A Case of Hyperbolic Billiards

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Introduction

The classic game of billiards has provided physics with an ingenious device to study chaotic behavior. In the scientific community, billiards is a dynamical system that is extremely simple to describe, and yet can lead to very complex behavior. Take for example the system described by Leonid Bunimovich and Gainluigi Del Mango in their paper entitled "Track Billiards." This paper describes a class of planar billiards easily created out of arcs and segments, but with the property that the system has non-zero Lyapunov exponents almost everywhere under proper conditions of the arcs and segments

The main point to this paper is in the creation of a class of hyperbolic billiards, as there exist few examples of exactly described hyperbolic billiards with several ergodic components. In the process of this description, however, the paper proves several related results for systems in $\mathbb{R}^3$ which are directly applicable to small-scale systems. These results can help provide details on the classical background behind many such systems, an area that has been neglected in the past.

Let us note that most of this paper explains, repeats, or derives from the paper [2], written by Bunimovich and Del Mango, whom shall hereafter be referred to as the authors. Their paper first describes motion of billiards, with a definition of planar billiards. They then prove the unidirectionality of the flow in the system by treating the flow of a system in $\mathbb{R}^3$, and using the results from that situation to prove the case in lower dimensions. After this result, the authors estimate and bound the motion in the circular guides of the track, and using these bounds give a bound on the focal lengths of the circular guides. The authors then prove that the system has an eventually strictly invariant measurable cone field, which from the results of [9] proves that the billiards is hyperbolic. The authors then generalize this motion to a similar system in $\mathbb{R}^3$.

Miscellaneous Definitions

Many technical terms appear in this proof and its results, the definitions of which are necessary to truly understand the paper. As an attempt to ease the reader's task of understanding, here is a short list of those terms found in the paper.

Probability Measure

The measure of a set a form of measurement on the set describing its size. For example, on the real line, the integers have zero measure, while the interval $[0, 1]$, has some nonzero measure, depending on the scale. If we also notice that the total probability of any complete set of events is 1, and that the probability of any one event occurring has a value in the interval $[0, 1]$, then we can intuitively define the idea of a probability measure.

Definition. *A measure of a set is a* ***probability measure*** *if the total measure of the set is 1. [4]*

Notice that if any set has a finite measure, then we can create a probability measure on the set from the previous measure through a proper rescaling.

"Almost Always" or "Almost Surely"

The ideas of "almost always" and "always" or "almost surely" and "surely are very closely related. The difference arises in that if something "always" occurs, there is no other possibility of occurrence, while if something "almost always" occurs, it happens with probability 1 [4]. As an example of this difference, if we ignore those people have birthdays on February 29, a person always has a birthday every year, while the current moment will "almost always" not be an integer number of years after your birth.

Definition. *An assertion is* ***"almost surely"*** *true on a set, if it is true for all points on the set, up to a set of zero measure.*

Groups

Groups are quite simply useful in the definitions for several maps of dynamical systems[5].

Definition. *A* ***group*** *is a set of elements for which a law of composition of two elements is well defined, which satisfy the following conditions*

- *If a and b are elements of the set, then so is the composition ab.*
- *Composition is associative.*
- *The set contains an identity element e, such that $ae = ea = 1$ for every element a in the set.*
- *If a is in the set, then the set contains an element b such that $ab = ba = e$. The element b is known as the inverse of a, and is denoted by a^{-1}.*

In addition, there exists another concept known as a semi-group, which is similar to a group with fewer constraints. Namely

Definition. *A* ***semi-group*** *is similar to a group, except that a semi-group need not contain an identity element, or an inverse for each of the elements.*

Dynamical System

The basic idea of a dynamical system is precisely what the name suggests; the study of a system that changes with time. If we have some space M, some non-empty set σ containing subsets of M that is closed under complementation, and finally a probability measure μ on M, we can then define several maps that together form the definition of a dynamical system [3].

