Geometry, Physics, and Harmonic Functions

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1 Introduction

Mathematics is a language of rigor and clarity. A plethora of symbols and words litter every student's math textbooks, creating arguments and generalizations. Pictures, on the other hand, are more commonly left out of the texts. There are many reasons for this: how is one to visualize 5-space? How can one prove a fact with a picture? Its cliche, but things really aren't always how they appear. Contrary to the rejection of visualization by the modern math community, Tristan Needham has produced an excellent book solely devoted to the visual side of one of the most beautiful topics in mathematics: complex analysis [3]. Another paper by the same author, [2], published in Mathematics Magazine, details a simple geometric solution to an algebraic problem. It is this paper that we will focus on. In The Geometry of Harmonic Functions, Needham takes a physical problem and its solution, generalizes it, and gives the reader multiple geometric interpretations. It is written with the assumption that the reader is familiar with complex analysis and hyperbolic geometry [1] (the latter will be briefly reviewed later). The fundamental concern of the paper is simple: how can we express values of a harmonic function in a region if we only know the values of the function on the boundary of that region? Before we can write down an equation to solve our problem, we must (as did the people who first discovered these facts) motivate the derivation of these formulae with a physical situation.

2 Physical Motivation of Dirichlet's Problem

Consider a thermally insulated sheet of metal. Heat can flow within the sheet, but will not leak out from above or below. At certain points, add heat at a consistent rate. At other points, subtract heat in the same way. These points are called sources and sinks, respectively. Naturally, heat will flow from the sources to the sinks. For a short time, the temperature at an arbitrary point in the sheet will vary over time. However, after a while, the heat flow (source to sink) will fall into a pattern. That is, if we consider the sheet of metal to be the complex plane, the temperature at a point z will have a constant value, $T(z)$. Elementary physics tells us that energy is conserved (heat put in at sources equals heat removed from sinks) and that in the steady state, $T(z)$ is harmonic. In other words, away from sources and sinks, T satisfies Laplaces equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Now consider a circle C, radius R, centered at the origin. Suppose that there are no sources or sinks in the interior of C. Hopefully, it seems likely to the reader (or at least possible) that $T(a)$ can be expressed in terms of $T(C)$.

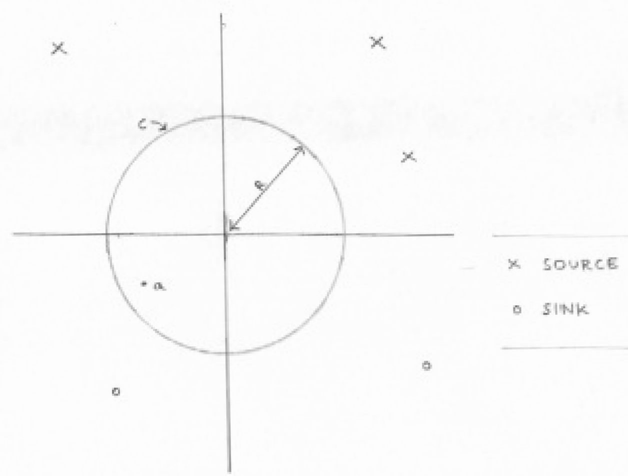


FIGURE #1

In 1820, Simon-Denis Poisson solved for $T(a)$ in terms of $T(C)$. By parameterizing $z = Re^{i\theta}$ and measuring temperature around ∂C, we can express the temperature at each boundary point by talking about $T(\theta)$. Now, Poisson's formula is

$$T(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_a(z) T(\theta) d\theta$$

where $P_a(z)$ is the Poisson Kernel, given by

$$\frac{R^2 - \|a\|^2}{\|z - a\|^2}$$

The exact value of the Poisson Kernel is not as important as the fact that it diminishes as the square of the distance between a and each point on C increases.

That makes sense: if you're in the kitchen with the oven open and the icebox open, and you're standing close to the oven and far from the icebox, youre going to feel the heat from the oven more than the cold air from the icebox. In terms of the figure below, $T(a)$ is more influenced by $T(b)$ than $T(c)$.

