v}\right)^2$ and $k^2 = \left(\frac{\omega}{c}\right)^2$ we find that

$$k^2 = \left(\frac{\omega}{c}\right)^2\epsilon\mu$$

In vacuum, $\epsilon = \mu = 1 \Rightarrow k = \frac{\omega}{c}$. Meaning that the poles of the doubly-transformed Green's Function are at $k = \pm\frac{\omega}{c}$. It is very interesting to consider how this expression changes in a non-vacuum environment, where ϵ and μ can depend on the momentum of the source $\mathbf{q}$ or its frequency (energy) ω. This is actually one of the branches of my currently ongoing research, and leads to interesting applications (discussed in Section 4).

Now that we have obtained an expression for the Green's function in momentum-frequency space, we must convert back to a more natural coordinate system. Transforming back to spatial coordinates:

$$\begin{aligned}
\tilde{G}(\mathbf{x}'-\mathbf{x},\omega) &= \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot(\mathbf{x}'-\mathbf{x})} \frac{1}{q^2-k^2} \\
&= -\int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{1}{q^2-k^2} \int d\Omega e^{i\mathbf{q}\cdot(\mathbf{x}'-\mathbf{x})} \\
&= -\int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{1}{q^2-k^2} 2\pi \int_0^{pi} \sin\theta d\theta \; e^{iq\cdot(\mathbf{x}'-\mathbf{x})\cos\theta} \\
&= -\int_0^\infty \frac{q^2 dq}{(2\pi)^2} \frac{1}{q^2-k^2} \int_{-1}^{1} du \; e^{iq\cdot(\mathbf{x}'-\mathbf{x})u} \\
&= -\int_0^\infty \frac{q^2 dq}{(2\pi)^2} \frac{1}{q^2-k^2} \frac{e^{iq|\mathbf{x}'-\mathbf{x}|} - e^{-iq|\mathbf{x}'-\mathbf{x}|}}{iq|\mathbf{x}'-\mathbf{x}|} \\
&= -\frac{2}{(2\pi)^2} \int_0^\infty \frac{q^2 dq}{q^2-k^2} \frac{\sin q|\mathbf{x}'-\mathbf{x}|}{q|\mathbf{x}'-\mathbf{x}|} \\
&= -\frac{1}{2\pi^2|\mathbf{x}'-\mathbf{x}|} \int_0^\infty \frac{q \sin q|\mathbf{x}'-\mathbf{x}| \; dq}{q^2-k^2} \\
&= -\frac{1}{2\pi^2|\mathbf{x}'-\mathbf{x}|} \frac{\pi}{2} e^{\pm ik|\mathbf{x}'-\mathbf{x}|} \\
&= -\frac{e^{\pm ik|\mathbf{x}'-\mathbf{x}|}}{4\pi|\mathbf{x}'-\mathbf{x}|}
\end{aligned}$$

Thus we have found that the Green's Function for the Helmoholz Equation is given by

$$\boxed{\tilde{G}(\mathbf{x}'-\mathbf{x},\omega) = -\frac{e^{\pm ik|\mathbf{x}'-\mathbf{x}|}}{4\pi|\mathbf{x}'-\mathbf{x}|}}$$

Where the $\pm$ is simply shorthand for the sum of the negative-power exponent and the positive-power exponent: $e^{\pm A} \equiv e^A + e^{-A}$

We Fourier transform one more time to get the Green's Function for the wave equation:

$$\begin{aligned}
G(\mathbf{x}'-\mathbf{x},t-t') &= \int_\infty^\infty \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \tilde{G}(\mathbf{x}'-\mathbf{x},\omega) \\
&= -\int_\infty^\infty \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{e^{\pm ik|\mathbf{x}'-\mathbf{x}|}}{4\pi|\mathbf{x}'-\mathbf{x}|} e^{\pm ik|\mathbf{x}'-\mathbf{x}|} \\
&= -\frac{1}{8\pi^2|\mathbf{x}'-\mathbf{x}|} \int_\infty^\infty d\omega e^{-i\omega((t-t')\pm|\mathbf{x}'-\mathbf{x}|/v)} \\
&= -\frac{\delta\left((t-t')\pm|\mathbf{x}'-\mathbf{x}|/v\right)}{4\pi|\mathbf{x}'-\mathbf{x}|} \\
&= -\frac{\delta\left(|\mathbf{x}'-\mathbf{x}|\pm v(t-t')\right)}{4\pi v|\mathbf{x}'-\mathbf{x}|}
\end{aligned}$$