Definition. *An* ***automorphism*** *of the measure space (M, σ, μ) is a injective map T of the space M onto itself such that for all $A \in \sigma$ we have $TA, T^{-1} \in \sigma$ and*

$$\mu(A) = \mu(TA) = \mu(T^{-1}A).$$

Further, the measure μ is a invariant measure *for the automorphism T.*

Definition. *An* ***endomorphism*** *of M is a surjective map T from M onto itself such that for any $A \in \sigma$ we have $T^{-1}A \in \sigma$ and*

$$\mu(A) = \mu(T^{-1}A),$$

where $T^{-1}A$ is the inverse image of the set A.

Definition. *Suppose $\{T^t\}$ is a one parameter group of automorphisms of $(M, \sigma, \mu), t \in \mathbb{R}$. Then $\{T^t\}$ is a* ***flow*** *if for any measurable function $f(x)$ on M, the function $f(T^t x)$ is measurable on the Cartesian product $M \times \mathbb{R}$.*

Definition. *Suppose $\{T^t\}$ is a one-parameter semi-group of endomorphisms of (M, σ, μ), $t \in \mathbb{R}^+$. Then $\{T^t\}$ is a* ***semiflow*** *if for any measurable function $f(x)$ on M, the function $f(T^t x)$ is measurable on the Cartesian product $M \times \mathbb{R}^+$.*

We then have that a "dynamical system" is a system under any of the above maps. In addition, the phase space of the system is given by M.

Definition. *Let the* ***orbits****, or trajectories, of a dynamic system be the different paths along which a particle can travel in the phase space. In addition, the progressive evolutions of a particle in continuous time over these different orbits is a* ***flow****, similar to how a person would describe the flow of water.*

Ergodic

Ergodic theory is the study of dynamical systems with invariant measure. More specifically, a main theorem of the theory is the idea that of Birkhoff-Khinchin Ergodic Theorem[3], which essentially states that time means and means along the trajectory exist. In addition, we can describe a dynamical system as ergodic if the system satisfies certain properties.

Definition. *A dynamical system with measure space (M, σ, μ) is said to be* ***ergodic*** *if the measure $\mu(A)$ of any invariant set $A \in \sigma$ equals 0 or 1.*

Lyapunov Exponents

Lyapunov exponents are a method of measuring the predictability of dynamical systems, which is related to the idea of "deterministic chaos." To determine the Lyapunov exponents for a system[1], we look at the linearization of the map that takes one point in the phase space to the next. Let us define the series of matrices representing these transformations as A^+. Let us consider the function $\chi^+(v) : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\chi^+(v) = \chi^+(v, A^+) = \lim_{m \to \infty} \frac{1}{m} \log |A_m v|$$

with the convention that $\log 0 = -\infty$. We then have that $\chi^+(v)$ is the **forward Lyapunov exponent of v**.

In addition, we have that there exists an integer p, $1 \leq p \leq n$ and real numbers $\chi_1^+ < \cdots < \chi_p^4$ such that

- The linear spaces $E_i^+ = \{v \in R^n : \chi^+(v) \leq \chi_i^+\}$ form a filtration

$$\{0\} = E_0^+ \subsetneqq E_1^+ \subsetneqq \cdots \subsetneqq E_p^+ = \mathbb{R}^n.$$

- If $v \in E_i^+ \setminus E_{i-1}^+$, then $\chi^+(v) = \chi_i^+$.
-

The spaces E_i form a filtration of $\mathbb{R}^n$ associated with χ^+. If we denote $k_i^+ = \dim E_i^+ - \dim E_{i-1}^+$, then we have that k_i is the multiplicity of χ_i.

Definition. *The **Lyapunov spectrum** of χ^+ is given by*

$$Sp\ \chi^+ = \{(\chi_i, k_i) : i = 1, \cdots, p\}$$

.

