THE LORENTZ TRANSFORMATION GROUP

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Contents

1. Introduction

The study of special relativity gives rise to the Lorentz transformation, which preserves the inner product in Minkowski space. It is very important within the theory of special relativity where we consider the Minkowski space $M(\Re, \Re^3)$, but there are more general Minkowski spaces which are of interest. In particular we will follow the work in Prof Ungar's paper entitled "The Abstract Lorentz Transformation Group"[5] and examine $M(\Re, V_\infty)$ space, where V_∞ is an arbitrary positive definite inner product space. Using basic results from theory of groups and objects called dyads, we are able to represent any Lorentz transformation by a pure Lorentz transform preceeded by a Thomas gyration, with pure Lorentz transformations can be defined by dyadics. This formulation allows consideration of the Lorentz transformation in very general Minkowski spaces. We will then focus on $\Re^3$, where the Lorentz transformation are typically represented as a 4×4 matrix. There is a simpler representation where the Lorentz transformation and Minkowski spacetime coordinates of $M(\Re, \Re^3)$ are encoded into 2×2 matrices called spinors. We will conclude this paper by considering a certain subgroup of the Lorentz transformation acting in $M(\Re, \Re^3)$ space, the abberation subgroup. There exists a representation of the abberation transformation as a linear fractional transformation which has a simple physical interpretation in terms of the well-known "headlight" effect.

2. Theory of Groups

2.1. **Isomorphisms and Homomorphisms.**

We call two sets S and T isomorphic with respect to an algebraic operation (which for our purposes we will call a product and denote by $ab = c$) if there is a one-to-one correspondence between their elements and the following holds: Given elements $a,b \in S$ corresponding to $x,y \in T$, if

$$ab = c, xy = z \tag{2.1}$$

then $c \in S$ corresponds to $z \in T$. An isomorphism between S and T is denoted by $S \cong T$. Two general properties of isomorphisms are

1. Symmetry, that is given $S \cong T$, it follows that $T \cong S$;
2. Transistivity, that is given $S \cong T$ and $T \cong R$, it follows that $S \cong R$.

Isomorphic sets generally differ from one another in the nature of their elements and in the operation defined on the two different sets. However, the properties of the operations are indistinquishable. That is, whatever can be proven about an operation in a given set S without reference to the properties of the elements of S is necessarily true for an operation in T which is isomorphic with S.

If we omit the requirement that two isomorphic sets S and T have a one-to-one correspondence between their elements, we can obtain a generalization of the concept called a homomorphism. More precisely, a mapping $\phi : S \to T$ is called a homomorphism if for any $a,b \in S$ it follows from

$$\phi a = c \qquad \phi b = d \tag{2.2}$$

that

$$\phi(ab) = cd \tag{2.3}$$

where $c,d \in T$, and we then call the set T a homomorphic image of S.

If T is a homomorphic image of S, the validity of the associative and commutative laws in S entail the validity of the corresponding laws in T. Additionally, if S has a unit element, e_S, then its image must be the unit element in T. The converse statements do not follow.[2]

It is possible to obtain all possible homomorphic images of a given set S with one algebraic operation. To accomplish this, we suppose there is a given partition P of S into disjoint subsets. We call call these subsets and denote them by $A\,B\,C\,\ldots$. The partition P is called regular if given $a_1\,a_2 \in A$ and $b_1\,b_2 \in B$, it follows that $a_1 b_1$ and $a_2 b_2$ both lie in the same class C. Thus the class C is completely determined by the classes A, B. Moreover, we can call C the product of A and B and we have an algebraic operation defined on the set F of all classes of the regular partition P. We call the set F found in this manner the factor set of S with respect to P. Notice that S can be mapped homomorphically to F, and is called the natural homomorphism. We now come to the following theorem:

Theorem 1. *If T is an arbitrary homomorphic image of S, and ϕ a homomorphic mapping of S onto T. then there exists a regular partition P of S into disjoint classes such that T is isomorphic to the factor set F with respect to P. Additionally, there exists an isomorphic mapping ψ of T onto F such that performing the mappings ψ and ϕ in succession is the natural homomorphism of S onto F.*

