Continued Fraction Approximations of the Riemann Zeta Function
MATH 336

Shawn Apodaca

1 Introduction

Continued fractions serve as a useful tool for approximation and as a field of their own. Here we will concern ourselves with results from Cvijovic and Klinowski from *Continued-Fraction Expansions for the Riemann Zeta Function and Polylogarithms* [3]. From the results, we will be capable of numerically approximating the Riemann zeta function ζ for integer values n, which are special cases of the polylogarithm.

2 Notation

We will denote the positive integers $\mathbb{N}$ and $\mathbb{N} \cup \{0\}$ as $\mathbb{Z}^+$. We will define the *polylogarithm function* as follows.

$$Li_\nu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu} \tag{1}$$

In particular, $Li_\nu(1) = \zeta(\nu)$ where $\zeta(\nu)$ is the *Riemann zeta function.* We will denote the set of all real-valued, bounded, monotone non-decreasing functions $\phi(t)$ with infinitely many values on $a \leq t \leq b$ as $\Phi(a,b)$ where a, b are elements of the extended reals $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$.

3 Preliminary Definitions and Results

Here we will give necessary definitions and some preliminary results.

3.1 Continued Fractions

We define a *continued fraction* as follows.

Definition 3.1. *An (infinite) continued fraction $K(a_k/b_k)$ is an expression of the form*

$$K(a_k/b_k) = \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k}{b_k} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$$

The nth approximate F_n is defined

$$F_n = \mathop{\mathrm{K}}_{k=1}^{n} \frac{a_k}{b_k} = \frac{A_n}{B_n}$$

We say $K(a_k/b_k)$ converges to F if the sequence of approximates converge F in the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. We call A_n the nth *numerator* and B_n the nth *denominator.* We say $K(a_k/b_k)$ diverges if the limit $\lim_{n\to\infty} F_n$ does not exist. We call each a_k and b_k the kth *numerator* and *denominator*, respectively. Note that we will be use the convention that $a_k \neq 0$. We say two continued fractions $K(a_k/b_k)$ and $K(a_k^*/b_k^*)$ are *equivalent*, written $K(a_k/b_k) \cong K(a_k^*/b_k^*)$, if each approximate $F_n = F_n^*$.

A continued fraction of the form

$$K(a_k/b_k) = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k z}{1} \tag{2}$$

Is called a *regular C-fraction* (regular corresponding fraction) and a continued fraction of the form

$$K(a_k/b_k) = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{1} \tag{3}$$

Is called a *modified regular C-fraction.* If each $a_k > 0$, then (2) and (3) are called *regular S-fraction* and *modified regular S-fraction* (Stieltjes fractions), respectively.

A finite continued fraction $\underset{k=1}{\overset{n}{\mathrm{K}}} \frac{a_k(z)}{b_k(z)}$ is said to correspond to the series $\sum_{k=0}^{\infty} \frac{c_k}{z^k}$ at $z = \infty$ if the following formal power series expansions are valid:

$$F_n(z) - \sum_{p=0}^{\lambda_n} \frac{c_p}{z^k} = \text{const} z^{-(\lambda_n+1)} + \cdots$$

Where $n = 1, 2, 3, \ldots$.

3.2 The Stieltjes-Riemann Integral

Here we will define the Stieltjes-Riemann integral of a function $f(x)$, denoted by $\int_a^b f(x)\, d\alpha(x)$, and give a few preliminary results. Here, we will use Apostol [1]. We define $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$ such that

$$\sum_{k=1}^{n} \Delta\alpha_k = \alpha(b) - \alpha(a)$$

We will also use the notion of a partition P of an interval $[a, b]$. This will be the same as that discussed in Foland [4]. We now define the Stieltjes-Riemann integral.

Definition 3.2. *Let $P = \{x_0, x_1, \ldots, x_k\}$ be a partition of $[a, b]$ and let $t_k \in [x_{k-1}, x_k]$. Then the Stieltjes-Riemann sum of f with respect to α is defined as*

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k$$

If there exists a unique number A such that for any $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that for every partition P finer than P_ϵ and for every choice of $t_k \in [x_{k-1}, x_k]$, we have that $|S(P, f, \alpha) - A| < \epsilon$. The number $A = \int_a^b f(x)\, d\alpha(x)$.

We state without proof that A is uniquely determined whenever it exists. For our proof the main theorem, we will need the following two theorems.

