Math 336 Final Exam, 8:30 am, June 6 , 2011

Name:____

One notebook-size page of notes is allowed (each side may be used).

1. Suppose f is analytic on $H = \{z = x + iy : y > 0\}$ and suppose $|f(z)| \leq 1$ on H and $f(i) = 0$. Prove

$$|f(z)| \leq \left|\frac{z-i}{z+i}\right|.$$

2. Let $m \geq 0$ be an integer. Compute the residue of $\Gamma(z)$ at $-m$.

3. Let $g(z, t)$ be a continuous function of z, t where $z \in D$, $t \in K$, D is an open subset of $\mathbb{C}$ and K is a compact subset of $\mathbb{R}^n$. Suppose that g is harmonic as a function z. Prove that $G(z) = \int_K g(z, t)dt$ is harmonic.

4. Let $f(z) = \dfrac{z-a}{1-\bar{a}z}$, where $|a| < 1$. Let $D = \{z : |z| < 1\}$. Prove that

$$\frac{1}{\pi}\int_D |f'(z)|^2 dxdy = 1.$$

5. Decide whether or not the following products converge.

   (a) $\displaystyle\prod_{j=1}^{\infty} e^{(-1)^j/\sqrt{j}}$

   (b) $\displaystyle\prod_{j=1}^{\infty} \left(1 + i\frac{(-1)^j}{\sqrt{j}}\right)$

6. Prove that all conformal maps from the upper half plane to the unit disk have the form

$$\alpha \frac{z-\beta}{z-\bar{\beta}},$$

where $|\alpha| = 1, \ Im(\beta) > 0$.

7. (a) Find a bounded harmonic function u defined on $D = \{z : |z| < 1\}$ so that

$$\lim_{z \to e^{it}} u(z) = \begin{cases} 1, & \text{if } 0 < t < \pi/2 \\ 0, & \text{if } \pi/2 < t < 2\pi \end{cases}$$

(b) Find an unbounded harmonic function with the same property.

8. Prove that if $|z| < 1$

$$\prod_{k=0}^{k=\infty} (1 + z^{2^k}) = \frac{1}{1 - z}$$

9. Let u be harmonic on the open set W. Assume that u is not constant. Let $D \subset W$ be an open subset of W. Prove that $u(D)$ is an open subset of $\mathbb{R}$.