We typically only talk about the largest χ_i, as this is usually the value that determines the large-scale behavior of the system. Typically, if $\chi_p > 0$, then we have that the system becomes chaotic, while if $\chi_p = 0$ then the system remains the same.

Definition. *We say that a system is **(non-uniformly) hyperbolic** if all of its Lyapunov exponents are non-zero almost everywhere on its phase space.*

Billiards

As previously stated, the dynamics governing billiards are extremely simple to describe. To accurately define the system, we need simply to describe the region in which the point particle is contained, the initial position, and the initial velocity, and from this information we can determine the position and velocity for almost all possible states, as the particle obeys the law of reflection.

Definition

Let us define the table of the billiard, Q, to be a bounded domain in $\mathbb{R}^2$ with δQ a finite union of piecewise differentiable curves.

Definition. *The **billiard in** Q is defined [2, 7] to be the dynamical system describing the motion of a point particle contained within the table, Q, with the motion constrained by the following rules:*

- *The particle moves along a straight line at unit speed until the particle collides with ∂Q at a point q_0,*
- *At q_0, the particle reflects with the the angle of incidence equal to the angle of reflection. Equivalently, decompose the velocity of the particle into components normal and tangent to ∂Q at q_0; after the collision, the tangential component of the velocity remains constant while the tangential component changes sign.*

Notice that in the above definition, the billiard system does not have a definition for collision incident at corners or cusps of the boundary; at these points the motion of the particle terminates and the billiard ends.

From these simple rules, all future motion of the particle has been determined.

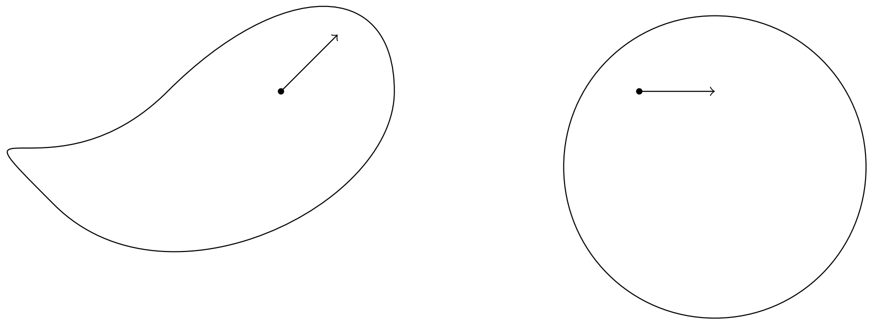

Figure 1: Two examples of billiards

Billiard Maps

If we simply wish to discus the dynamics of the system in discrete time intervals, as the authors do, we can look at the sequence of locations in which the particle collides with the boundary, and the resultant velocities after each collision. For this we will want to determine the mapping of the billiards called the billiard map, which takes one collision to the next.

To find this map, let M be the set of all vectors $(q, v) \in \mathbb{R}^2$, for which $q \in \partial Q$, and $\langle v, n(q) \rangle \geq 0$, where $\langle ., . \rangle$ is the standard dot product in $\mathbb{R}^2$, and $n(q)$ is the normal of ∂Q oriented toward the interior of Q. Notice that M is a smooth manifold with boundary. Additionally, if we interpret q as the position of the particle and v as the velocity, M represents all possible states of the particle after the collision with the boundary. As an ease to notation, define the projection $\pi(q, v) \equiv q$ for $(q, v) \in M$.

Next let us fix an orientation of ∂Q and define set of local coordinates of M by $M \ni x \mapsto (s(x), \theta(x))$, where s is the arclength parameter along the oriented boundary, and $0 \leq \theta\pi$ is the angle between the velocity and the oriented tangent of the boundary. For the remainder of this paper, elements of M will be noted as either $x = (q, v)$ or $x = (s, \theta)$. Let us now place a Riemmannian metric $ds^2 + d\theta^2$ and the probability measure $d\mu = (2|\partial Q|)^{-1} \sin\theta ds d\theta$ on M, with $|\partial Q|$ being the arclength of the boundary. Next we let ∂M be the set of all vectors $(q, v) \in M$ such that either $\langle v, n(q) \rangle = 0$ or q is the endpoint of a straight segment of ∂Q. We can then define $\operatorname{int} M = M \setminus \partial M$. We now have enough information to define the billiard map.

