# The Use of Function Series in the Analysis of Random Walks

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# 1 Introduction

An interesting topic in combinatorics involves analyzing the outcome of random walks. Typically, the random walk is represented with a directed graph with vertices specifying states and edges denoting the probability of getting from one state to the other. One may specifically study random walks on trees, in particular random walks on $\mathbb{Z}_+$ where each vertex represents an integer and each move of the random walk only allows going one integer forward or backward.

When the walks are on trees with infinitely many vertices (such as $\mathbb{Z}_+$), the analysis of what happens on a random walk on such trees involves infinite series. Bajunaid, Cohen, Colonna, and Singman [1] explore these random walks on trees and, by investigating these series, discuss a remarkable connection between the walks and an infinite sequence called the Catalan numbers. The paper is written to show deep connections between several topics in mathematics - starting from the random walks and the resulting series, to functions converging to those series and the Catalan numbers that result as the coefficients of the series, to the "figure-eight" sets on which series converge for generalizations of the Catalan numbers.

# 2 Graph Theory

Graph theory studies sets of vertices and edges, and has many ties with combinatorics. In this particular application, graphs are the catalyst for studying random walks: the path of a random walk is represented by a path on the graph.

## 2.1 Graphs

To proceed, several definitions are needed. Namely, we wish to define the notion of path.

**Definition 2.1.** A **graph** is a collection $V$ of points (commonly known as vertices or nodes) and $E$ of edges (or lines) connecting a (possibly empty) subset of them.

**Definition 2.2.** Two distinct vertices $u$ and $v$ are said to be **neighbors** if there exists an edge between them. We denote this relationship by writing $u \sim v$.

**Definition 2.3.** A **path** on a graph is a sequence of vertices $[v_0, v_1, \ldots]$ such that $v_i \sim v_{i+1}$ for $i \geq 0$.

**Definition 2.4.** A **cycle** on a graph is a path $[v_0, v_1, \ldots, v_k]$ such that $k > 0$ and $v_0 = v_k$.

## 2.2 Trees

A tree is a special form of graph. Trees have the nice property that paths which do not repeat edges are unique.

**Definition 2.5.** A **tree** is a graph containing no cycles, such that each vertex is the endpoint of only finitely many edges. We sometimes designate some vertex $e$ to be the **root** of the tree.

**Definition 2.6.** A **geodesic** is a path $[v_0, v_1, \ldots]$ such that $v_{i-1} \neq v_{i+1}$ for $i > 0$. If $u$ and $v$ are vertices of a tree $T$, we denote the unique geodesic $[w_0, w_1, \ldots, w_n]$ with $w_0 = u$ and $w_n = v$ by $[u, v]$. An infinite geodesic is called a **ray**.

### 2.3 Probabilities on Trees

We may also assign a specific number to each ordered pair of vertices in $T$, and treat that as a probability for studying random walks later on:

**Definition 2.7.** We define $p$ as the **nearest-neighbor transition probability** on a tree $T$ by $p(v, u) \geq 0$ if $v$ and $u$ are neighboring vertices, $p(v, u) = 0$ if $v$ and $u$ are not neighbors, and for all vertices $v$, $\sum_{u \sim v} p(v, u) = 1$.

**Definition 2.8.** If $\gamma = [v_0, \ldots, v_n]$ is a finite path, the **probability of** $\gamma$ **relative to p** is

$$p(\gamma) = \prod_{j=1}^{n} p(v_{j-1}, v_j),$$

with $p([v_0]) = 1$.

## 3 Random Walks

The study of random walks is essentially the study of infinite paths on trees with probabilities associated to the edges. Proceeding more formally, the notion of a random walk is discussed next.

### 3.1 Vital Definitions for Random Walks

If $W$ is the set of all infinite paths of a tree $T$, let $W(\gamma)$ denote the set of all paths in $W$ whose initial segment is $\gamma$. For a fixed vertex $v$, let $\mathcal{F}_v = \{W(\gamma) : \gamma \text{ begins with } v\}$. There exists a unique probability measure $P_v$ on $\mathcal{F}_v$ such that $P_v(W(\gamma)) = p(\gamma)$ for each $\gamma$ beginning at $v$.

**Definition 3.1.** The **random walk** on $T$ associated with $p$ is the set of all paths in $W$ together with each $P_v$.

