On The *Almost* Riemann Mapping Theorem in $\mathbb{C}^n$

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1 Introduction

Multivariable complex analysis is different from its single variable sister; there "is more involved than merely saying 'now let $n > 1$.' " Bell [1] goes over several basic results in multivariable complex analysis.

We examine one of them in particular, which is that the Riemann mapping theorem does not hold in dimension $n > 1$. We roughly sketch out the language and tools needed to go over this result, and look at three different proofs of the result.

2 Complex Analysis in Several Variables

2.1 Basic Terminology

We begin with some basic vocabulary and results.

Definition 2.1. *A complex variable $z \in \mathbb{C}^n$ is an n-tuple $(z_1, ..., z_n)$, with $z_j \in \mathbb{C}$. The z_j's are called the components of z.*

Definition 2.2. *Multiplication on $\mathbb{C}^n$ is defined componentwise; that is, for $z, w \in \mathbb{C}^n$, $zw = (z_1w_1, ..., z_nw_n)$. Multiplication by scalars is also what you would expect: for $z \in \mathbb{C}^n$, $\zeta \in \mathbb{C}$, $\zeta z = (\zeta z_1, ..., \zeta z_n)$.*

Definition 2.3. *A function $f \colon \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic (equivalently, analytic) if it satisfies the n-dimensional Cauchy-Riemann equations, i.e. it satisfies $\partial/\partial\bar{z}_j f = 0$, for $1 \leq j \leq n$.*

Definition 2.4. *A biholomorphic (equivalently, univalent) function $f \colon \Omega_1 \to \Omega_2$, $\Omega_1 \in \mathbb{C}^n$, $\Omega_2 \in \mathbb{C}^m$ is a function f which is bijective, holomorphic, and $\det u_{\mathbb{C}} f(z) \neq 0$, where $u_{\mathbb{C}}$ is the complex Jacobian with entries $u_{ij} = \frac{\partial f_i}{\partial z_j}$.*

Definition 2.5. *A biholomorphic map $f \colon \Omega \to \Omega$ which takes Ω to itself is called an automorphism.*

It will be useful that the complex Jacobian behaves like its real counterpart in the following respect:

Lemma 2.1 ([8, p. 11])**.** *$|u|^2$ is equal to the determinant of the Jacobian of f as a function of $2n$ variables on $\mathbb{R}^{2n}$.*

The proof follows from the Cauchy-Riemann equations. Here we will prove the one-dimensional case because it is easier to compute; however, the generalization to n dimensions should be clear.

Proof. Let $f(z) = u + iv$ be analytic on a domain Ω. We can treat Ω as a domain in $\mathbb{R}^2$ (by taking coordinates $z = (\Re(z), \Im(z)) = (x, y)$) and f as a map from Ω to $\mathbb{R}^2$, with components $f(x, y) = (u(x, y), v(x, y))$. The determinant of the Jacobian is

$$\det u = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

By the Cauchy-Riemann equations we replace the x derivatives with y derivatives:

$$\det u = \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left|\frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}\right|^2$$

By [3, p. 47 (3.2)], we have

$$\left|\frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}\right|^2 = |f'(z)|^2$$

$\square$

The generalization to n-dimensions should be clear.

Corollary 2.2. *The change of variables formula on $\mathbb{C}^n$ is then $\phi(w)dw = u(z) \cdot \phi(f(z))dz$. This follows from the formula for the real change of variables [2, Theorem 4.41] and the above lemma.*

Definition 2.6. *A multi-index α is an ordered n-tuple of non-negative integers $(\alpha_1, \alpha_2, ..., \alpha_n)$.*

The following notation is standard:

$$|\alpha| = \sum_{j=1}^{n} \alpha_j$$

$$\alpha! = \prod_{j=1}^{n} \alpha_j!$$

$$z^\alpha = \prod_{j=1}^{n} z_j^{\alpha_j}$$

Definition 2.7. *Let $\Omega \subset \mathbb{C}^n$. Suppose that $ze^{i\theta} \in \Omega$ for all $z \in \Omega$ and $\theta \in \mathbb{R}$. Then Ω is called circular.*

Definition 2.8. *The polydisc $P_n(a, r)$ is defined as the Cartesian product of n discs of radius r, each centered at a_i, i.e. the set $\{z \in \mathbb{C}^n : |z_i - a_i| < r, i = 1, 2, ..., n\}$.*

The ball $B_n(a, r)$ is defined as the ball of radius r centered at a, i.e. $\{z \in \mathbb{C}^n : \sum_{i=1}^n |z_i - a_i|^2 < r^2\}$.

It is generally obvious from context what n is, so we will drop the subscripts.

For $n = 1$, these obviously refer to the same set; however, in general they do not. By convention, the unit disc $B(0, 1) = P(0, 1)$ is denoted by D. The "unit (n)-polydisc" refers to the set $P(0, 1)$, and the "unit (n)-ball" refers to $B(0, 1)$.

Remark The unit polydisc and the unit ball are both clearly circular domains.

It turns out that it is in fact the polydisc which is easier to work with, as some of the theorems from one dimension are most easily adapted to the polydisc, rather than the ball. For example, the Cauchy integral theorem is defined as follows:

Theorem 2.3 (The Cauchy Integral Formula on polydiscs [6, p. 4])**.** *Suppose $f(z)$ is holomorphic in each component on the closure of the unit n-polydisc $\overline{P(0, 1)}$. Then for $\zeta \in P(0, 1)$,*

$$f(\zeta) = \frac{1}{(2\pi i)^n} \int_{|z_n|=1} \cdots \int_{|z_1|=1} f(z) \prod_{j=1}^{n} \frac{d\zeta_j}{z_j - \zeta_j}$$

There is an analogous integral over the unit ball, but it does not concern us here.

It is an easy result in [8, p. 4] that $f : \Omega_1 \to \Omega_2$, $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ holomorphic can be locally expanded as a power series

$$f(z) = \sum_{\alpha} c_\alpha z^\alpha$$

where α ranges over all multi-indices. There exists a neighborhood (for practical purposes it will be a small polydisc) around each z_0 such that the expansion around z_0 is absolutely and uniformly convergent for z in that neighborhood.

The coefficients c_α are given by

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_{P(z_0,\epsilon)} f(\zeta)\bar{\zeta}^\alpha d\zeta_1 \cdots d\zeta_n$$

Here, the integral is over a small polydisc of radius ϵ.