Proof. Consider all the elements whose images under the mapping ϕ coincide and consider them as a single class. In this way, we obtain a regular partition P of S into disjoint classes. That is, if a_1 and a_2 lie in class A and b_1 and b_2 lie in class B, we have

$$\phi a_1 = \phi a_2 = \alpha, \tag{2.4}$$

and

$$\phi a_1 = \phi a_2 = \beta. \tag{2.5}$$

where $\alpha\,\beta \in T$. By the holomorphism of ϕ, we have

$$\phi(a_1 b_1) = \phi(a_2 b_2) = \alpha\beta \tag{2.6}$$

which means that $a_1 b_1$ and $a_2 b_2$ lie in the same class C. Thus we obtain a factor set F, and we now have a one-to-one correspondence ϕ between all of the elements of T and all the elements of F. It remains to to associate each element of T with the clas sconsisting of all the originals of this element. Notice that ϕ is an isomorphism: If the elements α and β are linked with the classes A and B, respecitively, and if the elements a, b are chosen from these classes, then we find that AB is a class which contains ab. But

$$\phi(ab) = (\phi a)(\phi b) = \alpha\beta \tag{2.7}$$

which means that $\alpha\beta$ is associated to the class AB under the mapping ϕ. Now choose some $a \in S$ and let

$$\phi a = \alpha \qquad \psi\alpha = A. \tag{2.8}$$

Notice that the element a is contained in A. This implies that performing the mappings ψ and ϕ in succession is the natural homomorphism of S onto F, which is what we wanted to show.

□

2.2. **Definition of a Group and Basic Results.** We call a non-empty set G with one algebraic operation a group if:

(1) The operation is associative;
(2) The inverse operation can be performed in G.

Note that the operation defined on G does not need to be commutative. As we will see in section 3, the Lorentz Group does not have a commutative product.

We now quote without proof a few basic results from reference [2].

Lemma 1. *Given a group G, there exists a unique element $e \in G$ called the unit element with the property that $ae = ea = a$ for any $a \in G$.*

Lemma 2. *For any a in a group G, there exists a unique element a^{-1} called the inverse of a such that $aa^{-1} = a^{-1}a = e$.*

Notice that the unit element is its own inverse, and that the inverse of a^{-1} is a.

Lemma 3. *If a group G is mapped homomorphically onto a set H with one operation, then H is also a group.*

Theorem 1 can now be expressed as the following homomorphism theorem for groups:

Theorem 2. *(The Homomorphism Theorem) If ϕ is a homomorphic mapping of a group G onto a group H, then there exists a regular partition P of G such that H can be mapped isomoprhically onto the factor group F of G with respect to this partition. Additionally, the isomorphism ψ of H onto F can be chosen such that performing the mappings ϕ and ψ in succession is the natural homomorphism of G onto F.*

Notice that if a homomorphism ϕ of a group G onto a group H carries the element $a \in G$ into the element $\alpha \in H$, that is,

$$\phi a = \alpha, \tag{2.9}$$

then the image of the inverse of a is the inverse of α.[2]

For an element a in a group G, we can define the powers of a, a^n as a multiplied by itself n times for n positive, $a^0 = e$ where e is the unit element of G, and $a^{-n} = (a^n)^{-1}$ or $a^{-n} = (a^{-1})^n$. It is easily verified that the two definitions for negative powers of a are equivalent, that is $(a^n)^{-1} = (a^{-1})^n$.[2]

2.3. **Subgroups.** We call a non-empty subset H of a group G a subgroup if H itself is a group under the operation defined in G. We can determine that a subset H of G is a subgroup by verifying that[2]

(1) Given a,$b \in H$, $ab \in H$;
(2) Given $a \in H$, $a^{-1} \in H$.

The associative law must hold for the product in H as it holds in G, and similarly it follows from the two requirements that the unit element of G is in H. We call a subgroup H of G proper if H is distinct from G. Notice that under a homomorphic mapping ϕ of a group G onto a group F_G, a subgroup H of G is mapped onto a subset F_H of F_G. Since ψ is homomorphic for H, we know by the homomorphism theorem that F_H is a group with respect to the operation defined in F_G. That is, a given homomorphic mapping in G induces homomorphic mappings in all of its subgroups.

Suppose we have two groups, G and H. If H is a homomorphism to a subgroup A of G, we say that the group H can be embedded in G. In particular, if $A = G$, we say that H maps onto G.

Notice that if A and B are subgroups of G, then $A \bigcap B \neq \emptyset$ because both A and B must include the unit element of G. In fact, $C = A \bigcap B$ is itself a subgroup. To see this, suppose a,$b \in C$. Then $ab \in A$ and $ab \in B$ so that $ab \in C$ and similarly for the inverses of a and b. This can be extended further to show that the intersection of any set of subgroups of G is a subgroup of G. In particular, the intersection of all subgroups of G is the unit subgroup E which consists solely of the unit element $e \in G$.

Now choose an arbitrary non-empty subset M of a group G, and consider the intersection of all the subgroups of G for which M is a subset. The set formed in this way is itself a subgroup, and is called the subgroup generated by M, which denoted by $\{M\}$. For all $a \in M$, $\{M\}$ contains all powers of a and by extension all possible products of any finite number of these powers. However, all the elements of G which are the product of a finite number of powers of the elements of M form a subgroup of G which contains M. That is, we have shown that the subgroup generated by M consists of all elements in G that are the product of a finite number of powers of elements in M.