Definition. *The **billiard map**, $T : \operatorname{int} M \to M$, is the transformation given by $(q, v) \mapsto (q_1, v_1)$, where (q, v) and (q_1, v_1) are consecutive collisions of the particle.*

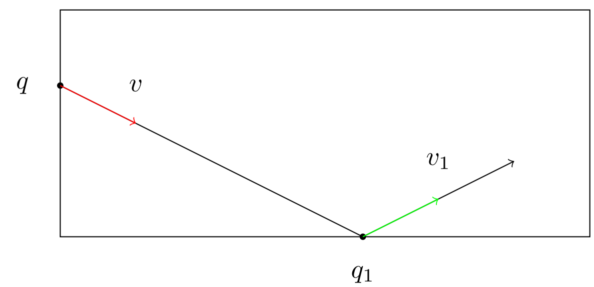

Figure 2: An example of a billiard mapping from (q, v) to (q_1, v_1)

Denote by S_1^+ the union of ∂M and the subset of M where T is not differentiable. Notice that T is differentiable for all points other than those that take int M to ∂M, as everything is well defined elsewhere. We then have that $S_1^+ = \partial M \cup T^{-1}\partial M$. Using general results of [6], we have that S_1^+ is a compact set consisting of finitely many smooth compact curves that can only intersect each other only at their endpoints. Notice that S_1^+ is a set of zero measure. Let us now define the set $S_1^- = M \setminus T(M \setminus S_1^+)$, then we have that T is a diffeomorphism (or a differentiable isomorphism with differentiable inverse) from $M \setminus S_1^+$ to $M \setminus S_1^-$, under which the measure is preserved.

Note that the billiard map T is time-reversible, as the only difference under time reversal is that $v \mapsto -v$ while q remains the same. We then have that the time reversal map $- : M \to M$ defined by $-(s, \theta) = (s, \pi - \theta)$ for all $(s, \theta) \in M$. Note that $- \circ T = T^{-1} \circ -$ everywhere on $M \setminus S_1^+$, and so the time reversal of the inverse is equal to the inverse of the time reversal, which makes sense.

Finally, for $n > 1$, define $S_n^+ = S_1^+ \cup T^{-1}S_1^+ \cup \cdots \cup T^{-n+1}S_1^+$ and $S_n^- = S_1^- \cup T^{-1}S_1^- \cup \cdots \cup T^{-n+1}S_1^-$. Essentially, we have that these sets are all of those for which T is not differentiable n times. Now when we take the limit as n approaches infinity, we create the sets S_∞^+ and S_∞^- for which we cannot apply T an infinite number of times (i.e. the map eventually takes the point to ∂M). We can also define $\tilde{M} = M \setminus S_\infty^+ \cup S_\infty^-$. Notice that T continues to preserve measure, and so we have that $\mu(S_\infty^+) = \mu(S_\infty^-) = 0$, while $\mu(\tilde{M}) = 1$.

As an example, let us look at the billiard system with a circular table and the particle starting along the boundary at a position in the phase space $x = (0, \pi/3)$. We then have that $T(x) = (2\pi/3, \pi/3)$, as the point in the direction $\pi/3$ is at the point $s = 2\pi/3$. Notice as well that the angle of incidence for this point is also $\pi/3$, and so the next collision is the same as the collision just described, except with a change in the arclength $\Delta s = 2\pi/3$. We then have that $T^2(x) = (4\pi/3, \pi/3)$. For the third collision, we again have the same situation, and so we have $T^3(x) = (2\pi, \pi/3) = x$, and so we have that this is a periodic orbit. We actually have from [7] that for *any* circular billiard, the θ for all collisions will be the same as the initial θ, and moreover, if $\theta = \pi q$, $q \in \mathbb{Q} \cap (0, 1)$, the motion will be periodic.