2.2 Homogeneous Functions

Definition 2.9. *A polynomial $P(z)$ is homogeneous of degree k if for every $\zeta \in \mathbb{C}, z \in \mathbb{C}^n$, $P(\zeta z) = \zeta^k P(z)$.*

Let

$$f(z) = \sum_{\alpha} c_\alpha z^\alpha$$

be a holomorphic function whose power series expansion is convergent in a neighborhood of the origin (for simplicity a disc centered at the origin). Now let

$$F_k(z) = \sum_{|\alpha|=k} c_\alpha z^\alpha$$

Each of these F_k's is homogeneous in z [8, p. 19]. Since $f(z)$ converges uniformly on some small polydisc, so we can rearrange terms and rewrite f as

$$f(z) = \sum_{k=0}^{\infty} F_k(z)$$

This is called the homogeneous expansion of f.

Lemma 2.4. *Let $f \colon \Omega \to \Omega$ be a holomorphic function, and Ω be a simply connected open subset of $\mathbb{C}^n$. Suppose the the homogeneous expansion of f is convergent on some small polydisc $P(\alpha, \epsilon)$, and $z \in P(\alpha, \epsilon)$. Then*

$$F_k(z) = \frac{1}{2\pi} \int_0^{2\pi} f(ze^{i\theta}) e^{-ik\theta} d\theta$$

Proof. Fix a $z \in \Omega$. By the homogeneity of $F_k(z)$,

$$f(ze^{i\theta}) = \sum_{j=0}^{\infty} e^{ij\theta} F_j(z)$$

Multiplication by $e^{-ik\theta}$ and integration over θ yields

$$\frac{1}{2\pi} \int_0^{2\pi} f(ze^{i\theta}) e^{-ik\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=0}^{\infty} e^{i(j-k)\theta} F_j(z) d\theta$$

The convergence of the sum is uniform inside a sufficiently small neighborhood, so we can move the integral inside the sum. $e^{i(j-k)\theta}$ integrates to zero for $j - k \neq 0$ so we obtain

$$\int_0^{2\pi} f(ze^{i\theta}) e^{-ik\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} F_k(z) d\theta = F_k(z)$$

□

2.3 The Automorphisms of B

Definition 2.10. *A Mobius transformation is an automorphism of the unit disc.*

Theorem 2.5. *Let $\zeta \in D$. Then there exists an automorphism of the unit disc $\phi_\zeta(z)$ such that $\phi_\zeta(0) = \zeta$.*

Proof. Let $\phi_\zeta(z) = (\zeta - z)/(1 - \bar{\zeta}z)$. Obviously, $\phi_\zeta(0) = \zeta$ (note also that $\phi_\zeta(\zeta) = 0$. It remains to show that this is an automorphism. First, we show it is injective

$$\phi'_\zeta(z) = \frac{1 - |\zeta|^2}{(1 + \bar{\zeta}z)^2}$$

Since this is nonzero inside the disc, ϕ_ζ is injective by the implicit mapping theorem.

Now, it is also surjective; explicitly, for any $w \in D$, let $z = (w + \zeta)/(1 + \bar{\zeta}w)$. $|z| = |w + \zeta|/|1 + \bar{\zeta}w| < 1$, so $z \in D$. Then $\phi_\zeta(z) = w$ so we are finished. □

Corollary 2.6. *The Mobius transformations act transitively on D. That is, for any $\zeta, \xi \in D$, there exists an automorphism of the unit disc ϕ which takes ζ to ξ.*

Proof. Take ϕ to be $\phi_\xi \circ \phi_\zeta$. □

Now we want to generalize the Mobius transformations of the disc to n dimensions.

Define $\phi_\alpha : B \to B$ as follows:

$$\phi_0(z) = -z$$

$$\phi_\alpha(z) = \frac{\alpha - \frac{z \cdot \alpha}{\bar{\alpha}} - \sqrt{1 - |\alpha|^2}\left(\alpha - \frac{z \cdot \alpha}{\bar{\alpha}}\right)}{1 - z \cdot \alpha};\ \alpha \neq 0, \alpha \in B$$

Here $z \cdot \alpha$ indicates the usual dot product on $\mathbb{C}^n$; explicitly, $z \cdot \alpha = \sum_{j=1}^n z_j \bar{\alpha}_j$. It is clear that this is a holomorphism for $|\alpha| < 1$, i.e. $\alpha \in B$. It is not obvious that this (behemoth of a definition) is the correct definition; however, Rudin [8] assures us that it is, although he writes it in a slightly more compact form. In particular, just like a Mobius transform, ϕ_α satisfies $\phi_\alpha(0) = \alpha$ and $\phi_\alpha(\alpha) = 0$. It also reduces to the Mobius transformation in the case $n = 1$.

Lemma 2.7 ([8, Theorem 2.2.2(v)])**.** *ϕ_α is an involution. That is, it satisfies: $\phi_\alpha(\phi_\alpha(z)) = z$*

Corollary 2.8. *ϕ_α is an automorphism of B.*

By the lemma, ϕ is an involution. It is a well known fact that involutions are bijections. It is also holomorphic; thus it is a biholomorphism from B to B.

Theorem 2.9 ([8, Theorem 2.2.3])**.** *The set of automorphisms on B acts transitively on B.*

Proof. For any $\zeta, \xi \in B$ take $\psi(z) = \phi_\xi(\phi_\zeta(z))$ is an automorphism of B which takes ζ to ξ. □

2.4 Schwarz

Now we briefly discuss the Schwarz lemma and the Schwarz-Pick theorem.

Theorem 2.10 (Schwarz lemma [3, Theorem, p. 260])**.** *Suppose $f\colon D \to D$ is a holomorphic mapping on the unit disc, $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D$. The Schwarz lemma guarantees us that, for $z \in D$,*

$$|f(z)| \leq |z|$$

and

$$|f'(0)| \leq 1$$

Furthermore, if $|f'(0)| = 1$ or $|f(z)| = |z|$ (for any $z \neq 0$, $z \in D$), then $f(z) = ze^{i\theta}$ for some real θ.

Theorem 2.11 (Schwarz-Pick theorem [5, Theorem 1.1])**.** *Suppose $f\colon D \to D$ is a holomorphic mapping from the unit disc into itself. Then for $z \in D$,*

$$\frac{|df|}{1 - |f(z)|^2} \leq \frac{|dz|}{1 - |z|^2}$$

Proof. In the Schwarz lemma, we would like to say something about $|f'(z)|$ on the whole disc. So, we want to get rid of the condition that $f(0) = 0$. We will want to fix an (arbitrary) point ζ and consider the following automorphisms:

$$g(z) = \frac{z + \zeta}{1 + \bar{\zeta} z}$$

$$h(z) = \frac{z - f(\zeta)}{1 - \overline{f(\zeta)} z}$$

and their derivatives

$$g'(z) = \frac{1 - |\zeta|^2}{(1 + \bar{\zeta} z)^2}$$

$$h'(z) = \frac{1 - |f(\zeta|^2}{(1 - \overline{f(\zeta)} z)^2}$$

It is clear that $g(0) = \zeta$ and $h(\zeta) = 0$. Then a new function defined by $F(z) = h(f(g(z)))$ indeed satisfies $F(0) = 0$.