In particular, we can consider M to be the set consisting of all the elements of a group G which lie in at least one of a given set of subgroups S of G. Then we note that $\{M\}$ is the smallest subgroup of G which contains S. We call this subgroup the subgroup generated by the given subsets and denote it by $\{A_1 \ldots A_N\}$. The subgroup generated in this way consists of all the elements of G which are the products of a finite number of elements in the given subgroups.

We call a subset M in a group G which generates the subgroup $\{M\}$ which is G itself a system of generators. Such a system of generators exists for every group; it is sufficient to take M to be G. It also follows that M is a system of gnerators of a group G if and only if every element of G is a product of a finite number of powers of elements of M. A system of generators M is called irreducible if there exists no proper subsystem of M which is a system of generators for G.

2.4. **Endomorphisms and Automorphisms.** An endomorphism of a group G is a homomorphic mapping of G onto one of its subgroups. An automorphism is an isomorphic mapping of G onto itself. There exists an endomorphism for every group called the null endomorphism which maps every element of G to the unit element. If H is a subgroup of G, then an endomorphism (resp. automorphism) induces a homomorphic (resp. isomorphic) mapping in H. Thus the image of H under an endomorphism χ is also a subgroup of G which we will denote as H_χ.

Suppose a group is given by a system of generators $M = \{a_n\}$. Then any endomorphism χ of G is completely determined by the images of all the generators. In particular, an automorphism of G maps the set of all the elements of M to another set of generators for G.

We can define multiplication of endomorphisms in the sense that they are performed in succession: For any $a \in G$ the product of two endomorphisms χ, η of G satisfies

$$(\chi\eta)a = \chi(\eta a). \tag{2.10}$$

Given $b \in G$, observe also that

$$(\chi\eta)(ab) = \chi(\eta(ab)) = \chi(\eta a \cdot \eta b) = (\eta a)\chi \cdot (\eta b)\chi = ((\eta\chi)a \cdot (\eta\chi)b, \tag{2.11}$$

which shows that the product of two endomorphisms is an endomorphism. It is trivially true that the product of two automorphisms is an automorphism. Notice that an inverse mapping only exists for automorphisms. Hence, we have found that the set Φ of all automorphisms of a group G is itself a group, and moreover a subgroup of all one-to-one mappings of G onto itself.

3. The Lorentz Group

3.1. **Minkowski Space.** In the study of special relativity, the logical result of the postulates that the speed of light is constant in all inertial reference frames and that the laws of physics are the same in all inertial reference frames is a spacetime called Minkowski space. Unlike Euclidean space, Minkowski space does not have a positive definite inner product. Rather it is the Cartesian product of two positive definite inner product spaces. One of these inner product spaces contributes negatively to the Minkowski inner product and is called the timelike dimensions, and the other inner product space contributes positively to the Minkowski inner product and is called the spacelike dimensions.

In particular, special relativity considers Minkowski space with one time-like Euclidean dimension and three spacelike Euclidean dimensions, denoted as $M(\Re, \Re^3)$. More generally, we can consider a Minkowski space $M = M(\Re, V_\infty)$ where V_∞ is some positive definite inner product space. A spacetime vector $s \in M$ can be denoted $s = (ct, x)$ where $t \in \Re$, $x \in V_\infty$, and c is an arbitrary real number. The physical interpretation of c is the speed of light.

The Minkowski inner product is defined as

$$s_1 \cdot s_2 = \langle x_1\, x_2 \rangle - c^2 \langle t_1\, t_2 \rangle \tag{3.1}$$

where the inner products on the right hand side are defined on their appropriate inner product spaces.

Minkowski spacetime forms a group under the operation of addition:

$$s_1 + s_2 = (c_1 + cT_2, X_1 + X_2) \tag{3.2}$$

and the Lorentz group can be formally defined as the set of all autmorphisms of Minkowski space which leave $s \cdot s$ invariant. We know from section 2.4 that the set of all automorphisms of a group is a group itself, and thus we can conclude that the Lorentz transformations from a group we call the Lorentz transformation group.

A general form for a Lorentz transformation acting in $M(\Re, \Re^3)$ space is a $4x4$ matrix. However, this representation is rather complicated, containing six parameters (three rotations and three boost directions) which define the transformation. There exist simpler representations of the Lorentz transformation that can be found by considering the properties of the Lorentz transformation as a group and which have the advantage of allowing for an arbitrary spacelike inner product space.