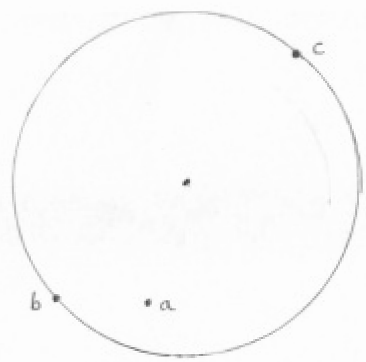


FIGURE #2

This formula for T(a) was an important result that drew attention (and controversy) from a veritable all-star team of mathematicians of that day: Riemann, Weierstrass, Schwarz, Klein, Poincare, and Hilbert. Dirichlet's problem is slightly trickier than the case mentioned before. It asks that given arbitrary (but piecewise continuous) boundary values on a simply connected region, does a harmonic function exist in that region that takes on the arbitrary values as the boundary is approached? Schwarz showed the existence of a solution in the case of the disk, and that it turned out to be (1). Basically, he showed that given piecewise continuous $T(\theta)$, we can construct $T(a)$ by Poisson's formula. Then he showed that $T(a)$ is always harmonic and that $T(a)$ approaches $T(\theta)$ as a approaches $Re^{i\theta}$.

3 Basics of Hyperbolic Geometry

$$\mathbb{H} = \{z \in \mathbb{C} : Im(z) \geq 0\}$$

Definition 1 *A* **hyperbolic line** *is either:*
a) a ray that is perpendicular to the real axis
b) the top half of a circle with center on the real axis

$\Longrightarrow$ Two distinct points p, q define a unique hyperbolic line.

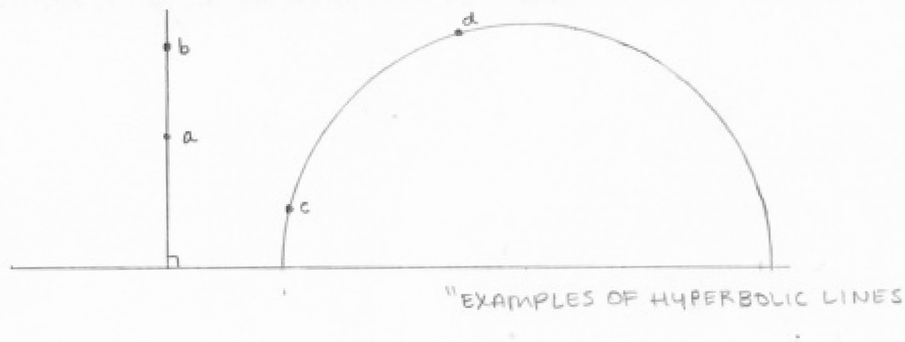


FIGURE #3

Definition 2 *Two hyperbolic lines are* **parallel** *if they are disjoint.*

Definition 3 *For piecewise differentiable* γ: $[a,b] \to \mathbb{H}$,

$$\textbf{length}_{\mathbb{H}}(\gamma) = \int_\gamma \frac{1}{Im(z)}|dz|$$

$$= \int_\gamma \frac{1}{Im(\gamma(t))}|\gamma'(t)|dt$$

Definition 4 *A* **metric** *on a set* $\mathcal{S}$ *is a function* d: $\mathcal{S} \times \mathcal{S} \to \mathbb{R}$ *that satisfies:*
a) $d(x,y) > 0 \;\forall\; x,y \in \mathcal{S}$ *where* $x \neq y$.
b) $d(x,y) = 0 \;\;,\;\; x = y$
c) $d(x,y) = d(y,x)$
d) $d(x,z) \leq d(x,y) + d(y,z)$

Note that part d) is the useful triangle inequality. If d is a metric for $\mathcal{S}$, then we say ($\mathcal{S}$, d) is a **metric space**. Now we will construct a metric for $\mathbb{H}$. Consider two points x,$y \in \mathbb{H}$. Let $\Gamma(x,y)$ be the set of all piecewise differentiable paths $\gamma_k : [a,b] \to \mathbb{H}$ such that $\gamma_k(a) = x$ and $\gamma_k(b) = y$ $\Gamma(x,y)$ is clearly nonempty by the postulate that there exists a line joining every two points. Now consider the function $d_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ given explicitly by

$$d_{\mathbb{H}}(x,y) = \inf\{length_{\mathbb{H}}(\gamma_k) : \gamma_k \in \Gamma(x,y)\}$$

$$= length(\gamma^*)$$

where γ^* is the particular γ_k that happens to have the smallest length.

Theorem 1 *(*$\mathbb{H}$, $d_{\mathbb{H}}$*) is a metric space.*

In our half-plane model of the hyperbolic geometry, γ^* will be a parametrization of the hyperbolic line segment joining x and y. Like Euclidian lines, hyperbolic lines will satisfy the four requirements listed in Definition 4.