Now, by the Schwarz lemma, we conclude that $|F'(0)| \leq 1$. But by the chain rule,

$$F'(0) = h'(f(\zeta))f'(\zeta)g'(0) = \left|\frac{df}{dz}\right| \frac{1 - |z|^2}{1 - |f(z)|^2} \leq 1$$

Now we formally multiply both sides by $\frac{|dz|}{1-|z|^2}$ and obtain

$$\frac{|df|}{1 - |f(z)|^2} \leq \frac{|dz|}{1 - |z|^2}$$

as desired. □

It is a little odd to see differentials being compared to each other; for the moment it is enough to understand this in terms of integrating over a curve. That is, for any curve $\gamma \in D$, the above inequality will imply

$$\int_{f(\gamma)} \frac{|df|}{1-|f(z)|^2} \leq \int_{\gamma} \frac{|dz|}{1-|z|^2}$$

We will see soon that this can be understood in terms of a metric on D.

3 Basic Geometry

3.1 The Distance Metric

We will not do any sort of justice to the subject of differential geometry here. A lot of our definitions may seem imprecise or ill-motivated; they can in fact be defined precisely, but all we need for this paper is a crude sketch of some concepts and properties.

Definition 3.1. *A metric on a domain Ω is a kind of notion of distance on Ω. Formally, it is a function $d\colon \Omega \times \Omega \to \mathbb{R}^+$ which satisfies*

1. $d(x,y) \geq 0$ *for all* $x, y \in \Omega$ *(it is positive semidefinite).*
2. $d(x,y) = 0$ *if and only if* $x = y$.
3. $d(x,y) = d(y,x)$ *(it is symmetric).*
4. $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in \Omega$ *(it satisfies the triangle inequality).*

The triangle inequality is important, because it intuitively says that the distance defined by the metric will tell you what the distance is over the "shortest" possible path between two points - that is, going from x to y is faster than stopping at z along the way.

Example 3.1. *The familiar Euclidean distance on $\mathbb{R}^n$ defined by*

$$d(x,y) = \sqrt{\sum_{1}^{n} (x_i - y_i)^2}$$

obviously satisfies all the properties above.

Example 3.2. *The Euclidean distance has an obvious generalization to $\mathbb{C}^n$ which also satisfies all the properties above; it is defined by*

$$d(x,y) = \sqrt{\sum_{1}^{n} |z_i - w_i|^2}$$

3.2 Line Elements

It will be useful to consider metrics which are defined by integration. That is, we will want to consider metrics roughly of the form:

$$d(x, y) = \int_x^y \sqrt{ds^2}$$

At this point, ds^2 is a mysterious looking thing; it is called sometimes called the 'line element' and sometimes simply referred to as 'the metric' (although that is shorthand). Roughly speaking, it defines the infinitesimal distance between z and $z + dz$. It is kind of like the 'infinitesimal' form of the distance function. Actually, the notation above with $\sqrt{ds^2}$ is sloppy; we will define it more precisely later, after we define the metric tensor.

Also, we don't know at this point how to get from x to y; obviously, going over different curves will give different results for this integral. What we really want is to consider all curves γ between x and y and find the smallest distance so that the metric will satisfy the triangle inequality. First we define the length (given a line element ds) over a smooth curve $\gamma\colon [0, 1] \to \Omega$ where $\gamma(0) = x, \gamma(1) = y$ to be

$$|\gamma|_{ds} = \int_\gamma ds$$

Now we take the distance between two points to be the infimum length over all smooth curves connecting the two points:

$$d(x, y) = \inf_\gamma |\gamma|_{ds}$$

Remark Note that such a curve may not exist. As a simple example, on $\mathbb{R}^2 - (0, 0)$, the distance (as defined above) between $(-1, 0)$ and $(1, 0)$ is 2, but there is no curve of length 2 connecting these points. However, we will not deal with such 'pathological' domains.

Example 3.3. *The Poincare line element is defined on D by*

$$ds_D^2 = \frac{|dz|^2}{(1 - |z|)^2}$$

Indeed, this is the differential we saw earlier at the end of our discussion of the Schwarz-Pick theorem. As an aside, this does produce a metric on D. It is not obvious from the definition, but induces a distance between two points:

$$d_P(z, w) = \tanh^{-1} \left| \frac{z - w}{1 - z\bar{w}} \right|$$

Now, by the Schwarz-Pick theorem we see that holomorphic maps never increase this metric. Indeed, let $f^*(ds_D^2)$ indicate the line element

$$\frac{|df|^2}{(1 - |f(z)|^2)^2}$$

The Schwarz-Pick theorem literally says in this notation that $f^*(ds_D^2) \leq ds_D^2$! Integration over a curve and its image tells us that the Poincare length of a curve is never increased under the action of a holomorphic function. More precisely,

$$|f(\gamma)|_P = \int_{f(\gamma)} \frac{|df|}{1-|f(z)|^2} \leq \int_\gamma \frac{|dz|}{1-|z|^2} = |\gamma|_P$$

Moreover, if f is an automorphism of D, then f is an isometry, or length-preserving map

Lemma 3.1. *If $f\colon D \to D$ is an automorphism, then any curve $\gamma \in D$ satisfies*

$$|\gamma|_P = |f(\gamma)|_P$$

In other words, the length of a curve in the Poincare metric is equal to the length of its image under an automorphism.

Proof. We use the Schwarz-Pick theorem on f and f^{-1} to obtain the two following inequalities:

$$f^*(ds_D^2) \leq ds_D^2$$

and

$$ds_D^2 = f^{-1*}(f^*(ds_D^2)) \leq f^*(ds_D^2)$$

Taking the two inequalities together we conclude that $f^*(ds_D^2) = ds_D^2$ and thus

$$\int_{f(\gamma)} f^*(ds_D^2) = \int_\gamma ds_D^2$$

□

3.3 Geodesics

Definition 3.2. *A geodesic for a metric M is a curve $\gamma_M\colon [0,1] \to \Omega$, which is a local minimum of distance between the two points $\gamma(0)$ and $\gamma(1)$, i.e. it locally minimizes the distance function*

$$d(x,y) = \int_x^y ds$$

As we noted before, there may not be such a curve connecting the two points. However, for all the 'easy' domains we are working with, this definition is fine. Also, note that such a curve is not necessarily unique - for example, see Example 3.5 below.