Thus the Green's Function for the Wave Equation is

$$\boxed{G(\mathbf{x}'-\mathbf{x},t-t') = -\frac{\delta(|\mathbf{x}'-\mathbf{x}|\pm v(t-t'))}{4\pi v|\mathbf{x}'-\mathbf{x}|}}$$

2.3 Analysis

Now that we have obtained the Green's funtions of some of these operators, it is generally quite easy to find solutions to specific problems. Before we look at some examples, notice that the limit of $k = 0$ in the Helmholtz equation is the Poisson Equation, suggesting that the Green's function of the ∇^2 operator is

$$G(\mathbf{x} - \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x}' - \mathbf{x}|}$$

Example: Consider a point charge, situated at $\mathbf{x}_0$. The charge distribution of such a construction is given by $\rho = q\delta(\mathbf{x}_0 - \mathbf{x})$, where q is the charge of the point charge. We wish to find potential generated by such a construction, where potential V is given as $\nabla^2 V = -\rho$. We recognize this as the Poisson Equation, allowing us to apply the Green's function determined above

$$V(\mathbf{x}) = -\int_{-\infty}^{\infty} d\mathbf{x}' \frac{-1}{4\pi|\mathbf{x}' - \mathbf{x}|} q\delta(\mathbf{x}_0 - \mathbf{x}) = \frac{q}{4\pi|\mathbf{x}_0 - \mathbf{x}|}$$

Therefore, the potentiall of a point charge is proportional to the inverse-distance between the point of evaluation and the charge location (as in Griffiths, page 50 or any other textbook in electrostatics)

We may also consider the Heat Equation $L = \frac{\partial}{\partial t} - \alpha\nabla^2$ as a member of this family. However, Fourier transforming the Wave equation to get rid of one of the derivatives with respect to t leaves us with the Heat Equation mixed with a frequency (energy component ω. Although analysis of this cross-dimensional problem and it's Fourier transform is possible, the result is fairly complicated and will not be persued here (for a full disclosure consider Byron, 453-469 [**?**]).

The power of Fourier transforms and Green's functions is that we are able to relate many different problems (Wave, Helmholtz, Poisson, Heat) to each other and unifying them with a general solution in the form of a set of Fourier-transform related Green's function solutions. With proper application of Green's function and Fourier transforms it is possible to solve a problem in whatever conjugate space yields the simplest algebra.

3 Electric Field of a Swift Ion

In this section it is my goal to show a practical application of the discussion of mathematical physics given i nthe previous two sections. Consider a single ion traveling with velocity $\mathbf{v}$, and, without loss of generatilty, suppose that the electron travels down the $\hat{\mathbf{z}}$ axis, meaning that $\mathbf{v} = v_z\hat{\mathbf{z}}$. We wish to determine the electric field generated by this ion. We begin with Ampere's and Farady's laws:

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{\dot{\mathbf{D}}}{c} \qquad \nabla \times \mathbf{E} + \frac{\dot{\mathbf{B}}}{c} = \mathbf{0}$$

Where $\mathbf{H}$ is the magnetizing field, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and $\mathbf{J}$ is the current density. $\mathbf{H} = \frac{\mathbf{B}}{\mu}$ and $\dot{\mathbf{D}} = \epsilon\dot{\mathbf{E}}$

Combining these two equations yields

$$\nabla \times \frac{\mathbf{B}}{\mu} = \frac{4\pi}{c}\mathbf{J} + \frac{\epsilon\dot{\mathbf{E}}}{c}$$

Taking the time derivative of both sides yields

$$\nabla \times \frac{\dot{\mathbf{B}}}{\mu} = \frac{4\pi}{c}\dot{\mathbf{J}} + \frac{\epsilon\ddot{\mathbf{E}}}{c}$$