4 Schwarz's Geometric Interpretation

Suppose we measure $T(0)$. If half of C were at unit temperature and half were at zero temperature, we would expect $T(0)$ to equal $1/2$. Further division of C into smaller equal arcs would lead us to expect $T(0)$ to be an average of all the temperature values on C. In general, this is Gausss Mean Value Theorem for harmonic functions:

$$T(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T(\theta) d\theta$$

This can be proved using the Cauchy Integral Formula. If $h(z)$ is a conformal (analytic) map, then $T(h(z))$ is automatically harmonic. From here, we will let $w = h(z)$. Suppose h maps the disk to itself. If z lies on C then so does $h(z)$. We have already measured the temperature at every point on C, so we know $T(h(z))$. Knowing the values of $T(h(z))$, we can now compute (2) and get

$$T(h(0)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T(h(\theta)) d\theta$$

Where the averaging is still taking place with respect to the angle of z, not its image under h. Now, we need to find an h that maps 0 to a, and we will be done constructing a function that allows us to find $T(a)$ by averaging the image of the boundary values. It is at this point that Needham constructs a second circle K, intersecting C at right angles, and a line L that connects 0 and a.

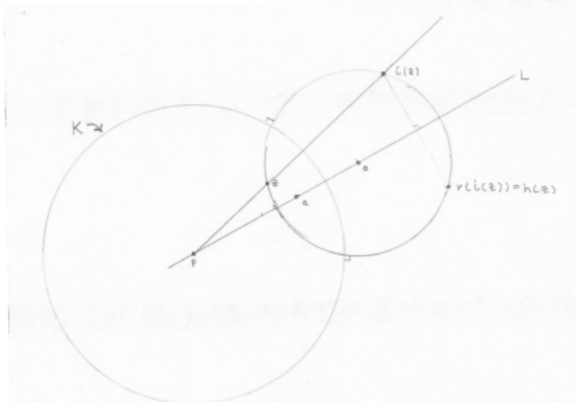


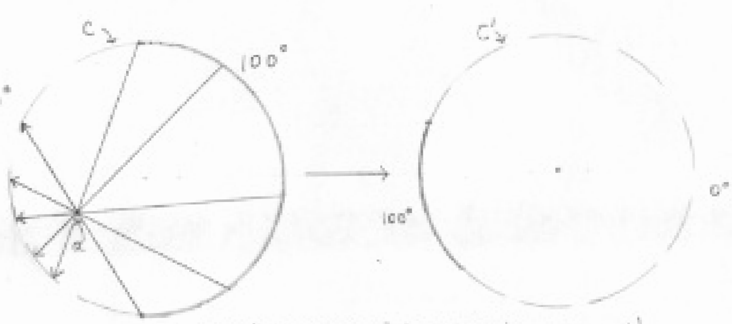
FIGURE # 4: h(z)

It turns out that h, geometrically speaking, is equivalent to inversion in K, and then reflection in L. Inversion of a point in the circle K ($i_K(z)$) and reflection of a point across the line L ($r_L(z)$) are both analytic (conformal) functions, so a composition of them will also be conformal. The intersection of K and L (two orthogonal hyperbolic lines) is the midpoint of the hyperbolic line segment connecting 0 and a. Inversion in K is equivalent to hyperbolic reflection in K. As in Euclidian geometry, reflection across two intersecting lines equates to rotation about their intersection point through two times the angle of their intersection. Therefore, since K and L are orthogonal, h is also a rotation of the hyperbolic plane by angle about the midpoint of $0a$, which explains why $h(0) = a$. In summary,

$$h = r_L \circ i_K$$

In a surprising turn of events, we need not invert, reflect, or use K at all. Actually, we can project each boundary point z through a and take the average of the new set of boundary values. In other words, Schwarz's result can be restated as:

To find the temperature at a, transplant each temperature on C to the point directly opposite it as seen from a, then take the average of the new temperature distribution on C



$T(a)$ = mean of temperatures on C'.
$= \frac{1}{2\pi} \int_{C'} T(z)\,dz\,,\ z \in C'$

FIGURE # 5: OUR RESULT

Figure 6 illustrates the elegant simplicity of our result. It affirms our intuition, and meets our demands. According to Needham, this geometric result is not found in any textbooks on the subject. This is a shame, since this is mathematically deep yet simple enough to be understood by a wide variety of students.

5 Conclusion

Clearly, this cute geometric result happens out of ideal circumstances. It offers no description of how to find T(a) on a non-circular region, and does not generally solve Dirichlet's Problem. However, it is a very simple and interesting geometric application of a seemingly uninteresting differential equation.

References

[1] J. W. Anderson, *Hyperbolic Geometry*, Springer, 1999.

[2] T. Needham, *The Geometry of Harmonic Functions*, Mathematics Magazine, 1994.

[3] T. Needham, *Visual Complex Analysis*, Oxford University Press, 1999.