We will use the notation $t \mapsto \gamma(t)$ to indicate an explicit parametrization of a geodesic. It should be clear from our definitions that the geodesic connecting two points *is* the curve we want to integrate over to obtain the distance between two points.

Because the notion of a geodesic is so intuitive, we present several examples without verifying that they are indeed the geodesics on the metric we want to look at.

Example 3.4. *On Euclidean n-space $\mathbb{R}^n$, the geodesics are simply the straight lines.*

Example 3.5. *On the sphere, the geodesic between two points is an arc of the great circle which goes through them; a great circle is a circle on the boundary of the sphere whose center is the same as the sphere's center.*

Note that this is not uniquely defined. In particular, there is a 'short' arc connecting any two points as well as a 'long' one (the one going the long way around the circle). Also, in the case that two points are antipodal, an arc of *any* great circle connecting them is a geodesic!

Example 3.6. *On the disc with the Poincare metric, the geodesic connecting two points is the arc of a circle which intersects the boundary of the disc orthogonally.*

3.4 The Metric Tensor

We now want to introduce the concept of the metric tensor. The metric tensor is a way of defining the inner product on a domain Ω; it also gives us a convenient way of defining the line element. It turns out that this is a natural thing to look at, because under certain natural types of transformations, lengths and angles are preserved.

Definition 3.3. *Suppose M is an n-dimensional surface, parametrized by coordinates $(u_1(z), ..., u_n(z))$ and appropriate smoothness conditions, and a vector field on the surface $r(u)$. We may want to parametrize r in t over a curve. A metric tensor g (often simply called 'the metric') is an $n \times n$ matrix whose entries are $g_{ij}(r) = \frac{\partial r}{\partial u_i} \cdot \overline{\frac{\partial r}{\partial u_j}}$. Note that these notions are all well-behaved in terms of their $\mathbb{R}^{2n}$ analog (i.e., identification of u_j with (x_j, y_j), etc.).*

In the above definition, it turns out that having the complex conjugates is the natural thing to do in order to get the correct things. This defines an invariant inner product on the surface:

$$a \cdot b = \sum_{i,j=1}^{n} a_i g_{ij} \bar{b}_j = \sum_{i,j=1}^{n} a_i \frac{\partial r}{\partial u_i} \cdot \frac{\partial r}{\partial u_j} \bar{b}_j$$

This definition appears to be somewhat circular since the definition involves a dot product in the first place; the point is though that you calculate g_{ij} in one basis and then this inner product holds in all bases. However, this is not our concern. What this leads us to is a natural way to produce the line element, given a metric tensor. We define the line element to be

$$ds^2 = \sum_{i,j=1}^{n} du_i g_{ij} \overline{du}_j$$

This may be formally interpreted as the square of the norm of the differential $|d\mathbf{u}|^2$.

Now we write the arclength of a curve (assume it is smooth) formally as:

$$|\gamma|_g = \int_\gamma ds$$

To actually calculate this and get the desired arclength, we "parametrize in t and take the square root" (assuming as usual that γ is a map from $[0,1]$ to Ω):

$$\begin{aligned} |\gamma|_g = \int_\gamma ds &= \int_0^1 \sqrt{\sum_{i,j=1}^n d\gamma_i g_{ij}(\gamma)\overline{d\gamma}_j} \\ &= \int_0^1 \left(\sqrt{\sum_{i,j=1}^n g_{ij}(\gamma(t))\gamma_i'(t)\overline{\gamma_j'(t)}} \right) dt \end{aligned}$$

where $d\gamma_i = \frac{\partial \gamma_i(t)}{\partial t} dt = \gamma_i'(t)dt$.

Example 3.7. *In Euclidean space, the metric we want is the identity matrix.*

This produces the familiar dot product:

$$a \cdot b = \sum_{i,j=1}^n a_i \delta_{ij} b_j = \sum_{i=1}^n a_i b_i$$

Then the line element we get is

$$ds^2 = \sum_{i=1}^n dx_i^2$$

In two dimensions, this is the familiar looking arclength for the curve $\gamma(t) = (x(t), y(t))$:

$$|\gamma| = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

Example 3.8 ([8, p. 51])**.** *The Poincare metric is given by*

$$g_{ij}(z) = \frac{2}{(1 - |z|^2)^2}, \ z \in D$$

This gives rise to the Poincare line element which we saw before.

3.5 Lebesgue Basics

We must begin by introducing the notion of an L^2 space. The theory of Lebesgue integration is more or less tangential to this paper, but certain ideas are essential. Roughly speaking, the Lebesgue measure on a subset of Euclidean space the generalization of the volume of that subset. For the open simply connected sets we always work with, it is essentially equivalent to the Euclidean volume.

Definition 3.4. *The L^2 inner product of two functions u and v on $\Omega \subset \mathbb{C}^n$ is indicated by $\langle u, v\rangle_\Omega$, and stands for the integral*

$$\int_\Omega u(z) \cdot \overline{v(z)} dV(z)$$

Here, $dV(z)$ is the "Lebesgue measure" on Ω. For the purposes of this discussion we can think of it as simply being the volume element $dz_1 dz_2 \cdots dz_n$.

Definition 3.5. *We say that $u(z) \in L^2(\Omega)$ or that is has a finite norm if $\|u\|^2_{L^2(\Omega)} = \langle u, u\rangle_\Omega = \int_\Omega u(z)\overline{u(z)}dV(z) < \infty$.*

Definition 3.6. *Two functions $u, v \in L^2(\Omega)$ are orthogonal if $\langle u, v\rangle_\Omega = \int_\Omega u(z)\overline{v(z)}dV(z) = 0$.*

Example 3.9. *The monomials are orthogonal on the polydisc.*

Lemma 3.2. *Let $f_\alpha(z) = z^\alpha$, $z \in \mathbb{C}^n$. Then the f_α are in $L^2(P(0,1))$ and are orthogonal.*

Proof. We want to calculate $\langle f_\alpha, f_\beta\rangle_{P(0,1)}$.

$$\langle f_\alpha, f_\beta\rangle_{P(0,1)} = \int_{P(0,1)} z^\alpha \bar{z}^\beta dV(z)$$