Taking the curl of both sides of Faraday's law gives

$$\nabla \times \nabla \times \mathbf{E} + \nabla \times \frac{\dot{\mathbf{B}}}{c} = \mathbf{0}$$

$$\nabla \times \nabla \times \mathbf{E} + \frac{4\pi\mu}{c^2}\dot{\mathbf{J}} + \frac{\epsilon\mu\ddot{\mathbf{E}}}{c^2} = \mathbf{0}$$

Recalling that $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$ and applying Gauss's Law in differential form ($\nabla \cdot \mathbf{E} = \frac{4\pi\rho}{\epsilon}$) yields

$$\begin{aligned}\mathbf{0} &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} + \frac{4\pi\mu}{c^2}\mathbf{J} + \frac{\epsilon\mu\dot{\mathbf{E}}}{c^2} \\ &= \frac{4\pi\;\nabla\rho}{\epsilon} - \nabla^2\mathbf{E} + \frac{4\pi\mu}{c^2}\mathbf{J} + \frac{\epsilon\mu\dot{\mathbf{E}}}{c^2}\end{aligned}$$

Therefore we have that

$$\nabla^2\mathbf{E} - \frac{\epsilon\mu\ddot{\mathbf{E}}}{c^2} = 4\pi\left(\frac{\nabla\rho}{\epsilon} + \frac{\mu\dot{\mathbf{J}}}{c^2}\right)$$

The format of the above expression is very similar to that of the wave equation:

$$\Box f(\mathbf{x}, t) = \left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) f(\mathbf{x}, t) = g(\mathbf{x}, t)$$

We can define a swift ion as a charged relativistic particle moving at constant velocity ($v \lesssim c$) in a straight line. Without loss of generality, suppose that the particle travels along the z axis, so we can represent its charge distribution and current as:

$$\rho(\mathbf{x}, t) = Q\delta\left(\mathbf{x} - [\mathbf{r} - \mathbf{v}t]\right)$$

$$\mathbf{J}(\mathbf{x}, t) = \mathbf{v}\rho(\mathbf{x}, t)$$

This means that we already have an expression for the Green's Function

$$\Box G = \delta(\mathbf{x'} - \mathbf{x})\delta(t' - t)$$

$$G(\mathbf{x'} - \mathbf{x}, t - t') = -\frac{\delta\left(|\mathbf{x'} - \mathbf{x}| \pm v(t - t')\right)}{4\pi v|\mathbf{x'} - \mathbf{x}|}$$

Thus we have

$$\begin{aligned}\Box\mathbf{E}(\mathbf{x}, t) &= -4\pi\left\{\frac{\nabla\rho(\mathbf{x}, t)}{\epsilon} + \frac{\mu\dot{\mathbf{J}}(\mathbf{x}, t)}{c^2}\right\} \\ \Rightarrow \mathbf{E}(\mathbf{x}, t) &= \int d^3x' \int_{\infty}^{\infty} dt'(-4\pi)\left\{\frac{\nabla'\rho(\mathbf{x'}, t')}{\epsilon} + \frac{\mu\dot{\mathbf{J}}(\mathbf{x'}, t')}{c^2} G(\mathbf{x} - \mathbf{x'}, t - t')\right\}\end{aligned}$$

Fourier transformation yields

$$\begin{aligned}\tilde{E}(\mathbf{x}, \omega) &= \int_{\infty}^{\infty} dt e^{i\omega t}\mathbf{E}(\mathbf{x}, t) \\ &= \int_{\infty}^{\infty} dt e^{i\omega t} \int d^3x' \int_{\infty}^{\infty} dt' G(\mathbf{x} - \mathbf{x'}, t - t')(-4\pi)\left\{\frac{\nabla'\rho(\mathbf{x'}, t')}{\epsilon} + \frac{\mu\dot{\mathbf{J}}(\mathbf{x'}, t')}{c^2}\right\}\end{aligned}$$

or

$$\tilde{\mathbf{E}}(\mathbf{x},\omega) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \int_{\infty}^{\infty} dt e^{i\omega t} \tilde{\mathbf{E}}(\mathbf{q},t)$$
$$= \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \tilde{\tilde{\mathbf{E}}}(\mathbf{q},\omega)$$