If $\Gamma_{v,w}$ is the set of all finite paths $[v = v_0, \ldots, v_n = w]$ such that $v_i \neq w$ for $0 < j < n$, and $W_{v,w}$ is the set of of all infinite paths $[v = v_0, v_1, \ldots, w, \ldots]$, then $W_{v,w}$ is the disjoint union of all $W(\gamma)$ where $\gamma \in \Gamma_{v,w}$. $W_{v,w}$ belongs to $\mathcal{F}_v$, so we may write $P_v(W_{v,w}) = F(v, w)$. $F$ is the probability that a random walk beginning at $v$ visits $w$ in positive time, and has the property

$$F(v, w) = \sum_{\gamma \in \Gamma_{v,w}} p(\gamma). \tag{1}$$

**Definition 3.2.** A random walk on a tree, $T$, is **recurrent** if $F(v, w) = 1$ for all vertices $v, w \in T$.

**Definition 3.3.** A **transient** random walk is one that is not recurrent.

## 3.2 Some properties of $F$

$F$ exhibits some interesting properties. In particular, the value of $F$ can determine if a walk is transient or recurrent. [1] states the following result:

**Proposition 3.4.** *Let $v$ and $w$ be distinct vertices of a tree $T$, and let $[v_0, \dots, v_n]$ be the geodesic path from $v_0 = v$ to $v_n = w$. Then the following hold:*

*(a)* $F(v, w) = \prod_{k=0}^{n-1} F(v_k, v_{k+1})$;
*(b)* $F(v, v) = \sum_{u \sim v} p(v, u)F(u, v)$;
*(c) If* $v \sim w$, *then* $F(v, w) = p(v, w) + \sum_{u \sim v, u \neq w} p(v, u)F(u, w)$.

**Proposition 3.5.** *If $F(v_0, v_0) = 1$ for some vertex $v_0$ of $T$, then the random walk on $T$ associated with $F$ is recurrent.*

## 3.3 Random Walks on $\mathbb{Z}_+^0$

Random walks on $\mathbb{Z}_+^0 = \mathbb{Z}_+ \cup \{0\}$ lead to some fascinating analysis. Because $\mathbb{Z}_+$ is not by itself a tree, a formal definition is first provided

**Definition 3.6.** A **nearest-neighbor random walk on** $\mathbb{Z}_+^0$ is a random walk on the tree with vertices $\{0, 1, 2, \dots\}$ and having the property that $u \sim v$ if and only if $v = u \pm 1$.

The tree defined above is an example of a homogeneous tree:

**Definition 3.7.** A **homogeneous tree** of degree $n \geq 2$ is a tree with every vertex having exactly $n$ neighbors.

It turns out that we may determine recurrence by the value of $F(1, 0)$ :

**Proposition 3.8.** *A nearest-neighbor random walk on $\mathbb{Z}_+^0$ is recurrent if and only if $F(1, 0) = 1$.*

A subclass of random walks is the "drunkard's walk," named so as to invoke the image of a drunkard moving forward with probability $p$ and backwards with probability $1 - p$, possibly "falling into the lake" by touching the point $0$ :

**Definition 3.9.** A **drunkard's walk** with parameter $p \in (0, 1)$ is a nearest-neighbor random walk on $\mathbb{Z}_+^0$ such that $p(n, n + 1) = p$ and $p(n, n - 1) = 1 - p$ when $n \geq 1$ and $p(0, 1) = 1$.

To analyze if the drunkard falls into the lake, the probability $F(1, 0)$ is calculated.

Let $A_n$ denote the set of finite sequences $[1 = c_0, c_1, \ldots, c_{2n} = 1]$ with $c_i \in \mathbb{Z}_+$, $|c_k - c_{k+1}| = 1$ for $k < 2n$. Let $a_n$ be the cardinality of $A_n$.

Consider a path of length $2n + 1$ with $n$ steps to the right (each having probability $p$) and $n + 1$ steps to the left (each having probability $1 - p$), so that the first vertex on the path is 1 and the last 0. Because these paths can be written as $[1 = c_0, c_1, \ldots, c_{2n} = 1, 0]$, there are $a_n$ such paths. From (1) it follows that

$$F(1, 0) = \sum_{n=0}^{\infty} a_n p^n (1 - p)^{n+1} = (1 - p) \sum_{n=0}^{\infty} a_n [p(1 - p)]^n.$$

It remains to find the coefficients $a_n$. The paper[1] discusses two methods.