Switch to polar coordinates and set $z_i^{\alpha_i} = re^{i\alpha_i\theta}\lambda_i^{\alpha_i}$ (where λ_i is the unit vector in the ith direction) and obtain

$$= \int_{P(0,1)} r^{|\alpha+\beta|} \prod_{i=1}^n e^{i(\alpha_i-\beta_i)\theta} d\lambda_i$$

The integral over each θ disappears unless $\alpha = \beta$. □

Lemma 3.3. *Let $f_\alpha(z) = z^\alpha$, $z \in \mathbb{C}^n$. Then the f_α are in $L^2(B(0,1))$ and are orthogonal.*

Proof. The integral over each sphere of radius r is zero by [8, Proposition 1.4.8] □

Lemma 3.4. *Suppose $h \in L^2(B(0,1))$ is holomorphic, with power series expansion $\sum_\beta a_\beta z^\beta = h(z)$ around 0. Then*

$$\langle h, z^\alpha\rangle_{(B(0,1))} = c_\alpha \frac{\partial^\alpha h}{\partial z^\alpha}(0)$$

for some constant c_α.

Proof. We simply calculate it.

$$\langle h, z^\alpha \rangle_{(B(0,1))} = \int_{B(0,1)} \left(\sum_\beta a_\beta z^\beta \right) \bar{z}^\alpha dV(z)$$

Since it is uniformly convergent, we can put the integral inside. Since monomials z^α are L^2 orthogonal on the ball (as just claimed), all the terms integrate to zero except for

$$\int_{B(0,1)} a_\alpha |z|^{2|\alpha|} dV(z) = c_\alpha \frac{\partial^\alpha h}{\partial z^\alpha}(0)$$

where $c_\alpha = \langle z^\alpha, z^\alpha \rangle_{B(0,1)}$ and $a_\alpha = \frac{\partial^\alpha h}{\partial z^\alpha}(0)$. □

Remark By [8, Proposition 1.4.9],

$$c_\alpha = \frac{n!\alpha!}{(n + |\alpha|)!}$$

Also, the above lemma turns out to be formally equivalent to Cauchy's integral formula.

We will also make reference to the following theorem.

Theorem 3.5 (Dominated Convergence Theorem)**.** *Suppose that $\{f_n\}$ is a sequence of Lebesgue measurable functions which converge pointwise to a function f. Suppose further that the function $|g|$ satisfies $\int_\Omega |g| dV < \infty$ and $|f_n(z)| \leq |g(z)|$ for all $n \in \mathbb{N}$, $z \in \Omega$. Then*

$$\lim_{n\to\infty} \int_\Omega f_n dV = \int_\Omega f dV$$

The following corollary follows obviously because boundedness implies domination:

Corollary 3.6. *Suppose that $f_n(x)$ is a Lebesgue measurable function, bounded in modulus for all $n \in \mathbb{N}$, $x \in \Omega$ and that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in \Omega$ (that is, f_n converges pointwise to f). Then*

$$\lim_{n\to\infty} \int_\Omega f_n dV = \int_\Omega f dV$$

Proof. Set $g(x)$ from the Dominated Convergence Theorem equal to $\sup_{x\in\Omega, n\in\mathbb{N}} |f_n(x)|$. □

3.6 The Bergman Kernel and Metric

Definition 3.7. *The Bergman space $A^2(\Omega)$ is the set of holomorphic L^2 integrable functions on Ω.*

Lemma 3.7 ([8, Lemma 1.4.2])**.** *The Bergman space is a Hilbert space with inner product* $\langle u, v\rangle = \int_\Omega f(z)\overline{g(z)}dV(z)$*.*

Now we introduce the Bergman kernel.

Definition 3.8. *The Bergman kernel* $K : \Omega \times \Omega \to \mathbb{C}^n$ *is a function which satisfies the following properties ([8, pp.40-41]):*

1. *It is holomorphic in one of its arguments. That is, for every fixed* $\zeta \in \Omega$*,* $K(z, \zeta)$ *as a function of* $z \in \Omega$ *is an element of* $A^2(\Omega)$*.*
2. *It is reproducing. That is, for every* $f(z) \in A^2(\Omega)$*,* $f(z) = \int_\Omega K(z, \zeta)f(\zeta)dV(\zeta)$*.*
3. *It is conjugate symmetric. That is,* $K(z, \zeta) = \overline{K(z, \zeta)}$*.*

By ([8, Proposition 1.4.6]), these properties uniquely determine the Bergman kernel on Ω. That is, for a given Ω, $K(z, \zeta)_\Omega$ can be calculated explicitly (in theory).

Definition 3.9 ([8, Definition 1.4.14])**.** *Take* $z \in \Omega$*. The Bergman metric is defined by*

$$g_{ij}(z) = \frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j} \log K(z, z)$$

Now, as before, if $\gamma\colon [0, 1] \to \Omega$ is smooth, define its arclength by

$$|\gamma|_{B(\Omega)} = \int_0^1 \left(\sqrt{\sum_{i,j=1}^{n} g_{ij}(\gamma(t))\gamma_i'(t)\overline{\gamma_j'(t)}}\right) dt$$

And again as before, for $\zeta, \xi \in \Omega$, define their Bergman length to be

$$d_{B(\Omega)}(\zeta, \xi) = \inf_{\gamma\colon \gamma(0)=\zeta, \gamma(1)=\xi, \gamma\in C^1} |\gamma|_{B(\Omega)}$$

The Bergman metric is important because it turns out to be the natural generalization of the Poincare metric in the sense that biholomorphic maps preserve distance on it, as evidenced by the following proposition:

Proposition 3.8 ([8, Proposition 1.4.15])**.** *Biholomorphic maps are isometries of the Bergman metric. That is, let* $\Omega_1, \Omega_2 \subseteq \mathbb{C}^n$ *be regions, and* $f\colon \Omega_1 \to \Omega_2$ *be biholomorphic. Then for all* $\zeta, \xi \in \Omega_1$*,*

$$d_{B(\Omega_1)}(\zeta, \xi) = d_{B(\Omega_2)}(f(\zeta), f(\xi))$$

Example 3.10 ([8, Proposition 1.4.24])**.** *The Bergman kernel on the polydisc is defined by*

$$K(z, \zeta) = \frac{1}{\pi^n}\prod_{j=1}^{n}\frac{1}{(1 - z_j\bar{\zeta}_j)^2}$$

Example 3.11 ([8, Proposition 1.4.23]). *The Bergman metric for the ball* $B(0,1) \subset \mathbb{C}^n$ *is:*

$$g_{ij}(z) = \frac{n+1}{(1-|z|^2)^2}\left((1-|z|^2)\delta_{ij} + \bar{z}_i z_j\right)$$

Restricting to $n = 1$, we see that the Berman metric on the unit disc is then

$$g_{ij}(z) = \frac{2}{(1-|z|^2)^2}$$

which as we saw before is the Poincare metric.