3.2. **Relativistic Velocity Addition.** Let $V_c = v \in V_\infty : \|v\| < c$. This is a subset of V_∞ called the open c-ball. Elements in V_c are called relativistically admissable velocities. Such v are constructed from the set of curves in Minkwoski space parameterized by t with a constant derivative less than c, and are in fact the derivatives of these curves. The reason for calling them velocities is thus clear - they represent time derivatives of linear curves through space and our focus on V_c is the condition in special relativity that the speed of light is constant in all inertial frames. Note that for our purposes when we reference $u, v \in V_c$, we are always referring to these velocities. Notice that V_c is not an inner product space because given two elements $v\,w \in V_c$, it does not follow that their inner product $v \cdot w$ is contained in V_c. Moreover, in order to have an algebraic operation on V_c, we define a new binary operation denoted by

$$u \oplus v = \frac{1}{1 + u \cdot v/c^2}[(u+v) - \frac{1}{c^2}\frac{\gamma_u}{1+\gamma_u}(u^2 v - (u \cdot v)u)] \tag{3.3}$$

where the binary operation $u+v$ is defined in V_∞ and $\gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}$. Note that the binary operation defined on V_c is not associative and therefore does not form a group. We call a set with an algebraic operation defined in the set a groupoid[1]; nothing else is assumed about the nature of the algebraic operation.

It is worth noting that when u, v are linearly dependent, that is u is proportional to v, then relativistic velocity addition reduces to:

$$\frac{u+v}{1 + u \cdot v/c^2} \tag{3.4}$$

which is the usual result obtained in simple treatments of special relativity.[3]

It is also clear that as $c \to \infty$, equation 3.3 reduces to

$$u \oplus v = u + v, \tag{3.5}$$

which is the expected result. What this means is that relativistic velocity addition reduces to the usual Galilean velocity addition when the speed of light is taken to be infinite[3].

3.3. **Dyads.** We now consider algebraic objects called dyads. A dyad, also known as the direct product uv of two vectors u and v in a real inner product space P, is a linear map of P into itself. Namely, uv acting on w is

$$uv(w) = u(v \cdot w). \tag{3.6}$$

For any $a, b, c, d, w \in P$, the following holds

$$ab \cdot cd(w) = ab(c[d \cdot w]) = a(b \cdot c)(d \cdot w) = (b \cdot c)ad(w). \tag{3.7}$$

Thus two successive dyad operations are equivalent to a single one. Dyads also have a very useful property: Let r be an automorphism of P. Then the following holds for a dyad uv:

$$r(uv) = (rurv)r. \tag{3.8}$$

We call a linear polynomial of dyads a dyadic. Consider a dyadic of the form $\Omega(u, v) = vu - uv$ for $u\,v \in P$, which will be useful in section 3.4. Observe that by formula 3.7, we have

$$\Omega(u,v)^3 = \Omega(u,v)\cdot\Omega(u,v)\cdot\Omega(u,v) = (vu\cdot vu - vu\cdot uv - uv\cdot vu + uv\cdot uv)\cdot\Omega(u,v) \tag{3.9}$$

$$\Omega(u,v)^3 = (v(u\cdot v)(u\cdot v) - v(u\cdot u + v\cdot v) - u(v\cdot v)(u\cdot u) + u(v\cdot v)(u\cdot u))\Omega(u,v) \tag{3.10}$$

$$\Omega(u,v)^3 = [(u \cdot v)^2 - |u|^2\,|v|^2]\Omega(u,v). \tag{3.11}$$

From property 3.8 of dyads, we notice that it can be extended to dyadics to obtain the identity:

$$r\Omega(u,v)^3 = \Omega(ru,rv)r. \tag{3.12}$$

3.4. **Thomas Gyration.** The set of all automorphisms of a groupoid is a group[5]. Thus we can consider the set of automorphisms of V_c denoted by $Aut(V_c)$, which is a group by the preceeding remark. We can then define a map

$$gyr[u,v] : V_c \times V_c \to Aut(V_c), \tag{3.13}$$

which we call the Thomas gyration, by the equation

$$gyr[u,v] = E + a(u,v)\Omega(u,v) + \beta(u,v)\Omega(u,v)^2, \tag{3.14}$$

where E is the unit transformation, and $\Omega(u,v)$ is the dyadic defined in section 3.3,

$$\alpha(u,v) = -\frac{1}{c^2}\frac{\gamma_u\gamma_v(1+\gamma_u+\gamma_v+\gamma_{u\oplus v})}{(1+\gamma_u)(1+\gamma_v)(1+\gamma_{u\oplus v})} \tag{3.15}$$

and

$$\beta(u,v) = \frac{1}{c^4}\frac{\gamma_u^2\gamma_v^2}{(1+\gamma_u)(1+\gamma_v)(1+\gamma_{u\oplus v})}. \tag{3.16}$$