## 3.4 A Combinatorial Approach

The notion of a Dyck path is used for this calculation:

**Definition 3.10.** A **Dyck path** of length $k$ is a finite sequence $(a, b_0), (a + 1, b_1), \ldots (a + k, b_k) \in \mathbb{Z} \times \mathbb{Z}$ such that $b_{i+1} = b_i \pm 1$.

Notice that there is a one-to-one correspondence between the paths of the form $[c_0, \ldots, c_{2n}]$ and the respective Dyck paths $(0, c_0), (1, c_1, ), \ldots (2n, c_{2n})$. Hence, $a_n$ is the number of Dyck paths from $(0, 1)$ to $(2n, 1)$ that are restricted to $\mathbb{Z}^0_+ \times Z_+$.

We may also consider the number of such Dyck paths restricted to $\mathbb{Z} \times \mathbb{Z}$ and subtract the number of those that touch the $x-$axis. This latter form of Dyck path corresponds to a Dyck path from $(0, -1)$ to $(2n, 1)$ by reflecting the part from $(0, -1)$ to the point where the path first touches the $x-$axis across the $x-$axis.

As noted before, the number of Dyck paths between $(0, 1)$ to $(2n, 1)$ is the number of ways to move up $n$ times and down $n$ times in $2n$ moves, or $\binom{2n}{n}$. Similarly, the number of Dyck paths from $(0, -1)$ to $(2n, 1)$ is the number of ways to move up $n + 1$ times and down $n - 1$ times, or $\binom{2n}{n-1}$. Hence we have

$$\begin{aligned} a_n = \binom{2n}{n} - \binom{2n}{n-1} &= \binom{2n}{n} - \frac{(2n)!}{(n+1)!(n-1)!} \\ = \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

Notice that $a_n = \frac{1}{n+1}\binom{2n}{n} = C_n$, by definition the $n-$th Catalan number.

## 3.5 An Analytic Approach

First we consider a recurrence approach. First, let $\sigma A_n \to A_{n+1}$ be defined as such:

$$\sigma([c_0, c_1, \ldots, c_{2n}]) = [1, c_0 + 1, c_1 + 1, \ldots, c_{2n} + 1, 1].$$

Notice that $\sigma$ is a one-to-one relationship between $A_n$ and $A_{n+1}$ with image the set of paths in $A_{n+1}$ that revisit 1 only at the last step.

We now define the concatenation $c \cdot d$ for $c = [c_0, \ldots, c_{2n}] \in A_n$ and $d = [d_0, \ldots, d_{2m}] \in A_m$ to be $c \cdot d = [c_0, c_1, \ldots, c_{2n}, d_1, \ldots, d_{2m}] \in A_{n+m}$. A result [1] follows.

**Theorem 3.11.** *The sets $A_n$ are defined inductively as follows: $A_0 = \{[1]\}$ and for each nonnegative integer $n$ $A_{n+1}$ is the disjoint union of the sets of paths $\sigma(A_k) \cdot A_{n-k}$ for $k = 0, 1, \ldots, n$.*

**Example 3.12.** *By the theorem, $A_1 = \sigma(A_0) \cdot A_0 = \{[1,2,1] \cdot [1]\} = \{[1,2,1]\}$. Also, $A_2 = \sigma(A_0) \cdot A_1 \cup \sigma(A_1) \cdot A_0 = \{[1,2,1] \cdot [1,2,1], [1,2,3,2,1][1]\} = \{[1,2,1,2,1], [1,2,3,2,1]\}$.*

*Proof.* Any $c$ in $A_{n+1}$ can be written as $[c_0, c_1, \ldots, c_{2k+2}, c_{2k+3}, \ldots, c_{2n+2}]$, where $c_{2k+2}$ is the first occurrence of 1 after $c_0$ (it is possible that $2k+2 = 2n$.) If $c' = [c_0, \ldots, c_{2k+2}]$ and $c'' = [c_{2k+2}, \ldots, c_{2n+2}] \in A_{n-k}$. But notice that because none of $c_i \in c'$ is 1 except for $c_0, c_{2k+2}$, $c' = \sigma([c_1 - 1, \ldots, c_{2k+1} - 1])$. Notice that $c$ is represented uniquely as $c' \cdot c''$. Thus, any $c$ may be represented as some element of $\sigma(A_k) \cdot A_{n-k}$.