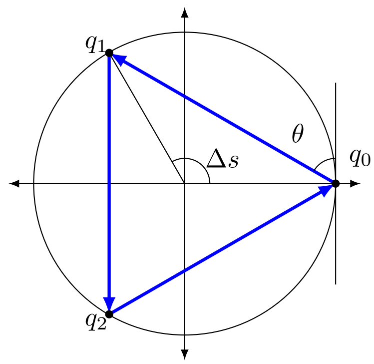

Figure 3: The billiard mapping from the example

Track Billiards

The authors define track billiards to be a subset of billiards in which the table, Q, is of a special type, namely a tubular neighborhood around a special type of Jordan curves. However, the authors provide another definition of tracks that does not require a discussion of topology, and the definition also provides an easier way of visualizing the track.

Definition

This alternate definition of track billiards revolves around two basic building blocks, namely

Definition. *A* ***circular guide*** *is a region of an annulus with circles of radii* $r_1 > r_2 > 0$ *contained within a sector with central angle* $0 < \alpha < 2\pi$*. In addition, a* ***straight guide*** *is a rectangle with transverse width* w*.*

With these two guides, we can then define a track.

Definition. *A* ***track*** *is a compact, connected region formed from a finite union of straight and circular guides in which all circular guides are separated by at least one straight guide. In addition, there exists a number,* δ*, such that for all straight guides in the track,* $w = \delta$*, and for all circular guides* $r_1 - r_2 = \delta$*. Finally, notice that other than along the cross-sectional boundary connecting two consecutive guides, no two of the guides can intersect.*

We shall define Q to be a track for the rest of this paper, unless otherwise specified.

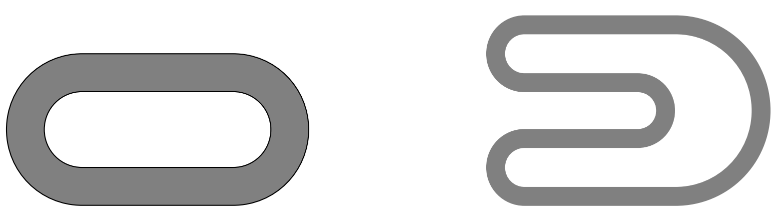

Figure 4: Two examples of tracks

Billiards in a Circular Guide

As we study the characteristics of track billiards, we will obviously want to understand the properties of the collisions at all points within the track. Notice that the movement of a point particle within a straight track is rather obvious; the particle simply collides off opposite walls with a constant velocity tangent to the direction of the track. The motion of the particle within a circular guide, however, is not as easy to describe. The curved boundary alters the direction of the particle, and the interior boundary prevents a generalization to that of a simple circular boundary.

To discover the motion within a circular guide, let us consider a guide with outer and inner radii $r_1 = 1$ and $0 < r_2 = r < 1$, respectively. Notice that under the proper rescaling, all circular guides can be represented as such. We shall denote by M_1 the set of all collisions (q, v) such that q belongs to the outer circle. In other words, we define M_1 to be the set of collisions incident upon the outer guide. Now notice that the path of a particle leaving from a point $x = (s, \theta) \in M_1$ is tangent to the full circle containing the inner arc of the guide if and only if $\theta(x) \in \{\bar{\theta}, \pi - \bar{\theta}\}$, where $\bar{\theta}$ is defined as $\bar{\theta} = \cos^{-1} r \in (0, \pi/2)$. Finally, let us define the set $D_1 = \{(s, \theta) : (s, \theta) \in M_1$ and $\theta \notin \{0, \bar{\theta}, \pi - \bar{\theta}, \pi\}\}$ such that D_1 is the set of all collisions off the outer arc that are tangent to neither of the full circles containing the inner or outer arcs.