But we know that

$$\{\nabla^2 - \frac{1}{v^2}\}\mathbf{E}(\mathbf{x},t) = -4\pi \left\{ \frac{\nabla\rho(\mathbf{x},t)}{\epsilon} + \frac{\mu\dot{\mathbf{J}}(\mathbf{x},t)}{c^2} \right\}$$
$$\Rightarrow \{-q^2 + k^2\}\tilde{\tilde{\mathbf{E}}}(\mathbf{q},\omega) = \int_{\infty}^{\infty} dt e^{i\omega t} \int d^3x e^{-i\mathbf{q}\cdot\mathbf{x}} (-4\pi) \left\{ \frac{\nabla\rho(\mathbf{x},t)}{\epsilon} + \frac{\mu\dot{\mathbf{J}}(\mathbf{x},t)}{c^2} \right\}$$
$$= 4\pi \left\{ \frac{-i\mathbf{q}}{\epsilon} \tilde{\tilde{\rho}}(\mathbf{q},\omega) + \frac{\mu(i\omega)}{c^2} \tilde{\tilde{\mathbf{J}}}(\mathbf{q},\omega) \right\}$$

Assuming non-magnetizable media ($\mu = 1$) we have

$$\tilde{\tilde{\mathbf{E}}}(\mathbf{q},\omega) = \frac{1}{-q^2 + k^2\epsilon} 4\pi i \left\{ \frac{-\mathbf{q}}{\epsilon} + \frac{\omega\mathbf{v}}{c^2} \right\} \tilde{\tilde{\rho}}(\mathbf{q},\omega)$$
$$= \tilde{\tilde{G}}(\mathbf{q},\omega) 4\pi i \left\{ \frac{-\mathbf{q}}{\epsilon} + \frac{k\mathbf{v}}{c} \right\} \tilde{\tilde{\rho}}(\mathbf{q},\omega)$$

Therefore

$$\tilde{\mathbf{E}}(\mathbf{x},\omega) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \tilde{\tilde{G}}(\mathbf{q},\omega) 4\pi i \left\{ \frac{-\mathbf{q}}{\epsilon} + \frac{k\mathbf{v}}{c} \right\} \tilde{\tilde{\rho}}(\mathbf{q},\omega)$$
$$= \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} 4\pi i \frac{\frac{-\mathbf{q}}{\epsilon} + \frac{k\mathbf{v}}{c}}{q^2 - k^2\epsilon} \tilde{\tilde{\rho}}(\mathbf{q},\omega)$$

At this point it is useful to compute $\tilde{\tilde{\rho}}$

$$\tilde{\tilde{\rho}}(\mathbf{q},\omega) = \int d^3x e^{-i\mathbf{q}\cdot\mathbf{x}} \int_{\infty}^{\infty} dt e^{i\omega t} \rho(\mathbf{x},t)$$
$$= \int d^3x e^{-i\mathbf{q}\cdot\mathbf{x}} \int_{\infty}^{\infty} dt e^{i\omega t} Q\delta(\mathbf{x} - [\mathbf{r} + \mathbf{v}t])$$
$$= Q \int_{\infty}^{\infty} dt e^{i\omega t} e^{-i\mathbf{q}\cdot[\mathbf{r}+\mathbf{v}t]}$$
$$= Q e^{-i\mathbf{q}\cdot\mathbf{r}} \int_{\infty}^{\infty} dt e^{it[\omega - \mathbf{q}\cdot\mathbf{v}]}$$
$$= 2\pi Q e^{-i\mathbf{q}\cdot\mathbf{r}} \delta(\omega - \mathbf{q}\cdot\mathbf{v})$$

Continuing with the previous integral yields

$$\tilde{\mathbf{E}}(\mathbf{x},\omega) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} 4\pi i \frac{\frac{-\mathbf{q}}{\epsilon} + \frac{k\mathbf{v}}{c}}{q^2 - k^2\epsilon} \left[2\pi Q e^{-i\mathbf{q}\cdot\mathbf{r}} \delta(\omega - \mathbf{q}\cdot\mathbf{v})\right]$$

Assuming that the charge Q crosses the origin $\mathbf{r} = \mathbf{0}$ at time $t = 0$ lets us simplify this expression to

$$\tilde{\mathbf{E}}(\mathbf{x},\omega) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} 4\pi i \frac{\frac{-\mathbf{q}}{\epsilon} + \frac{k\mathbf{v}}{c}}{q^2 - k^2\epsilon} \left[2\pi Q \delta(\omega - \mathbf{q}\cdot\mathbf{v})\right]$$