Conversely, any elements of $\sigma(A_k)$ and $A_{n-k}$ can be concatenated to yield an element in $A_{n+1}$. $\square$

It follows that $a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k}$.

Now we consider a power series to determine the value of $F(1,0)$ :

**Theorem 3.13.** *The power series $\sum_{n=0}^{\infty} a_n z^n$ converges when $|z| \leq 1/4$, and its sum $g(z)$ is given by*

$$g(z) = \begin{cases} \frac{1-\sqrt{1-4z}}{2z} & \text{if } 0 < z \leq 1/4, \\ 1 & \text{if } z = 0, \end{cases}$$

*where $\sqrt{w}$ denotes the principal branch of the square root of $w$.*

*Proof.* Let $G = \frac{1-\sqrt{1-4z}}{2z}$, defined to be 1 at the origin. $G$ is analytic in the disk $D = \{z : |z| < 1/4\}$, continuous on $\bar{D}$, and $G(z) \to 1$ as $z \to 0$. 0 Let $\{b_n\}$ be the sequence of Taylor coefficients of $G$ at the origin. Then, $b_0 = 1$.

Notice that $G$ satisfies the functional relation

$$G(z)^2 = (G(z) - 1)/z :$$

$$\left(\frac{1-\sqrt{1-4z}}{2z}\right)^2 = \frac{1 - 2\sqrt{1-4z} + 1 - 4z}{4z^2} = \frac{1 - \sqrt{1-4z} - 2z}{2z^2}$$

$$= \frac{\frac{1-\sqrt{1-4z}}{2z} - 1}{z} = \frac{G(z) - 1}{z}.$$

Recalling that $f(z)g(z) = \sum_{k=0}^{\infty} c_k z^k$ with $c_k = a_k b_0 + a_{k-1} b_1 + \cdots + a_1 b_{k-1} + a_0 b_k$, we have

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} b_k b_{n-k}\right) z^n = \sum_{n=0}^{\infty} b_n z^n \sum_{n=0}^{\infty} b_n z^n = \sum_{n=1}^{\infty} b_n z^{n-1} = \sum_{n=0}^{\infty} b_{n+1} z^n.$$

From this it follows

$$b_{n+1} = \sum_{k=0}^{n} b_k b_{n-k}.$$

Since $a_0 = b_0$ and both sequences obey the same recurrence relation, we conclude that $\{a_k\} = \{b_k\}$.

Thus, $g(z) = G(z)$ for $z$ in $D$. The binomial expansion gives

$$\sqrt{1-4z} = 1 - 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} z^n,$$

so

$$g(z) = \sum_{n=1}^{\infty} \binom{2n-2}{n-1} z^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} z^n = \sum_{n=0}^{\infty} a_n z^n. \tag{2}$$

By Stirling's formula, $n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$. Then, $a_n \sim \pi^{-1/2} 4^n n^{-3/2}$,

so the series in (2) converges when $z = 1/4$, which implies that it converges absolutely and uniformly on $D$. Hence $g$ is also analytic in $D$ and continuous on $\bar{D}$. This ensures that $g = G$. $\square$

Recall that $F(1,0) = (1-p)\sum_{n=0}^{\infty} a_n [p(1-p)]^n = (1-p)g(p(1-p))$. But $\sqrt{1-4p(1-p)} = |2p-1|$ when $0 \le p \le 1$, and thus

$$F(1,0) = \begin{cases} 1 & \text{if } 0 \le p \le 1/2, \\ \frac{1-p}{p} & \text{if } 1/2 \le p \le 1. \end{cases}$$

# 4 Generalized Results

There are more cases to consider. Foremost, the $n$-th generalized Catalan number with parameter $k$ (note that if $k = 2$ we regain the original Catalan number sequence) is written as

$$a_{n,k} = \frac{1}{(k-1)n+1}\binom{kn}{n}$$

and appears in the following interesting example:

Consider the random walk on $\mathbb{Z}_+^0$ with transition probabilities $p(0,1) = 1$, $p(1,2) = 1$, $p(1,0) = 1-p$, $p(n,n+1) = q$ and $p(n,n-1) = 1-q$ for $n > 1$. It turns out [1] that taking $p = q$ yields

$$F(2,1) = \begin{cases} 1 & \text{if } 0 \le q \le 1/2, \\ \frac{1-q}{1} & \text{if } 1/2 \le q \le 1. \end{cases}$$