4 The Riemann Mapping Theorem for $n > 1$

4.1 Preliminaries

Theorem 4.1 (Riemann Mapping Theorem). *If* B *is an open simply-connected subset of* $\mathbb{C}$*, then it is biholomorphic to the unit disc* $D = \{z : |z| < 1\}$.

A standard proof is given in [3].

We are now prepared to go over the fact that the polydisc and the n-ball are not biholomorphic for $n > 1$.

Claim 1. *The Riemann Mapping Theorem does not hold in n dimensions. We provide an example.*

Theorem 4.2. *There is no map* $\phi : D(0,1) \to B(0,1) \subseteq \mathbb{C}^2$ *that is biholomorphic.*

4.2 A Proof by Krantz

Proof. This proof is from [6, p. 54]. The proof is by contradiction. The proof holds for $\mathbb{C}^2$, although it generalizes easily to arbitrary n. Let D^2 denote the unit 2-polydisc and B^2 denote the unit 2-ball.

Suppose there is such a biholomorphism $\phi : D^2 \to B^2$. For convenience, we will want to have $\phi(0,0) = (0,0)$. This is easy to ensure. We are guaranteed that there is a unique $a \in D^2(0,1)$ such that $\phi(a) = 0$, by the biholomorphicity of ϕ. However, we can always transform each D into itself by means of a Mobius transformation; so, we can find a Mobius transformation M such that $M(0) = a$. Then compose M with ϕ, and we might as well call this new function ϕ.

Now we would like to examine geodesics in the Bergman metric, as biholomorphisms are isometries of the Bergman metric - geodesics in the Bergman metric are mapped to geodesics under biholomorphic maps (Lemma 3.8). On B^2, by symmetry, the geodesics are $t \mapsto t\beta$, $\beta \in \partial B$.

On D^2, by symmetry, the geodesic from 0 in the direction $(1,1)$ is $t \mapsto (t,t)$. The Cartesian product of two rotations - $(z_1, z_2) \mapsto (e^{i\theta_1}z_1, e^{i\theta_2}z_2)$ - is obviously biholomorphic, so it carries geodesics on D^2 to geodesics on D^2. So, we just map $(1,1)$ to $(e^{i\theta_1}, e^{i\theta_2})$ and we see that the geodesic from 0 to that point is $t \mapsto (te^{i\theta_1}, te^{i\theta_2})$. Let $\alpha = (e^{i\theta_1}, e^{i\theta_2}) \in \partial D^2$.

We want to show that $\lim_{t\to 1^-} \phi(t\alpha)$ exists. But ϕ takes geodesics to geodesics, so $\phi(t\alpha)$ is a geodesic on the ball; we just saw that the geodesics on the ball are those of the form $t \mapsto t\beta$. Moreover, ϕ is an isometry of of the Bergman metric, so if $t_1\alpha$ is further from the origin than $t_2\alpha$ then $\phi(t_1\alpha)$ is also further from the origin than $\phi(t_2\alpha)$. So, the limit does in fact exist, and also must lie on ∂B.

One implication of this step is that points in ∂D^2 map to points in ∂B. Let us compose ϕ with a rotation of the ball so that $\phi'(0,1) = (0,1)$. We may as well call this new function ϕ.

Now, take $f\colon B \to \mathbb{C}$ be defined by $f(z,w) = (z+1)/2$. f must be holomorphic on B, but it is also holomorphic on a neighborhood of $\overline{B}$. Also, note that $|f| \leq 1$ on $\overline{B}$, with equality in $\overline{B}$ only attained at $(1,0)$.

Now, we take $0 < r < 1$ and consider the integral

$$\frac{1}{2\pi}\int_0^{2\pi} f(\phi(r, re^{i\theta}))d\theta$$

As we concluded earlier, each of the curves $r \mapsto (r, re^{i\theta})$ is a geodesic in D^2, and so $r \mapsto \phi(r, re^{i\theta})$ is a distinct geodesic in B for each distinct θ. Then as $r \to 1$, $f(\phi(r, re^i\theta)) \to f(\phi(1, e^{i\theta}))$. Because of this pointwise convergence and f is bounded, we take the limit as $r \to 1$ and pass the limit inside the integral due to the Dominated Convergence Theorem and obtain

$$\lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} f(\phi(r, re^{i\theta}))d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\phi(1, e^{i\theta}))d\theta$$

Its modulus must be less than 1, because $|f(\phi(1, e^{i\theta}))| < 1$ for $\theta \neq 0$.

However, we can calculate this integral explicitly. Let $(\phi_1, \phi_2) = \phi(z,w)$. See that $f(\phi(r, re^{i\theta})) = (\phi_1(r, re^{i\theta}) + 1)/2$.

So,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(\phi(r, re^{i\theta}))d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\phi_1(r, re^{i\theta}) + 1}{2}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{u_1(r, re^{i\theta}) + v_1(r, re^{i\theta}) + 1}{2}d\theta\end{aligned}$$

Where u_1 and v_1 are the real and imaginary parts of ϕ_1, respectively. Since they are both harmonic in each of the arguments, the mean value property of harmonic functions tells us that this integrates to

$$\frac{u_1(r,0) + v_1(r,0) + 1}{2} = f(\phi(r,0))$$

Taking the limit as $r \to 1$ tells us that this integral should evaluate to $f(\phi(1,0)) = 1$. We just concluded, however, that the modulus of the integral must be less than 1, so we have a contradiction. $\square$

The proof is easily generalized to n dimensions (just stick n everywhere instead of 2). We now present another proof by [1] and a very slick proof by [8].

4.3 A Proof by Bell

Proof. This proof is from [1, Theorem 2.1]. It is again by contradiction.

Suppose there exists $f\colon P(0,1) \to B(0,1)$ that is biholomorphic. As before, we can ensure that $\phi(0) = 0$ by composing it with the appropriate Mobius transformation.

Now, let $u = \det \frac{\partial f_i}{\partial z_j}$ be the determinant of the Jacobian of f, and let $U = \det \frac{\partial f_i^{-1}}{\partial z_j}$ be the determinant of f^{-1}. Now let $\phi \in L^2(B(0,1))$, calculate its norm and change variables:

$$\int_{B(0,1)} |\phi(w)|^2 dV(w) = \int_{P(0,1)} |u(z)|^2 |\phi(f(z))|^2 dV(z) \tag{1}$$

If the integral on the left is finite, the integral on the right is as well and so $\phi \in L^2(B(0,1))$ implies $u \cdot \phi(f(z)) \in L^2(P(0,1))$. Note that these integrals could have been written as inner products on $L^2(\Omega)$.