As a matter of notation, let us say that a collision $x \in D_1$ "leaves" the guide if the last collision of a particle with state x on the outer arc occurs at q; i.e., we state that a collision leaves the guide if it never again collides the outer arc before leaving the guide. We also say that a collision $x \in D_1$ is "entering" the guide if $-x$ is leaving the guide, precisely the intuitive definition for entering the guide. Now let us denote the number of times a particle with initial state $x \in D_1$ collides with the outer arc before leaving the guide by $n_1(x) \geq 0$.

To study the motion in the guide, let us now turn our attention to the map $T_1 : D_1 \to D_1$ that takes a collision on the outer circle to the next collision on the outer circle, or takes a collision x to itself if $n_1(x) = 0$. More precisely, if we denote by $\delta(\theta)$ half the central angle of the arc bounded by two consecutive collisions of the particle with the outer circle, we have for all $(s, \theta) \in D_1$ that

$$T_1(s, \theta) = \begin{cases} (s + 2\delta(\theta), \theta), & \text{if } n_1(x) > 0, \\ (s, \theta), & \text{if } n_1(x) = 0. \end{cases}$$

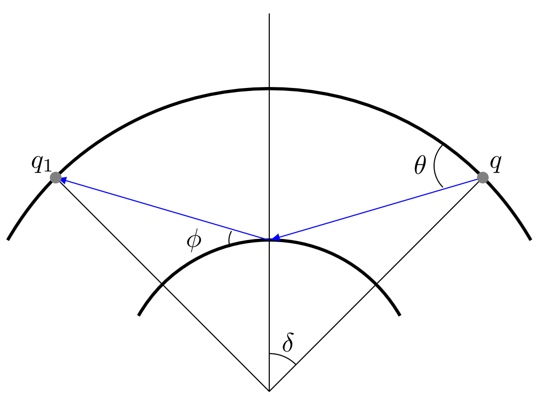

Figure 5: A diagram of the map T on a circular guide

Note that for $\theta \in (0, \bar{\theta}) \cup (\pi - \bar{\theta}, \pi)$, we have that the $2\delta(\theta)$ is the inner angle of the isosceles triangle with vertices x, $T(x)$, and the center of the circle. Notice that the two outer angles can be denoted by $\pi/2 - \theta$, and so we have that $2\delta(\theta) = \pi - (\pi/2 - \theta) - (\pi/2 - \theta) = 2\theta$, and so we have that $\delta(\theta) = \theta$ in this range.

Now for $\theta \in (\bar{\theta}, \pi - \bar{\theta})$, we have that the triangle formed from the collision at x, the resultant point of contact along the inner arc and the center of the circles has angles $\delta(\theta)$, $\phi(\theta) + \pi/2$, and $\pi/2 - \theta$, where $\phi(\theta)$ is the angle of the collision with the inner arc. We then have that $\pi = \delta + \phi + \pi/2 + \pi/2 - \theta \Rightarrow \delta = \theta - \phi$. Realize as well that we can relate θ and ϕ by looking at the geometry of the system and equivalent triangle, and so we have that $\cos\theta = r\cos\phi$. We then have that for θ in this range, that $\delta = \theta - \cos^{-1}(\cos\theta/r)$. Altogether, we have that the function for δ is given by

$$\delta(\theta) = \begin{cases} \theta - \cos^{-1}\left(\frac{\cos\theta}{r}\right), & \theta \in \left(\bar{\theta}, \pi - \bar{\theta}\right), \\ \theta, & \theta \in (0, \bar{\theta}) \cup (\pi - \bar{\theta}, \pi). \end{cases}$$

And since $\delta(\theta)$ is a differentiable function on $(0, \pi) \backslash \{\bar{\theta}, \pi - \bar{\theta}\}$, we have

$$\delta'(\theta) = \begin{cases} 1 - \frac{\sin\theta}{\sqrt{r^2 - \cos^2\theta}}, & \theta \in \left(\bar{\theta}, \pi - \bar{\theta}\right), \\ \theta, & 1 \in (0, \bar{\theta}) \cup (\pi - \bar{\theta}, \pi). \end{cases}$$