And from the relation $F(1,0) = (1-p) + pF(2,1)F(1,0)$,

$$F(1,0) = \begin{cases} 1 & \text{if } 0 \le q \le 1/2, \\ \frac{(1-p)q}{q-p+pq} & \text{if } 1/2 \le q \le 1. \end{cases}$$

[1] alludes to the following theorem, the proof of which is filled in below:

**Theorem 4.1.** *For $k \ge 2$, the series $(1-z)\sum_{n=0}^{\infty} a_{n,k}[z(1-z)^{k-1}]^n$ converges on $R = \{z : |z(1-z)^{k-1}| \le (k-1)^{k-1}/k^k\}$ and its sum $f(z)$ is given by*

$$f(z) = \begin{cases} 1 & \text{if } z \in D_0, \\ h(z) & \text{if } z \in D_1, \end{cases}$$

*where $h(z)$ is an analytic function satisfying $h(z)^{k-1} + \cdots + h(z) + 1 = \frac{1}{z}$; and $D_0$ and $D_1$ are the two disjoint bounded regions in the complement of $\partial R : D_0$ is the one containing the origin, $D_1$ is not.*

*Proof.* First we consider a specific real-valued case. Let $F(1,0)$ be determined as above. Let $k$ be any integer bigger than 1, and fix $q \ge 1/2$. Choose $p$ to be the unique solution to the equation

$$\frac{(1-p)q}{q-p+pq} = \sqrt[k-1]{\frac{1-q}{q}}.$$

In this case, $F(2,1) = \frac{1-q}{q} = F(1,0)^{k-1}$ and thus $F$ is a decreasing function of $q$. Write

$$s = \sqrt[k-1]{\frac{1-q}{q}}$$

and solve for $p$ to get

$$p = \frac{q(1-s)}{q(1+s)-s}.$$

When $q = 1$, $s = 0$ and $p = 1$. As $q \to 1/2$ from the right, $s \to 1$ and $p \to 1/k$ by l'Hôpital's rule. Thus this relationship gives a one-to-one correspondence $q \mapsto p$ between $(1/2, 1)$ and $(1/k, 1)$.

It follows that if $q > 1/2$, $p > 1/k$, which implies that the random walk is not recurrent, and thus $F(1,0) < 1$ by (3.8).

We saw that $F(1,0)$ satisfies $F(1,0)^{k-1} = F(2,1)$, or equivalently (from the relation stated above), $F(1,0) = 1 - p + pF(1,0)^k$.

Let $f(p) = F(1,0)$ where $p$ and $q$ are related as above. Then $f(p)$ obeys

$$f(p) = 1 - p + pf(p)^k.$$

If $f(p) \neq 1$, by the well-known formula for the partial sum of a geometric series,

$$f(p)^{k-1} + \cdots + f(p) + 1 = \frac{1-f(p)^k}{1-f(p)}$$

and from the functional relation restriction, this is equal to

$$\frac{1-f(p)^k}{1-(1-p+pf(p)^k} = \frac{1}{p}.$$

If $f(p)=1$, then $f(p)^{k-1}+\cdots+f(p)+1=k$, and otherwise the sum is less than $k$. From this, if we assume that $0 \le p \le 1/k$ and $f(p) \neq 1$, we reach the contradiction that the sum is bigger than $k$, so it follows that $f(p)=1$ when $0 \le p \le 1/k$. Since $F(1,0)=f(p)<1$ when $p>1/k$, we have

$$f(p) = \begin{cases} 1 & \text{if } p \in [0,1/k], \\ y(p) & \text{if } p \in [1/k,1], \end{cases}$$

where $y(p)$ satisfies $y(p)^{k-1}+\cdots+y(p)+1=\frac{1}{p}$.

Now we consider the complex case, by calling on the Lagrange-Bürmann inversion formula, which states[1] that for a smooth function $\varphi$, if $\zeta$ satisfies $\zeta = w+\alpha\varphi(\zeta)$, then

$$\zeta = w + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!}[\varphi^{(n-1)}(w)^n]$$

for $\alpha$ sufficiently small.