Theorem 3.3. *Suppose f is continuous on $[a, b]$ and α is any monotonic, increasing function. Then f is integrable with respect to α over $[a, b]$.*

For a proof, see [2]. We now give criteria where a Stieltjes-Riemann integral simplifies to a Riemann integral.

Theorem 3.4. *Suppose f is integrable with respect to α on $[a,b]$. If α is continuously differentiable on $[a,b]$, then $\int_a^b f(x)\alpha'(x)\,dx$ exists. Further*

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx$$

For proof, see Apostol [1].

3.3 The Markov Theorem

We will state the Markov theorem, without proof, since it will be used the proof of the main theorem. For a proof, see Perron [6]. However, we will state it as found in Jones and Thron [5].

Theorem 3.5. *Suppose $\phi \in \Phi(0,a)$. Then there is a modified S-fraction which corresponds to the series*

$$\sum_{k=0}^{\infty} \frac{(-1)^k \mu_k}{z^k} \quad \text{where} \quad \mu_k = \int_0^a t^k\,d\phi(t) \tag{4}$$

at $z = \infty$, converges to the function

$$\int_0^a \frac{z}{z+t}\,d\phi(t) \tag{5}$$

for all $z \in \mathbb{C} \setminus [-a,0]$.

3.4 Hankel Determinants

Definition 3.6. *Suppose $\{c_k\}_{k=0}^{\infty}$ is a sequence. Then the Hankel determinants $H_m^{(r)}$ associated with $\{c_k\}$, where $r \in \mathbb{Z}^+$ and $m \in \mathbb{N}$ are given by*

$$H_0^{(r)} = 1, \qquad H_m^{(r)} = \begin{vmatrix} c_r & c_{r+1} & \cdots & c_{r+m-1} \\ c_{r+1} & c_{r+2} & \cdots & c_{r+m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r+m-1} & c_{r+m} & \cdots c_{r+2m-2} & \end{vmatrix}$$

4 The Main Theorem

Theorem 4.1. *Suppose that $r \in \mathbb{Z}^+$ is a non-negative integer and $m,n \in \mathbb{N}$. For any fixed r,m,n, define $A_m^{(r)}(n)$ as the determinant of an $m \times m$ matrix*

$$A_m^{(r)}(n) = \det \left\| \frac{(-1)^{i+j+r}}{(r+i+j-1)^n} \right\|_{1 \le i,j \le m}$$

Where we define $A_0^{(r)}(n) = 1$. Then

$$-Li_n(-z) = \operatorname*{K}_{k=1}^{\infty} \frac{a_{n,k}z}{1} \tag{6}$$

With

$$a_{n,1} = 1, \quad a_{n,2m} = -\frac{A_m^{(1)}(n)A_{m-1}^{(0)}(n)}{A_m^{(0)}(n)A_{m-1}^{(1)}(n)}, \quad a_{n,2m+1} = -\frac{A_{m-1}^{(1)}(n)A_{m+1}^{(1)}(n)}{A_m^{(0)}(n)A_m^{(1)}(n)} \tag{7}$$

Proof. Consider the function

$$\phi_n(t) = \begin{cases} 0, & t = 0 \\ \frac{1}{(n-1)!}\int_0^t \left(\log\left(\frac{1}{x}\right)\right)^{n-1} dx, & 0 < t \le 1 \\ 1, & t > 1 \end{cases}$$

For $n = 1$, the integrand is just 1, so it is clearly integrable and $\phi_n(t)$ is continuous. Where $n \in \mathbb{N}$. Prudnikov [7] gives us

$$\int_\epsilon^t \left(\log\left(\frac{1}{x}\right)\right)^{n-1} dx = \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!}\left(t(\log t)^k - \epsilon(\log\epsilon)^k\right)$$

We apply L'Hôptial's rule to get that $\epsilon \log^k \epsilon \to 0$ as $\epsilon \to 0$. So

$$\int_0^t \left(\log\left(\frac{1}{x}\right)\right)^{n-1} dx = \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!} t \log^k t \tag{8}$$

L'Hôpital's rule gives that $\phi_n(t) \to 0$ as $t \to 0^+$ and $\phi_n(t) \to 1$ as $t \to 1^-$. For $0 < t \le 1$, $\log\left(\frac{1}{x}\right) \ge 0$ and continuous and, thus, integrable, so the integral is monotonically increasing and continuous on $[0, 1]$. Further, $\phi_n(t) \in \Phi(0, \infty)$.