The next step requires some measure theory which we have not gone over. Suffice to say that Bell [1, Eq. (2.1)] (quickly) shows that for ϕ and ψ in L^2 with “compact support”,

$$\langle u \cdot (\phi \circ f), \psi \rangle_{P(0,1)} = \langle \phi, U \cdot (\psi \circ f^{-1}) \rangle_{B(0,1)}$$

Now we come to the main part of the proof: we would like to show that f must be linear. Plugging in $\phi = 1$, $\psi = z^\alpha$ into (1), we have

$$\begin{aligned} \langle u, z^\alpha \rangle_{P(0,1)} &= \langle 1, U(f^{-1})^\alpha \rangle_{B(0,1)} \\ &= c_0 U(0)(f^{-1}(0))^\alpha \end{aligned}$$

The last step is by Lemma 3.4. But since $f(0) = 0$ by assumption, for $|\alpha > 0|$, $\langle u, z^\alpha \rangle_{P(0,1)} = 0$. Thus u projected against any nonconstant monomial is szero and so in the Taylor expansion all the nonconstant terms must be zero. Since f is biholomorphic, u can't be zero, so it must be a nonzero constant.

We use [1, Eq. (2.1)] again, this time with $\phi = z_i$, $\psi = z^\alpha$:

$$\langle u f_i, z^\alpha \rangle_{P(0,1)} = \langle z_i, U(f^{-1})^\alpha \rangle_{B(0,1)}$$

and again by Lemma 3.4,

$$= c_i \partial U(f^{-1})^\alpha / \partial z_i(0)$$

This is zero if $|\alpha| > 1$. Thus uf_i projected against any monomial z^α, $|\alpha| > 1$ must be zero, so its series expansion must be linear. Since we just concluded that u is a nonzero constant, the expansion of f must be linear and so f is a linear transformation.

However, invertible linear transformations of B map it to ellipsoids. Thus a biholomorphism between B and $P(0,1)$ would seem to imply that the unit polydisc is an ellipsoid, which it is clearly not for $n > 1$. □

4.4 A Proof by Rudin

There is yet another way of doing this proof - which is very slick - but it will require the introduction of a couple more (easy) theorems.

Definition 4.1. *Define the kth iterate of f to be f composed with itself k times as follows:*

$$f^k(z) = f \circ \overset{k}{\cdots} \circ f(z)$$

Lemma 4.3. *Suppose $f : \Omega \to \Omega$ satisfies $f(0) = 0$ is holomorphic function whose power series expansion is convergent in some ball centered at the origin. Suppose further that $f(z)$ has the homogeneous expansion*

$$f(z) = z + \sum_{i=j}^{\infty} F_i(z)$$

(Note that j not necessarily equal to 2).

Then, $f^k(z)$ has the homogeneous expansion

$$f^k(z) = z + kF_j(z) + O(z^{j+1})$$

where $O(z^{j+1})$ is a (haphazard) way to denote the error which contains terms of the form z^α, $|\alpha| \geq j+1$. Because of the uniform convergence of the series we could move the terms around and write it equivalently as $O(z^{j+1} = \sum_{j+1}^{\infty} G_i(z)$; note that the G_i's are not necessarily the same as the F_i's.

Proof. The proof is by induction on k. The base case is clearly true from the way we have set things up. Now suppose it is true for some k. We want to calculate the homogeneous expansion of f^{k+1}.

Now,

$$f^{k+1}(z) = f^k(f(z)) = f(z) + kF_j(f(z)) + \sum_{i=j+1}^{\infty} F_i(z)$$

$$= f(z) + k \sum_{|\alpha|=j} c_\alpha f(z)^\alpha + O(z^{j+1})$$

Note that Cauchy's inequalities [4, Theorem 2.2.7] guarantee us that this error decays faster in modulus than $C/(j+1)!$, where C may depend on f and z.

Now let us examine this sum. We can expand $f(z)$ around 0 and each of the sums will have good enough convergence properties for our purposes. Explicitly,

$$k \sum_{|\alpha|=j} c_\alpha f(z)^\alpha$$

$$= k \sum_{|\alpha|=j} c_\alpha \left(z + \sum_{|\beta|\geq j} c_\beta z^\beta \right)^\alpha$$

We are interested in the terms with order $|\alpha| = j$. In fact, if we look at the partial sums

$$\left(z + \sum_{j \le |\beta| \le m} c_\beta(z) z^\beta\right)^\alpha$$

it is clear that only the lowest order terms in the expansion of this polynomial are of order $|\alpha| = j$, so we rewrite this sum as

$$z^\alpha + O_m(z^{j+1})$$

where $O_m(z^{j+1})$ may depend on m.

However, we can make an estimate on our sum; to be really explicit,

$$\left|\left(z + \sum_{j \le |\beta| \le m} c_\beta(z) z^\beta\right)^\alpha\right| = \left|z + \sum_{j \le |\beta| \le m} c_\beta(z) z^\beta\right|^j$$
$$\le \left(|z| + \left|\sum_{|\beta| \ge j} c_\beta(z) z^\beta\right|\right)^j \le \left(|z| + \frac{C}{(j+1)!}\right)^j$$

where the last inequality is by the estimate on the remainder provided us by Cauchy's estimate. Note that it is not dependent on m. In fact, even when we take the limit as $m \to \infty$, we are guaranteed

$$\lim_{m \to \infty} |z^\alpha + O_m(z^{j+1})| = |z^\alpha + O(z^{j+1})| \le \left(|z| + \frac{C}{(j+1)!}\right)^j$$