Notice that if $\varphi=\zeta^k$, $\zeta=f(p)$, $w=1-p$, and $\alpha=p$, then we are examining precisely the functional relation $f(p)=1-p+pf(p)^k$ and the Lagrange-Bürmann inversion formula yields

$$f(p) = 1-p+\sum_{n=1}^{\infty}\frac{p^n}{n!}\frac{(kn)!}{(kn-n+1)!}(1-p)^{kn-n+1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(k-1)n+1}\binom{kn}{n}p^n(1-p)^{(k-1)n+1}$$

$$= (1-p)\sum_{n=0}^{\infty} a_{n,k}[p(1-p)^{k-1}]^n,$$

and replacing $p$ with the complex variable $z$ yields the series being investigated,

$$(1-z)\sum_{n=0}^{\infty} a_{n,k}[z(1-z)^{k-1}]^n.$$

The convergence of this series can be determined by using the root test.

$$\lim_{n\to\infty}\left|\sqrt[n]{a_{n,k}[z(1-z)^{k-1}]^n}\right| = \lim_{n\to\infty}\sqrt[n]{a_{n,k}}\left|[z(1-z)^{k-1}]\right|.$$

But by Stirling's formula,

$$a_{n,k} \sim \sqrt{\frac{k}{2\pi n^3(k-1)^3}}\left(\frac{k^k}{(k-1)^{k-1}}\right)^n,$$

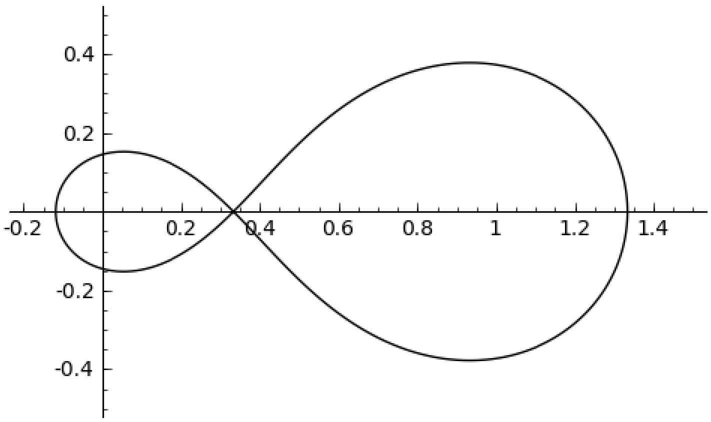


Figure 1: $\partial R$ for $k = 4$, rendered in SAGE

so $\sqrt[n]{a_{n,k}} \to \frac{k^k}{(k-1)^{k-1}}$. Thus the series converges for $z$ satisfying

$$\frac{k^k}{(k-1)^{k-1}} |[z(1-z)^{k-1}]^n| = 1,$$

or in the region

$$R = \{z : |z(1-z)^{k-1}| \leq (k-1)^{k-1}/k^k\},$$

as desired.

The boundary of this region is a figure-eight shape, pictured above for $k = 4$.

The complement of this boundary is a union of three disjoint regions, $D_0$, $D_1$, and $D_2$, with $D_2$ the unbounded one and $D_0$ the one containing the origin.

To gain uniform convergence, consider the series

$$g(w) = \sum_{n=0}^{\infty} a_{n,k} w^n.$$

The power series $\sum a_{n,k} w^n$ converges by the root test, as above, for $w \leq (k-1)^{k-1}/k^k$ and thus converges uniformly for $w \leq (k-1)^{k-1}/k^k - \varepsilon$, $\epsilon > 0$. $g(w)$ is therefore analytic for this disk.

Setting $w = z(1-z)^{k-1}$, which is an analytic function of $z$ for $k \geq 2$, so $f(z) = (1-z)g(w(z)$ is analytic in closed subsets of $R$ not containing points in $\partial R$.

Because $f(z) = 1$ on the real line in $D_0$ and $f(z) = h(z)$ on the real line in $D_1$, and because $f(z)$ must be analytic, the desired conclusion is reached:

$$f(z) = \begin{cases} 1 & \text{if } z \in D_0, \\ h(z) & \text{if } z \in D_1, \end{cases}$$

with $h(z)$ an analytic function satisfying $h(z)^{k-1} + \cdots + h(z) + 1 = \frac{1}{z}$. $\square$

Notice that the $k = 2$ case is consistent with (3.13) above, as the curve $\{|z(1-z)| = 1/4\}$ intersects itself at the point $z = 1/2$, so the function

$$f(z) = \begin{cases} 1 & \text{if } z \in D_0, \\ h(z) & \text{if } z \in D_1, \end{cases}$$

with $h(z) = \frac{1-z}{z}$ (which is analytic in $D_1$) clearly satisfying $h(z) + 1 = \frac{1}{z}$ :

$$\frac{1-z}{z} + 1 = \frac{1-z+z}{z} = \frac{1}{z}$$

obeys the theorem.