Recalling that the ion only travels along the z axis, we can write $\mathbf{v} = v\hat{\mathbf{e}}_z$, $\delta(\omega - \mathbf{q}\cdot\mathbf{v}) = \delta(\omega - q_z v) = \delta\left(v\left[\frac{\omega}{v} - q_z\right]\right) = \frac{1}{v}\delta\left(q_z = \frac{\omega}{v}\right)$. This will alow us to evaluate the q_z integral:

$$
\begin{aligned}
\tilde{\mathbf{E}}(\mathbf{x},\omega) &= \int \frac{d^3q}{(2\pi)^2} e^{i\mathbf{q}\cdot\mathbf{x}} 4\pi i \frac{(q_x\hat{\mathbf{e}}_x + q_y\hat{\mathbf{e}}_y + q_z\hat{\mathbf{e}}_z)/\epsilon - k\mathbf{v}/c}{q_x^2 + q_y^2 + q_z^2 - k^2\epsilon} Q\delta(\omega - \mathbf{q}\cdot\mathbf{v}) \\
&= \frac{4\pi i Q}{v} \int \frac{d^2q}{(2\pi)^2} e^{iq_x x} e^{iq_y y} e^{i\frac{\omega}{v}z} \frac{(q_x\hat{\mathbf{e}}_x + q_y\hat{\mathbf{e}}_y + \frac{\omega}{v}\hat{\mathbf{e}}_z)/\epsilon - k\mathbf{v}/c}{q_x^2 + q_y^2 + (\omega/v)^2 - k^2\epsilon} \\
&= \frac{4\pi i Q}{v} e^{i\frac{\omega}{v}z} \left\{ \int \frac{d^2q}{(2\pi)^2} e^{iq_x x} e^{iq_y y} \frac{(q_x\hat{\mathbf{e}}_x + q_y\hat{\mathbf{e}}_y)/\epsilon}{q_x^2 + q_y^2 + (\omega/v)^2 - k^2\epsilon} - \int \frac{d^2q}{(2\pi)^2} e^{iq_x x} e^{iq_y y} \frac{k\mathbf{v}/c - \omega/(v\epsilon)}{q_x^2 + q_y^2 + (\omega/v)^2 - k^2\epsilon} \hat{\mathbf{e}}_z \right\}
\end{aligned}
$$

Now, if we notice that

$$
\begin{aligned}
\left(\frac{\omega}{v}\right)^2 - k^2\epsilon &= \left(\frac{\omega}{v}\right)^2 - \left(\frac{\omega}{c}\right)^2 \epsilon \\
&= \left(\frac{\omega}{v}\right)^2 \left[1 - \frac{(\omega/c)^2\epsilon}{(\omega/v)^2}\right] \\
&= \left(\frac{\omega}{v}\right)^2 \left[1 - \left(\frac{v}{c}\right)^2 \epsilon\right] \\
&= \left(\frac{\omega}{v}\right)^2 \frac{1}{\gamma^2} \\
&= \left(\frac{\omega}{\gamma v}\right)^2
\end{aligned}
$$

where γ is the Lorentz factor from special relativity. Similarly,

$$
\frac{kv}{c} - \frac{\omega}{v\epsilon} = -\frac{\omega}{v\epsilon}\left[1 - \epsilon\left(\frac{v}{c}\right)^2\right] = -\frac{\omega}{v\epsilon\gamma^2}
$$

Thus our expression for $\tilde{\mathbf{E}}$ becomes

$$
\begin{aligned}
\tilde{\mathbf{E}}(\mathbf{x},\omega) = \frac{4\pi i Q}{v} e^{i\frac{\omega}{v}z} &\int \frac{d^2q}{(2\pi)^2} e^{iq_x x} e^{iq_y y} \frac{(q_x\hat{\mathbf{e}}_x + q_y\hat{\mathbf{e}}_y)/\epsilon}{q_x^2 + q_y^2 + (\omega/v\gamma)^2} \\
+ \frac{4\pi i Q}{v} &\int \frac{d^2q}{(2\pi)^2} e^{iq_x x} e^{iq_y y} \frac{\omega/v\epsilon\gamma^2}{q_x^2 + q_y^2 + (\omega/v\gamma)^2} \hat{\mathbf{e}}_z
\end{aligned}
$$