Now realize that $\delta'(\theta) \to -\infty$ as $\theta \to \bar{\theta}^+$, and $\delta'(\theta) \to \infty$ as $\theta \to (\pi - \bar{\theta})^-$. As an ease of notation, let us define $\delta(x) = \delta(\theta(x))$ and $\delta'(x) = \delta(\theta(x))$ for all $x \in D_1$.

From these definitions and observations, we now have that for all $x \in D_1$ that

$$D_x T_1^{n_1(x)} = \begin{pmatrix} 1 & 2n_1(x)\delta'(x) \\ 0 & 1 \end{pmatrix}.$$

Unidirectionality

The unidirectionality property of a billiard track is precisely what the intuitive idea for it would be; namely that a particle traveling in one direction down the track will continue to travel down the same path. More precisely, we have:

Definition. *A billiard in a track Q has the **unidirectionality** property if every billiard trajectory not contained within a cross section of Q moves through every cross section of Q with the same direction.*

In the proof of this property for all tracks, we will look at a specific type of tubular domain in $\mathbb{R}^3$, from which we can then simplify to $\mathbb{R}^2$.

Let us define the "skeleton" or "backbone" of the tubular neighborhood $\tilde{Q}$ as a closed, simple curve $\phi : S^1 \to \mathbb{R}^3$ parameterized by the arclength, s. Let us assume that ϕ is piecewise C^2, or more precisely, we have that there exist $a_0 < b_0 = a_1 < b_1 < \cdots < b_{n-1} = a_n < b_n = a_0$ such that $S^1 = \cup_{1 \le i \le n} [a_i, b_i]$, and then $\phi \in C^2$ on each interval $[a_i, b_i]$. Let us also assume that the curvature on each interval $[a_i, b_i]$ is either identically zero - a straight segment - or never zero. In the case for the non-zero curvature, for each $s \in [a_i, b_i]$, let $\{T(s), N(s), B(s)\}$ be the Frenet frame of ϕ, where $T(s), N(s)$, and $B(s)$ are the tangent, normal, and binormal vectors of ϕ at $\phi(s)$. In the case for the identically zero curvature, let us define $T(s)$ to be the tangent of the curve, and further define $N(s)$ and $B(s)$ to be some fixed basis such that the set $\{T(s), N(s), B(s)\}$ is an orthonormal basis of $\mathbb{R}^3$. In addition, let the cross section of $\tilde{Q}$, Ω, be a compact, convex subset of $\mathbb{R}^2$ whose boundary is a piecwewise regular simple closed curve $\zeta : S^1 \to \mathbb{R}^2$, with $|\zeta(\alpha)| > 0$ for all $\alpha \in S^1 = [0, 2\pi)$.

Definition. *The* ***tubular neighborhood*** *of ϕ with cross section Ω is the domain $\tilde{Q}$ bounded by the surface $\psi(s, \alpha) = \phi(s) + F(s)\zeta(\alpha)$, with $(s, \alpha) \in S^1 \times S^1$, where F is the 3×2 matrix with column vectors given by $N(s) and B(s)$.*

We assume that a tubular neighborhood, $\tilde{Q}$, does not intersect itself, or more specifically that the diameter of Ω is sufficiently small such that the map $\Phi : S^1 \times \Omega \to \tilde{Q}$ given by $(s, p) \mapsto \phi(s) + p$ is a diffeomorphism. For our case, we can assume that $\max_\alpha |\zeta(\alpha)| < (\max_s |\kappa(s)|)^{-1}$, where $\kappa(s)$ is the curvature of $\phi(s)$.