It turns out that for the case $k = 3$,

$$h(z) = \frac{\sqrt{\frac{4}{z} - 3} - 1}{2},$$

which is analytic in $D_1$ (the point $z = 4/3$ is in $\partial R$ and branching does not occur in $R$) and satisfies $h(z)^2 + h(z) + 1 = \frac{1}{z}$ :

$$\left(\frac{\sqrt{\frac{4}{z} - 3} - 1}{2}\right)^2 + \frac{\sqrt{\frac{4}{z} - 3} - 1}{2} + 1 = \frac{\frac{2}{z} - 1 - \sqrt{\frac{4}{z} - 3} + \sqrt{\frac{4}{z} - 3} - 1 + 2}{2} = \frac{1}{z}.$$

An explicit formula for $h(z)$ is not given, and the cases for $k > 3$ get more and more complex.

# 5 Further Study

The authors [1] aim to show the inherent beauty that comes from connecting seemingly unrelated topics of mathematics. Random walks and function series complement one another in analysis, and some interesting sequences (the generalized Catalan numbers) and curves (the boundary of $R$ above) appear, as well. Naturally, the desire to discover more about these topics arises.

The paper [1] discusses even more general random walks. An example is a walk treating the odd and even numbers as separate tracks, with probability of moving from an odd number to the next odd number or an even number to the next even number as $p$, and respectively the probability of moving from a number to the number preceding it as $1-p$. This walk ties in with the generalized Catalan numbers $a_{n,3}$.

Another study was of a homogeneous tree of degree 3, with probabilities defined by a parameter $p$. This tree may be transient or recurrent depending on the parameter.

In addition, using $G(w) = \sum_{n=0}^{\infty} b_n w^n$ with $b_n$ defined by the recurrence relation, $b_0 = 1$ and $b_{n+1} = \sum_{i_1+\cdots+1_k=n} a_{i_1} \cdots a_{i_k}$, it can be shown [1] that the generalized Catalan numbers obey the recurrence relation

$$a_{n+1,k} = \sum_{i_1+\cdots+i_k=n} a_{i_1,n} \cdots a_{i_k,n}.$$

This follows from the derivation of the relation $G(w)^k = \frac{1}{w}(G(w)-1)$ and results seen in the proof of (4.1) above.

The Catalan numbers also appear in many other topics in combinatorics.

A particularly interesting example was demonstrated by Stanley and Chapman[2], and relates to series and winding numbers:

**Theorem 5.1.** *Let* $f(n) = \sum_P (-1)^{w(P)}$*, where* $P$ *ranges over all lattice paths in the plane with* $2n$ *steps, starting and ending at the origin, with steps* $(1,0)$*,* $(0,1)$*,* $(-1,0)$*,* $(0,-1)$*, and where* $w(P)$ *denotes the winding number of* $P$ *with respect to the point* $(1/2, 1/2)$*. It follows that* $f(n) = 4^n C_n$*, where* $C_n$ *is the* $n-$*th Catalan number.*

Other random walks are also intriguing. Busquet-Méelou[3] shows that the number of Lattice walks (with moves North, East, South, West) of path length $2n$ in the quarter-plane starting and ending at the origin (this corresponds to Random Walks on homogeneous trees of degree 4) is

$$\frac{1}{(2n+1)(2n+2)} \binom{2n+2}{n+1}^2.$$

## References


[1] I. Bajunaid, J. M. Cohen, F. Colonna, and D. Singman, "Function Series, Catalan Numbers, and Random Walks on Trees", *The American Mathematical Monthly*, 112.9 (2005), 765-785.

[2] B. Bousquet-Mélou. "Walks in the Quarter Plane: Kreweras' Algebraic Model", *The Annals of Applied Probability*, 15.2 (2005), 1451-1491.

[3] R. P. Stanley and R. Chapman. "An Unexpected Appearance of the Catalan Numbers", *The American Mathematical Monthly*, 110.7 (2003), 640-642.