The electric field due to a single ion moving down the z axis is, once we have integrated out the time dependence, cylindrically symetric (you can imagine that the Fourier Transform smeared out the electron over its entire trajectory). Therefore we can choose to look at

$\mathbf{x} = (b, 0, z)$. Now we can compute the above two integrals seperately:

$$\begin{aligned}
\int \frac{d^2q}{(2\pi)^2} e^{iq_x x} e^{iq_y y} \frac{(q_x \hat{\mathbf{e}}_x + q_y \hat{\mathbf{e}}_y)/\epsilon}{q_x^2 + q_y^2 + (\omega/v\gamma)^2} &= \int \frac{d^2q}{(2\pi)^2} e^{iq_x b} \frac{(q_x \hat{\mathbf{e}}_x)/\epsilon}{q_x^2 + q_y^2 + (\omega/v\gamma)^2} \\
&= \int_{-\infty}^{\infty} t \frac{dq_x}{(2\pi)^2} e^{iq_x b} \frac{q_x \hat{\mathbf{e}}_x}{\epsilon} \int_{-\infty}^{\infty} \frac{dq_y}{q_x^2 + q_y^2 + (\omega/v\gamma)^2} \hat{\mathbf{e}}_x \\
&= \int_{-\infty}^{\infty} \frac{dq_x}{(2\pi)^2} e^{iq_x b} \frac{q_x \hat{\mathbf{e}}_x}{\epsilon} \frac{\pi}{\sqrt{q_x^2 + (\omega/v\gamma)^2}} \hat{\mathbf{e}}_x \\
&= -\frac{i}{\epsilon} \frac{\partial}{\partial b} \int_{-\infty}^{\infty} \frac{dq_x}{(2\pi)^2} e^{iq_x b} \frac{q_x \hat{\mathbf{e}}_x}{\epsilon} \frac{\pi}{\sqrt{q_x^2 + (\omega/v\gamma)^2}} \hat{\mathbf{e}}_x \\
&= -\frac{2\pi i}{\epsilon (2\pi)^2} \hat{\mathbf{e}}_x \frac{\partial}{\partial b} \frac{1}{2} \int_{-\infty}^{\infty} dq_x e^{iq_x b} \frac{eiq_x b}{\sqrt{q_x^2 + (\omega/v\gamma)^2}} \\
&= -\frac{i}{2\pi\epsilon} \hat{\mathbf{e}}_x \frac{\partial}{\partial b} K_0\left(\frac{\omega b}{v\gamma}\right) \\
&= -\frac{i}{2\pi\epsilon} \hat{\mathbf{e}}_x \frac{\omega}{v\gamma} \frac{\partial}{\partial u} K_0(u) \\
&= -\frac{i\omega}{2\pi\epsilon v\gamma} \hat{\mathbf{e}}_x K_1\left(\frac{\omega b}{v\gamma}\right)
\end{aligned}$$

Here, K_i represents the modified Bessel function of the second kind - they are basically the basis functions for cylindrical corrdinates (notice that this makes sense, as our electric field has cylindrical symmetry). I have also used the back cover of Jackson [**?**] to compute the dq_y integral and have used the fact that $\frac{\partial}{\partial u} K_0(u) = K_1$, as in Boas, Chapters 12.12-12.18 [**?**].