Ungar[5] remarks that α and β satisfy the relationship

$$\alpha^2 + [u^2v^2 - (u\cdot v)^2]\beta^2 - 2\beta = 0 \tag{3.17}$$

for any $u\, v \in V_c$. We also have

$$\gamma_{u\oplus v} = \gamma_u\gamma_v(1 + u\cdot v/c^2). \tag{3.18}$$

Since the Thomas gyration $gyr[u,v]$ is a dyadic, it is a linear map of V_∞ into itself. And from property 3.8, we find that the Thomas gyaration has the property that

$$r\ gyr[u,v] = gyr[ru,rv]r \tag{3.19}$$

for any $u\, v \in V_c$ and $r \in Aut(V_c)$. Notice that if the r and $gyr[u,v]$ commute, that is $r\ gyr[u,v] = gyr[u,v]r$, then we find that equation 3.19 becomes

$$gyr[u,v] = gyr[ru,rv]. \tag{3.20}$$

It is immediatley evident that $gyr[u,v]$ commutes with itself, so by equation 3.25 we know that

$$gyr[u,v] = gyr[gyr[u,v]u, gyr[u,v]v]. \tag{3.21}$$

We notice that the Thomas gyration $gyr[v,u]$ is both the inverse and the adjoint for $gyr[u,v]$. That is,

$$gyr[u,v]gyr[v.u] = E \tag{3.22}$$

and

$$\langle gyr[u,v]a, b\rangle = \langle a, gyr[v,u]b\rangle \tag{3.23}$$

where $u, v \in V_c$, $a, b \in V_\infty$, and E the unit element. As we see from the two previous equations, Thomas gyration preserves the inner product in V_∞ and as such is an orthogonal transformation. Thomas gyration also respects the binary operation defined in V_c, that is

$$gyr[u,v](a\oplus b) = gyr[u,v]a \oplus gyr[u,v]b \tag{3.24}$$

Ungar notes that Thomas gyration is special, that is of unit determinant[5]. These properties are enough for the Thomas gyration to be an automorphism both of the groupoid V_c and the group V_∞. We now come to the following theorem:

Theorem 3. *Let $SO(A)$ be the set of all special orthogonal transformations on a set A with respect to a given algebraic operation defined on A, and let $gyr[u,v]$ be the Thomas gyration, and V_∞, V_c be as previously defined. Then*

$$gyr[u,v] \in SO(V_\infty) \subset SO(V_c) \subset Aut(V_c) \tag{3.25}$$

Proof. To see that $SO(V_\infty) \subset SO(V_c)$ is justified, let $W \in SO(V_\infty$. W must be linear and preservesthe inner product in V_∞ So observe that

$$W(u \oplus v) = W(\frac{1}{1+u\cdot v/c^2}[(u+v) - \frac{1}{c^2}\frac{\gamma_u}{1+\gamma_u}(u^2 v - (u\cdot v)u)]) \tag{3.26}$$

$$W(u \oplus v) = (\frac{1}{1+u\cdot v/c^2}[(Wu+Wv) - \frac{1}{c^2}\frac{\gamma_u}{1+\gamma_u}((Wu)^2 Wv - (Wu\cdot Wv)Wu)]) \tag{3.27}$$

$$W(u \oplus v) = (\frac{Wu+Wv}{1+Wu\cdot Wv/c^2}[(Wu+Wv) - \frac{1}{c^2}\frac{\gamma_u}{1+\gamma_u}(u^2 Wv - (u\cdot v)Wu)]) \tag{3.28}$$

$$W(u \oplus v) = Wu \oplus Wv \tag{3.29}$$

which of course implies that W respects the binary operation defined in V_c. Moreover, since W is a special orthogonal transformation on $V_\infty \supset V_c$, we have that $W \in SO(V_c)$, which is what we wanted to show. □

3.5. **General Representation of the Lorentz Group Using Dyads.** We will define the pure Lorentz transformation (a Lorentz transformation with no spacetime rotation) $L(v) : M \to M$ by

$$B(v) = E + \gamma_v b(v) + \frac{\gamma_v^2}{1+\gamma_v} b^2(v) \tag{3.30}$$

where $v \in V_c$, E is the unit transformation of TX, and

$$b(v) = \begin{pmatrix} 0 \\ v \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} v. \tag{3.31}$$