This estimate, combined with [2, Theorem 2.77] guarantees us that this is the Taylor expansion of $f(z)^\alpha$ when z is real. By the identity theorem for holomorphic functions this then implies that this is the correct power series expansion of $f(z)^\alpha$ when on the neighborhood we are looking at. This provides the motivation for rearranging the expansion of $f(z)^\alpha$ as

$$z^\alpha + O(z^{j+1})$$

Now we can expand and rearrange the terms in the original expansion of f^{k+1} as follows:

$$f^{k+1} = f(z) + k \sum_{|\alpha| = j} c_\alpha f(z)^\alpha + O(z^{j+1})$$

$$z + \sum_{|\alpha| = j} c_\alpha z^\alpha + O(z^{j+1}) + \left(k \sum_{|\alpha| = j} c_\alpha z^\alpha + O(z^{j+1})\right) + O(z^{j+1})$$
$$= z + (k+1) F_j(z) + O(z^{j+1})$$

where in the last step we have used $\sum_{|\alpha|=j} c_\alpha z^\alpha = F_j(z)$. It is legal for us to stick all these error terms together, as our estimates on the errors along with [2, Theorem 2.77] again guarantee us that they die quickly enough and that this is an expansion of $f^{k+1}(z)$ for z real. Again by the identity theorem we can conclude that this is the correct expansion of f^{k+1} and we are done. □

Theorem 4.4 (Cartan's Uniqueness Theorem [8, Theorem 2.1.1])**.** *Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded connected open set, $f\colon \Omega \to \Omega$ is holomorphic, and there is a $\zeta \in \Omega$ such that $f(\zeta) = \zeta$ and $f'(\zeta) = I$, where I is the identity. Then $f(z) = z$ for $z \in \Omega$.*

Proof. Without loss of generality, suppose $\zeta = 0$ (and naturally that $0 \in \Omega$). Since Ω is open and connected, there is an r_1 such that $z \in \mathbb{C}^n$, $|z| < r_1 \Rightarrow z \in \Omega$, and since it is bounded there is an r_2 such that $|z| > r_2 \Rightarrow \notin \Omega$.

Since f is holomorphic, it has a power series expansion which is convergent inside some radius of convergence; set r_1 so that it is smaller than the radius of convergence (if it isn't already). Thus, f can be expanded homogeneously as

$$f(z) = z + \sum_{i=2}^{\infty} F_i(z)$$

Now we prove by induction on m that $F_m(z) = 0$. Assume that $F_i(z) = 0$ for $2 \le i \le m-1$. The base case $m = 2$ is trivially true. Now by our lemma we just proved, $f^k(z)$ has the homogeneous expansion

$$f^k(z) = z + kF_m(z) + \sum_{j=m+1}^{\infty} c_j F_j(z)$$

Now fix a z so that $|z| < r_1$; by 2.4 we have

$$\frac{1}{2\pi}\int_0^{2\pi} f^k(ze^{i\theta})e^{-im\theta}d\theta = kF_m(z)$$

Now because f^k maps into Ω and thus $|f^k(z)| < r_2$ we can place an estimate on the integral on the left:

$$\left|\frac{1}{2\pi}\int_0^{2\pi} f^k(ze^{i\theta})e^{-im\theta}d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f^k(ze^{i\theta})||e^{-im\theta}|d\theta \le r_2$$

Thus, $|F_m(z)| < r_2/k$; since this holds for all k and for any $|z| < r_1$, we conclude that $F_m(z) = 0$ and thus $f(z) = z$ for all $|z| < r_1$. Since Ω is connected, the identity theorem for holomorphic functions implies that $f(z) = z$ for all $z \in \Omega$.

□

Theorem 4.5 ([8, Theorem 2.1.3])**.** *Suppose Ω_1 and Ω_2 are circular regions in $\mathbb{C}^n$ which both contain the origin, Ω_1 is bounded, and $f\colon \Omega_1 \to \Omega_2$ is biholomorphic and fixes the origin. Then, f is a linear transformation.*

Proof. Let $g = f^{-1}, A = f'(0)$. By the chain rule, $g(f(0))' = g'(0)f'(0) = Ag'(0)$. Since $g(f(z)) = z$, $Ag'(0) = I$ so $g'(0) = A^{-1}$.

Now fix a real θ and let $h\colon \Omega_1 \to \Omega_1$ be defined as $h(z) = g(e^{-i\theta} f(e^{i\theta} z))$. Since Ω_1 and Ω_2 are both circular, h is defined for all $z \in \Omega_1$ and since it is the composition of products of holomorphic functions it is also holomorphic.

Since $h(0) = g(f(0)) = 0$ and $h'(0) = g'(0)f'(0) = I$, it satisfies the conditions of the Cauchy Uniqueness Theorem, and so $h(z) = z$. Now,

$$e^{i\theta} f(z) = e^{i\theta} f(g(e^{-i\theta} f(e^{i\theta} z))) = f(e^{i\theta} z)$$

This holds for every $z \in \Omega_1, \theta \in \mathbb{R}$. Then in the power series expansion, we require that $e^{i\theta} c_\alpha z^\alpha = c_\alpha z^\alpha e^{i|\alpha|\theta}$ (in multi-index notation, $(e^{i\theta})^\alpha = e^{i\alpha_1\theta} e^{i\alpha_2\theta} \cdots e^{i\alpha_n\theta} = e^{i(\alpha_1+\cdots+\alpha_n)\theta} = e^{i|\alpha|\theta}$). Clearly then c_α must be zero for all $|\alpha| > 1$ and f must be linear. □

Theorem 4.6. *Suppose $\Omega \subset \mathbb{C}^n$ is a circular region which contains the origin, and there is a biholomorphism f which takes it to B. Then there is a linear transformation of $\mathbb{C}^n$ which maps B onto Ω*

Proof. Set $\alpha = f^{-1}(0), g(z) = f(\phi_\alpha(z))$ Then $g(0) = 0$ and g is a biholomorphic from B to Ω. By the Cauchy Uniqueness Theorem g is linear. □

Now we can disprove the Riemann mapping theorem again.

Rudin's proof. From [8, Corollary, p.27]. A biholomorphic map between the ball and the polydisc would be linear as we just proved. As we noted before, a linear transformation cannot map the ball to the polydisc. □

4.5 Dessert

While the Riemann mapping theorem itself does not hold in n dimensions, a weaker theorem is true.

Theorem 4.7 (Almost Riemann n-Mapping Theorem [1, Theorem 6.1])**.** *Every bounded domain Ω in $\mathbb{C}^n$ which has a transitive automorphism group and a C^2 boundary is biholomorphic to the unit n-ball*

To have a transitive automorphism group means that, for any two points $z, w \in \Omega$, there exists an automorphism $f\colon \Omega \to \Omega$ such that $f(z) = w$.

Proof. It's all French to me. See [7]. □

Remark Polydiscs do not have transitive automorphism groups.

5 Conclusion

Sadly, there was neither time nor space to give a full and detailed exposition of the basic results in multvariable complex analysis. However, there are many interesting results which seem to bring together complex analysis and geometry; [5] is one example which goes over such results. We hope to have presented a compelling exposition of this interesting result, and that we have sparked the reader's interest.

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