Now we turn to the second integral. We find

$$\begin{aligned}
\int \frac{d^2q}{(2\pi)^2} e^{iq_x x} e^{iq_y y} \frac{\omega/v\epsilon\gamma^2}{q_x^2 + q_y^2 + (\omega/v\gamma)^2} \hat{\mathbf{e}}_z &= \int \frac{d^2q}{(2\pi)^2} e^{iq_x b} \frac{(q_x \hat{\mathbf{e}}_x + q_y \hat{\mathbf{e}}_y)/\epsilon}{q_x^2 + q_y^2 + (\omega/v\gamma)^2} \\
&= \int_{-\infty}^{\infty} \frac{dq_x}{2\pi} e^{iq_x b} \int_{-\infty}^{\infty} \frac{dq_y}{2\pi} \frac{\omega/v\epsilon\gamma^2}{q_x^2 + q_y^2 + (\omega/v\gamma)^2} \\
&= \frac{\omega}{v\epsilon\gamma^2} \hat{\mathbf{e}}_z \int_{-\infty}^{\infty} \frac{dq_x}{2\pi} e^{iq_x b} \frac{\pi}{q_x^2 + (\omega/v\gamma)^2} \\
&= \frac{2\omega\pi}{(2\pi)^2 v\epsilon\gamma^2} \hat{\mathbf{e}}_z \frac{1}{2} \int_{-\infty}^{\infty} dq_x \frac{e^{iq_x b}}{q_x^2 + (\omega/v\gamma)^2} \\
&= \frac{\omega}{(2\pi v\epsilon\gamma^2} \hat{\mathbf{e}}_z K_0\left(\frac{\omega b}{v\gamma}\right)
\end{aligned}$$

This leaves us with our final, neat expression for the electric field:

$$\boxed{\tilde{\mathbf{E}}(\mathbf{b}, \omega) = \frac{2Q\omega}{v^2\epsilon\gamma} e^{i\frac{\omega}{v}z} \left[\frac{i}{\gamma} K_0\left(\frac{\omega b}{v\gamma}\right) \hat{\mathbf{z}} - K_1\left(\frac{\omega b}{v\gamma}\right) \hat{\mathbf{R}}\right]}$$

Where K_0 and K_1 are modified bessel functions of the second kind, $\hat{\mathbf{R}} = \hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y$ is the unit vector pointing in the radial direction away from the z axis along which the ion travels. Notice that here we have recovered the generality of $\mathbf{x} = (b, b, z)$ using the cylindrical symmetry of the problem. One may be interested in further Fourier transforming this solution to obtain the electric field in terms of both time and space, but this will eliminate said symmetry and generate a fairly complicated expression for the electric field. In practice, the above form is much easier to use.

4 The Electron Energy Loss Experiment

The Electron Energy Loss Spectroscopy (EELS) uses a Scanning Electron Microscope to fire single, high-energy relativistic electrons at a target. In the past decade advances have been made that allow experimentalists to determine the slightest changes in the energy of such an electron as it flys by a target. These energy losses (on the order of 1-5 electronvolts of the electron's original 300 kilo-electonvolts) have been attributed to the electron field generated by the electron coupling into the electron cloud of the target, which is often a metal nanoparticle of 10-200 nm diameter. Although the electron is practically a point particle and its electric field is very localized (the field is almost absent just 5 nm away from the electron) the EELS experiment detects that the electron cloud of the nanoparticle experiences a global, wave-like oscillation due to the electron's perturbation. Such oscillations are known in the field as *surface plasmons* and have the same nodal structure as many of the waves we are accostomed to (guitar strings or wave-like light). As an analogy, consider poking a water balloon with a stick (the swift electron). The water inside the balloon will oscillate, causing the walls of the balloon (the electron cloud) to expand and contract in a wave-like manner.

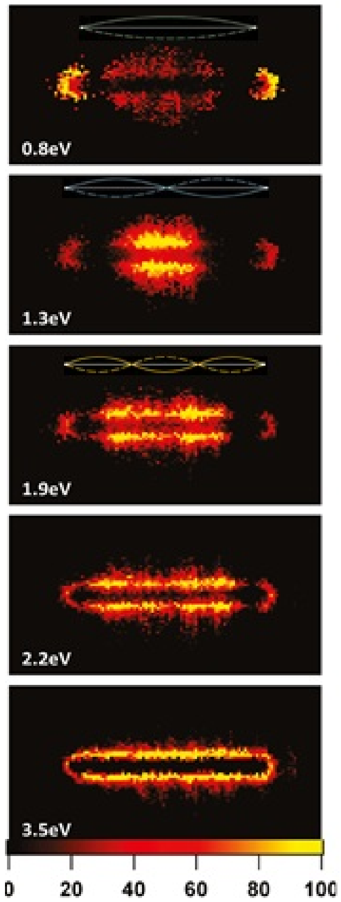

Figure 4: The above plots show the wave-like excitation of the electron cloud of a metal nano-rod due to different energies being coupled into the cloud by the swift electron. Orange and yellow areas represent the locations where the electron was most likely to couple to the electron cloud. The images on the left are the experimentally obtained plots, while the images on the right are the results of our theoretical computations. Notice the wave-like structure consistent throughout both experimental and theoretical results