Consider the following integral, called the Stieltjes transform of $\phi_n(t)$.

$$f_n(z) = \int_0^\infty \frac{d\phi_n(t)}{z + t} \tag{9}$$

Where $z \notin [-\infty, 0]$. Then by (3.3), the integrand is integrable with respect to $\phi_n(t)$. Further, since $\phi_n(t)$ is continuously differentiable on $[0, \infty)$, by theorem (3.4), we have that

$$f_n(z) = \frac{1}{(n-1)!}\int_0^1 \frac{1}{z+t}\left(\log\left(\frac{1}{t}\right)\right)^{n-1} dt$$

We then substitute $x = \log\left(\frac{1}{t}\right)$. This gives us that $t = e^{-x}$ and $dt = -e^{-x}dx$. So

$$f_n(z) = \frac{1}{(n-1)!}\int_\infty^0 \frac{x^{n-1}}{z + e^{-x}}(-e^{-x})\, dx = \frac{\frac{1}{z}}{(n-1)!}\int_0^\infty \frac{x^{n-1}}{e^x + \frac{1}{z}}\, dx$$

This a form of the *Fermi-Dirac integral*, which has a known polylogarithm representation. In our case

$$f_n(z) = -Li_n\left(-\frac{1}{z}\right)$$

Using the series representation of the polylogarithm (1), we get, for $|z| > 1$, the following.

$$f_n(z) = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k^n z^k} \iff zf_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^n z^k} = \sum_{k=0}^{\infty} \frac{c_{n,k}}{z^k}$$

Where we have let $c_{n,k} = \frac{(-1)^k}{(k+1)^n}$. Further, Markov tells us there exists a corresponding modified S-fraction that converges to $zf_n(z)$ for all $z \in \mathbb{C} \setminus [0, -1]$ and even tells us that

$$c_{n,k} = (-1)^k \mu_{n,k} \tag{10}$$

Where $\mu_{n,k} = \frac{1}{(n-1)!} \int_0^1 t^k \left(\log\left(\frac{1}{t}\right)\right)^{n-1} dt$.

Jones and Thron [5] give us that whenever a series $S = \sum_{k=0}^{\infty} \frac{c_k}{z^k}$ corresponds to a modified C-fraction $C = \underset{k=0}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{1}$ at $z = \infty$, we know that

$$a_1 = c_0, \qquad a_{2m} = -\frac{H_m^{(1)} H_{m-1}^{(0)}}{H_m^{(0)} H_{m-1}^{(1)}}, \qquad a_{2m+1} = -\frac{H_{m-1}^{(1)} H_{m+1}^{(0)}}{H_m^{(0)} H_m^{(1)}}$$

Which is exactly what we have, except

$$a_{n,1} = 1, \qquad a_{n,2m} = -\frac{A_m^{(1)}(n) A_{m-1}^{(0)}(n)}{A_m^{(0)}(n) A_{m-1}^{(1)}(n)}, \qquad a_{n,2m+1} = -\frac{A_{m-1}^{(1)}(n) A_{m+1}^{(0)}(n)}{A_m^{(0)}(n) A_m^{(1)}(n)}$$

With each $A_m^{(r)}(n)$ as described in the main theorem. We then have

$$zf_n(z) = \cfrac{a_{n,1}}{1 + \cfrac{a_{n,2}}{z + \cfrac{a_{n,3}}{1 + \cfrac{a_{n,4}}{z + \cdots}}}}$$

Dividing both sides by z and simple factoring gives us

$$f_n(z) \cong \cfrac{a_{n,1}(1/z)}{1 + \cfrac{a_{n,2}(1/z)}{1 + \cfrac{a_{n,3}(1/z)}{1 + \cfrac{a_{n,4}(1/z)}{1 + \cdots}}}} = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_{n,k}(1/z)}{1}$$

Thus, $-Li_n(-1/z) = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_{n,k}(1/z)}{1}$. So

$$-Li_n(-z) = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_{n,k}z}{z}$$

And we are done. □

5 Additional Results

We conclude with some calculations. Using our results, we may immediately use our results for $-Li_1(-z) = \log(1+z)$ and $-Li_n(-1) = (1-2^{1-n})\zeta(n)$, for integers $n \geq 2$. Cvijović works out the first of these for us.