By linearity of dyadics, $b(v)$ and $B(v)$ are linear transformations of TX. Note that bv is defined for all $v \in V_\infty$ whereas $B(v)$ is only defined for $v \in V_c$. Notice also that

$$b(v)\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} v\cdot x \\ vtc^2 \end{pmatrix} \tag{3.32}$$

By definition of $b(v)$, it is immediately evident that $b(-v) = -b(v)$. Moreover, it is straightforward to see that $b(v)$ satisfies the identity

$$b^3(v) = \frac{v^2}{c^2} b(v) = \frac{\gamma_v^2 - 1}{\gamma_v^2} b(v) \tag{3.33}$$

for $v \in V_c$. It then follows that the inverse of a pure Lorentz transformation $B(v)$ is $B(-v)$. As has been noted previously in section 3.1, the Lorentz transformation keepings the inner product of Minkowski space invariant. More explicity, from equations 3.31 and 3.32 we have

$$B(v)\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} ct \\ x \end{pmatrix} + \gamma_v \begin{pmatrix} v \cdot x \\ vtc^2 \end{pmatrix} + \frac{\gamma_v^2}{1+\gamma_v} b(v) \begin{pmatrix} v \cdot x \\ vtc^2 \end{pmatrix} \tag{3.34}$$

or

$$B(v)\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma_v(c^2 + v \cdot x) \\ c^2 x + \gamma_v vtc^2 + \frac{\gamma_v^2}{1+\gamma_v}(v \cdot x)v \end{pmatrix} = \begin{pmatrix} ct' \\ x' \end{pmatrix} \tag{3.35}$$

from which we obtain

$$\langle B(v)s_1, B(v)s_2 \rangle = \langle s_1, s_2 \rangle \, . \tag{3.36}$$

Examing equation 3.34, we notice that a pure Lorentz transform can be written in the form

$$B(u)\gamma_v \begin{pmatrix} 1 \\ v \end{pmatrix} = \gamma_{u \oplus v} \begin{pmatrix} 1 \\ u \oplus v \end{pmatrix}. \tag{3.37}$$

This equation can be interpreteted as a boost defined by the velocity parameter u. There is infact a unique boost B(u) which takes $\gamma_v(1, v)^t$ to $\gamma_w(1, w)^t$ for any given $v\, w \in V_c$, namely that u must satisfy the equation

$$w = u \oplus v \tag{3.38}$$

for which there is one unique solution,

$$u = w \oplus (-gyr[w, v]v), \tag{3.39}$$

which follows directly from the relationship $v \oplus (-v) = E$ where E is the null element.

We now consider the map $\rho(W)(ct, x) = (ct, Wx)$ for any $W \in SO(V_\infty)$. Interpreting W as a rotation of the spacelike vector x, we see that $\rho(W)$ is a rotation of Minkowski spacetime. By equation 3.32 we notice that $\rho(V)b(v)\rho(V^{-1} = b(Vv)$ for any $V \in SO(V_\infty)$. So for all $(ct, x)^t \in M$, we find that

$$\rho(V)b(v)\rho(V^{-1})(ct, x) = \rho(V)b(v)(ct, V^{-1}x) = \rho(V)(v \cdot V^{-1}x, vtc^2) \tag{3.40}$$

$$\rho(V)b(v)\rho(V^{-1})(ct, x) = (v \cdot V^{-1}x, Vvtc^2) = (Vv \cdot x, Bvtc^2) = b(Vv)(ct, x). \tag{3.41}$$

Applying this result to the definition of the pure Lorentz transformation, we obtain

$$\rho(V)B(v)\rho(V^{-1}) = B(Vv) \tag{3.42}$$

for $v \in V_c$, and thus

$$V \in SO(V_\infty) \subset SO(V_c). \tag{3.43}$$

Ungar[5] remarks that it can be shown that the product of two pure Lorentz transformations is equivalent to a single Lorentz transformation preceded by a thomas gyration, that is, the following holds:

$$B(u)B(v) = B(u \oplus v)\rho(gyr[u, v]). \tag{3.44}$$

If we parameterize the general Lorentz transformation, $L : M \to M$, by $v \in V_c$ and $V \in SO(V_\infty)$, we can define it as

$$L[v, V] = B(v)\rho(V). \tag{3.45}$$

This parametrization is unique, as shown in the following theorem:

Theorem 4. *Given a Lorentz transformation L, there is a unique decomposition of L into a pure Lorentz $B(v)$ and a spacetime rotation $\rho(V)$.*