Curiously, such excitations have been attributed to plasmons only very recently, and it is my group's goal to construct a computational model that will accurately predict which plasmons (how many nodes in the wave) can be excited by a swift electron and how such plasmons will evolve in time. We have found numerous results that spike our interest. For instance, we have shown that placing two nanometals within several nanometers of each other and shooting a single electron near one such metal can lead to a plasmonic excitation that travels from one metal to the other, creating a 'hot spot' in the junction between them. Such a hot spot experiences a 100-fold magnification of the incident electric field (due to the single electron). Applications of such hot spots have been suggested in a variety of fields ranging from solar cell enhancement to single-molecule spectroscopy.

The computation model that we use relies on the **Discrete Dipole Approximation**, where the nanometal is broken into hundreds of thousands of small dipoles (arrows that point in the direction of electric flow) that, starting at rest, react to the electric field generated by the electron. While at first only the dipoles near the electron are influenced by the perturbing field (remember that the electron's field is very localized), as some dipoles re-orient to accommodate the change in the electric field, nearby dipoles are also perturbed, causing them to also re-orient. This effect propogates throughout the metal, creating the

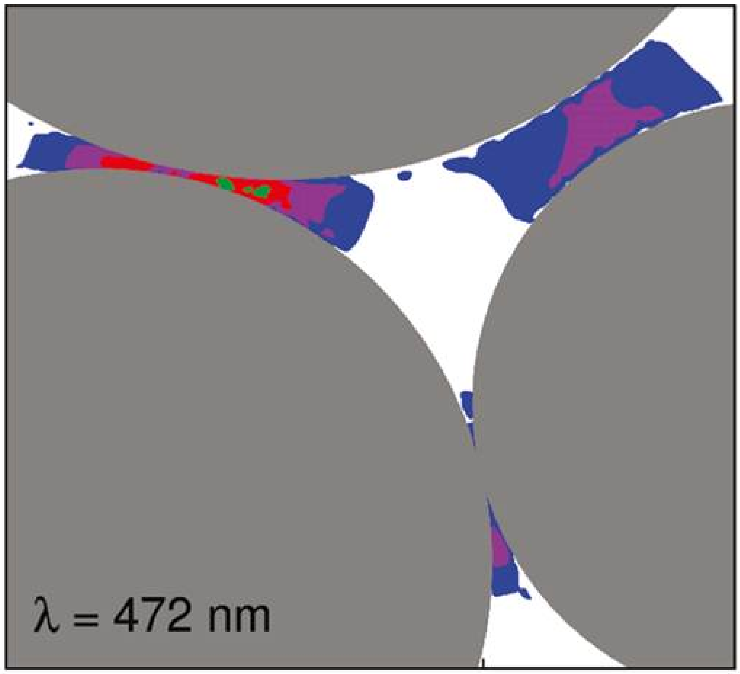

Figure 5: A close-up on a computational model of a hot spot. Three circular metals, placed very close together, focus the electric field generated by the swift electron into the junctions that seperate them, magnifying the field up to 100-fold. This effect has been used extensively in chemical spectroscopy [**?**] and has been both a problem and a possible way to enhance solar cells [**?**]

wave-like oscillations that we observe.

The purpose of the derivation within this paper - the electric field of a swift ion - is clear. The explicit form of the electric field is used directly in computing the electron's effect on the metal's electron cloud. To make derivations simplest the model assumes that the electron travels though a vacuum, allowing $\epsilon = \mu = 1$. Future research will involve studying how a non-vacuum medium $\epsilon \neq 1$, and non-constant medium $\epsilon = \epsilon(\mathbf{x})$ affect this calculation. If our model can correctly predict the scattering of a swift electron off a general medium, we will be able to use the EELS experiment to *determine* the dielectric constant ϵ of any material (even one consisting of many different materials in random proportions).

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