$$\log(1+z) = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_{1,k}z}{1}$$

Where

$$a_{1,1} = 1, \qquad a_{1,2m} = \frac{m}{2(2m-1)}, \qquad a_{n,2m+1} = \frac{m}{2(2m+1)}$$

Take $z = 1$. Then we should have an approximation for $\log(2)$. We have

$$\{a_{1,k}\}_{k=1}^{11} = \left\{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{3}, \frac{1}{5}, \frac{3}{10}, \frac{3}{14}, \frac{2}{7}, \frac{2}{9}, \frac{5}{18}, \frac{5}{22}\right\}$$

So $\log(2) \approx 0.69314721238833921$. The more precise value is $\log(2) \approx 0.6931471805599453$. For $z = 2$, that is, $\log(3)$, we multiply each of the $a_{1,k}$ by 2. This gives us an approximation $\log(3) \approx 1.0986132368628543$ as compared to the more precise $\log(3) \approx 1.0986122886681098$. More appropriately, let $z = e - 1$. Using this value will, of course, give us an exact value for $\log(1+z) = \log(e) = 1$ to compare to. We get $\log(e) \approx 1.0000003205889758$. We may even let $z = ei - 1$. This gives $\log(ie) \approx 1.0003005411960346 + 1.5698739723830915i$. The exact value of $\log(ie) = 1 + i\frac{\pi}{2}$.

Using *Mathematica* v.6, we calculate the first 6 numerators of $a_{n,k}$ for $1 \leq n \leq 10$ (attached).

$n \setminus k$	1	2	3	4	5	6
1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{3}{10}$
2	1	$\frac{1}{4}$	$\frac{7}{36}$	$\frac{17}{63}$	$\frac{647}{2975}$	$\frac{294777}{1099900}$
3	1	$\frac{1}{8}$	$\frac{37}{216}$	$\frac{217}{999}$	$\frac{30271}{143375}$	$\frac{1566514917}{6568807000}$
4	1	$\frac{1}{16}$	$\frac{175}{1296}$	$\frac{493}{2835}$	$\frac{2081687}{10784375}$	$\frac{10844839670553}{51313584550000}$
5	1	$\frac{1}{32}$	$\frac{781}{7776}$	$\frac{26281}{189783}$	$\frac{10916749081}{64142065625}$	$\frac{76450170047272869}{409861546568230000}$
6	1	$\frac{1}{64}$	$\frac{3367}{46656}$	$\frac{38471}{350649}$	$\frac{1367030273}{9326890625}$	$\frac{862205866965201969}{5259102163258300000}$
7	1	$\frac{1}{128}$	$\frac{14197}{279936}$	$\frac{2685817}{31048839}$	$\frac{370008803686537}{2978948746015625}$	$\frac{142619835103283115759605121 9}{99377593509096374572900000000}$
8	1	$\frac{1}{256}$	$\frac{58975}{1679616}$	$\frac{5253901}{77386995}$	$\frac{148194215774887}{1423937552734375}$	$\frac{48713739188597525561552252793}{389298869226947292093500000000}$
9	1	$\frac{1}{512}$	$\frac{242461}{10077696}$	$\frac{253202761}{4772359863}$	$\frac{1481600973702225343}{17129406985162109375}$	$\frac{3706881030016093368309746961803079}{34104132476517440790161093000000000}$
10	1	$\frac{1}{1024}$	$\frac{989527}{60466176}$	$\frac{219367507}{5311870893}$	$\frac{13770332030069337287}{192711355734365234375}$	$\frac{28418187774129835063167253670879192 67}{302076340799855955779133350900000000000}$

With the above table, we calculate the 6th approximants F_6 for a given n of the continued fraction expansion of the Riemann zeta function $\zeta(n)$. Below is a table of values for $2 \leq n \leq 10$ accompanied by the values found using *Mathematica's* internal command.

n	F_6	*Mathematica*
2	1.6448969002937126	1.6449340668482262
3	1.2020463030724917	1.2020569031595942
4	1.082320277569941	1.082323233711138
5	1.0369270009498681	1.0369277551433698
6	1.0173428854434825	1.017343061984449
7	1.008349239007861	1.0083492773819227
8	1.004077348357326	1.004077356197944
9	1.0020083913041948	1.0020083928260821
10	0.9990395073157157	1.0009945751278178

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