Proof. Suppose we have a Lorentz transformation L such that

$$L = B(u)\rho(U) = B(v)\rho(V) \tag{3.46}$$

for some $u, v \in V_c$ and $U, V \in SO(V_\infty)$. Then we have

$$B(-v)B(u) = \rho(VU^{-1}). \tag{3.47}$$

But we have shown that the product of two pure Lorentz transformations is

$$B(-v)B(u) = B(-v \oplus u)\rho(gyr[-v, u] = B(w) = \rho(W) \tag{3.48}$$

where $w = -v \oplus u$ and $W = VU^{-1}gyr^{-1}[-u, v]$. On the otherhand, $\rho(W)$ is an automorphism of TX which must leave t unchanged. So we find that $B(w)$ must leave t entact. This is true if and only if $B(w) = B(0)$ so that W is the identity transformation, because $B(W)$ preserves the Minkowski inner product, and thus we can conclude that $w = 0$. By definition of w, this implies that $u = v$ and furthermore $U = V$. Thus we conclude that the decomposition of the Lorentz transformation into a boost preceeded by a spacial rotation is unique. □

3.6. **Spinor Representation.** When we limit ourselves to a Minkowski space with space-like dimension $\Re^3$, the typical representation is with the use of a 4×4 matrix, but a simplification exists in which 2×2 complex matrices are used to represent the Minkowski space and the corresponding Lorentz transformations. Here we can denote the coordinates of Minkowski space by a vector $X = (x, y, z, t)$ where t is the time-like dimension and x, y, z are the coordinates of the space-like dimension.

We propose a mapping of this vector to a 2×2 matrix of the following form:

$$\chi = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \tag{3.49}$$

Notice that this is an isomorphic mapping that takes a vector $X \in M(\Re, \Re^3)$ to a Hermitian 2×2 matrix. We call this representation of the Minkowski space a spinor.

It is easily shown[4] that the transformation defined as

$$\begin{pmatrix} t'+z' & x'+iy' \\ x'-iy' & t-z' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \tag{3.50}$$

keeps the Minkowski inner product invariant when $|Det(\Lambda)| = 1$ where $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. That is, under specific conditions the above transformation represents a Lorentz transformation. Reference [4] also points out that multiplying all of the matrix elements of Λ by a phase factor $e^{i\phi}$ where $\phi \in \Re$ also satisfies $|Det(\Lambda)| = 1$, and does not change the transformation so that we can require without loss of generality that $Det(\Lambda) = 1$, that is,

$$ad - bc = 1 \tag{3.51}$$

Since multiplying each of these numbers by -1 satisfies this equation[6] as well, we have found a holomorphic mapping of the Lorentz transformation the general dyadic representation for $M(\Re, \Re^3)$ to a spinor representation which is simpler in this particular Minkowski space. It is also a simplification of the 4×4 matrix representation typically used in the formulation of special relativity[3].

More explicitly, we perform the multiplications of the right-hand side of equation 3.6 and find that the the coordinates under a Lorentz transform are given by

$$x' = (Re(a\bar{d}+Re(c\bar{b}))x-(Im(b\bar{c})+Im(d\bar{a}))y+(Re(a\bar{c}-Re(b\bar{d}))z+(Re(a\bar{c}+Re(b\bar{d}))t \tag{3.52}$$

$$y' = (Im(c\bar{b}+Im(d\bar{a}))x-(Re(c\bar{b})-Re(a\bar{d}))y+(Im(c\bar{a}+Im(b\bar{d}))z+(Im(c\bar{a}+Im(d\bar{b}))t \tag{3.53}$$

$$z' = (Re(a\bar{b}-Re(c\bar{d}))x-(Im(b\bar{a})+Im(c\bar{d}))y+\frac{1}{2}(a\bar{a}+d\bar{d}-c\bar{c}-b\bar{b})z+\frac{1}{2}(a\bar{a}+b\bar{b}-c\bar{c}-d\bar{d})t \tag{3.54}$$

$$t' = (Re(a\bar{b}+Re(c\bar{d}))x-(Im(b\bar{a})+Im(d\bar{c}))y+\frac{1}{2}(a\bar{a}+c\bar{c}-b\bar{b}-d\bar{d})z+\frac{1}{2}(a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d})t \tag{3.55}$$

where it is immediately evident that all coefficients are real[4][6]. From the above equations, the relationship betweeen the typical 4×4 matrix representation and the spinor representation characterized by a, b, c, d with $ad - bc = 1$ is clear.

3.7. **The Abberation Transformation Subgroup.** We now restrict our attention to a particular subgroup of the Lorentz transformations called the abberation transformation. This subgroup is the set of Lorentz transformations which preserve a null Minkowski inner product, and for our purposes we will assume that the time-like component satisfies $t > 0$. This subgroup can be represented by a linear fractional transformation. Newman[4] notes that in principle this result can be derived as a direct consequence of the equations 3.52, 3.53, 3.54, and 3.55 and the condition that $x^2 + y^2 + z^2 - t^2 = 0$, but we will take another approach which is more instructive and which avoids fairly complicated algebraic manipulations. Note that since $t > 0$ by assumption, $t - z$ and $t + z$ are both nonnegative, and we have $t + z = \frac{x^2+y^2}{t-z}$. We now claim

$$\begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} = \begin{pmatrix} \frac{x+iy}{\sqrt{t-z}} \\ \sqrt{t-z} \end{pmatrix} \begin{pmatrix} \frac{x+iy}{\sqrt{t-z}} & \sqrt{t-z} \end{pmatrix}. \tag{3.56}$$

This is straightforward to see because

$$\begin{pmatrix} \frac{x+iy}{\sqrt{t-z}} \\ \sqrt{t-z} \end{pmatrix} \begin{pmatrix} \frac{x+iy}{\sqrt{t-z}} & \sqrt{t-z} \end{pmatrix} = \begin{pmatrix} \frac{(x+iy)(x-iy)}{t-z} & x+iy \\ x-iy & t-z \end{pmatrix} = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix}. \tag{3.57}$$

Thus we can write the abberation transformation, A, as

$$\begin{pmatrix} \frac{x'+iy'}{\sqrt{t'-z'}} \\ \sqrt{t'-z'} \end{pmatrix} \begin{pmatrix} \frac{x'+iy'}{\sqrt{t'-z'}} & \sqrt{t'-z'} \end{pmatrix} = A \begin{pmatrix} \frac{x'+iy'}{\sqrt{t'-z'}} \\ \sqrt{t'-z'} \end{pmatrix} \left[\begin{pmatrix} \frac{x'+iy'}{\sqrt{t'-z'}} & \sqrt{t'-z'} \end{pmatrix} \overline{A^T} \right], \tag{3.58}$$

and so we find that

$$\begin{pmatrix} \frac{x'+iy'}{\sqrt{t'-z'}} \\ \sqrt{t'-z'} \end{pmatrix} = A \begin{pmatrix} \frac{x+iy}{\sqrt{t-z}} \\ \sqrt{t-z} \end{pmatrix} \tag{3.59}$$

because given two matrices A and B such that

$$A\overline{A^T} = B\overline{B^T}, \tag{3.60}$$

we must have $A = B$.

Two equations result from the equation above. Taking their ratio, we obtain

$$\zeta' = \frac{a\zeta + b}{c\zeta + d} \tag{3.61}$$

where $\zeta = \frac{x+iy}{t-z}$ and similarly for ζ'. That is, we have found that the abberation subgroup can be represented as a linear fractional transform, which moreover is unique with condition 3.51 and reference [6].

We now notice that there is a simple geometric interpretation of ζ. The ratios $\frac{x}{t}$, $\frac{y}{t}$, and $\frac{z}{t}$ are the direction cosines for a null vector in Minkowski space. So we can also define ζ as

$$\zeta = \frac{x+iy}{t-z} = \frac{\frac{x}{t} + i\frac{y}{t}}{1 - \frac{z}{t}} = \frac{\sin\theta(\cos\phi + i\sin\phi)}{1-\cos\theta} = e^{i\phi}\cot\frac{\theta}{2}. \tag{3.62}$$

Written in this way, it becomes clear that ζ is uniquely determined by the angles ϕ and θ. It is also clear why the name of the abberation subgroup was chosen. Null vectors in Minkowski space correspond to paths which light can travel in spacetime. Light travels at a speed which is constant in all inertial reference frames (that is, it is invariant under Lorentz transformations). The angles θ and ϕ then describe the angular direction of light, so that the abberation transformation describes the relation between the angular direction of light observed in two different reference frames. This variance is known as abberation of light, or the "headlight" effect.

4. Conclusion

We have examined the structure of Minkowski space using the theory of groups. From this, we have found a velocity addition operation under the constraint that the velocity is bounded by some constant. This allows the construction of a general representation of the Lorentz transformation acting in $M(\Re, V_\infty)$ Minkowski space where V_∞ is an arbitrary positive definite inner product space, namely as a Thomas gyration preceeding a pure Lorentz transformation parameterized by some velocity v with $|v| < c$. When we restrict our focus to $M(\Re, \Re^3)$, a simplification is possible in which the Lorentz transformation and Minkowski space are represented by 2×2 matrices called spinors with a single algebraic constraint on the components of the Lorentz transformation spinor. Further restricting our focus to the abberation subgroup, we are able to connect the theory of special relativty to complex analysis by showing that the abberation transformation can be represented as a linear fractional transformation parameterized by a number ζ which has a simple geometric interpretation in terms of angles and which describes simply the abberation of light.

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