Proposition 1. *Consider the tubular neighborhood $\tilde{Q}$ of $\mathbb{R}^3$, and assume that its cross section is a circular disk, or that each curve $\phi([a_i, b_i])$ is planar. Then the billiard inside $\tilde{Q}$ has the unidirectionality property.*

Proof. Let $s \in [a_i, b_i]$ and $\alpha \in [0, 2\pi)$. We then have that the vectors $\delta_s\psi = T(s) + F'(s)\zeta(\alpha)$ and $\partial_\alpha\psi = F(s)\zeta'(\alpha)$ span the tangent plane of ∂Q at $\psi(s, \alpha)$. Let us consider the vector $\tilde{n}(s, \alpha) = \partial_s\psi \wedge \partial_\alpha\psi$. If we recall that the derivatives of the Frenet frames are

$$T'(s) = \kappa(s)N(s) \qquad N'(s) = -\kappa(s)T(s) + \tau(s)B(s) \qquad B'(s) = -\tau(s)N(s)$$

where κ and τ are the curvature and the torsion elements of ϕ, respectively. We then have that

$$\begin{aligned}
\tilde{n}(s\alpha) &= \partial_s\psi \wedge \partial_\alpha\psi \\
&= (T(s) + F'(s)\zeta(\alpha)) \wedge (F(s)\zeta'(\alpha)) \\
&= T(s) \wedge F(s)\zeta'(\alpha) + F'(s)\zeta(\alpha) \wedge F(s)\zeta'(\alpha) \\
&= T(s) \wedge F(s)\zeta'(\alpha) + [(N'(s))\,(B'(s))]\,\zeta(\alpha) \wedge F(s)\zeta(\alpha) \\
&= T(s) \wedge F(s)\zeta'(\alpha) + [(-\kappa(s)T(s) + \tau(s)B(s))\,(-\tau(s)N(s))]\,\zeta(\alpha) \wedge F(s)\zeta(\alpha) \\
&= T(s) \wedge F(s)\zeta'(\alpha) - T(s)\zeta_1(\alpha) \wedge F(s)\zeta(\alpha) - FJ\zeta(\alpha) \wedge F(s)\zeta(\alpha) \\
&= (1 - \kappa(s)\zeta_1(\alpha))T(s) \wedge F(s)\zeta'(\alpha) - FJ\zeta(\alpha) \wedge F(s)\zeta(\alpha)
\end{aligned}$$

where $\zeta_1(\alpha)$ is the component of ζ in the N direction and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since ζ is piecewise regular, it follows that $\tilde{n}(s, \alpha)$ is parallel to the normal line to ∂Q through $\psi(s, \alpha)$. We then simply have that $\langle \tilde{n}(s, \alpha), T(s) \rangle = -\tau \langle \zeta(\alpha), \zeta'(\alpha) \rangle$. By our hypothesis on the structure of the region, we have that either $\tau(s) = 0$, or $\langle \zeta(\alpha), \zeta'(\alpha) \rangle = 0$, and so we have that $\langle \tilde{n}(s, \alpha), T(s) \rangle = 0$, and so the plane orthogonal to $T(s)$ is orthogonal to boundary to $\tilde{Q}$. We therefore have that the set $N = \{(q, v) \in M : \langle v, n(q) \rangle = 0\}$ is invariant. In other words, we have that if set of collisions where the resultant velocity is parallel to the boundary is invariant.

Now let us consider the billiard flow $t \mapsto (q(t), v(t))$ inside $\tilde{Q}$. Let $v_*(t) = \langle v(t), T_*(t) \rangle$, where $T_*(t) = T(s(t))$ and $s(t)$ is given by $(s(t), p(t)) = \Phi^{-1}(q(t))$. Let us suppose that the particle motion is defined for $t \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$, and on this time interval, only one collision occurs at

$t = 0$. We claim that v is continuous on the time interval. Let us first note that T_* is continuous on $(-\varepsilon, \varepsilon)$ and that v is continuous on the intervals $(-\varepsilon, 0) \cup (0, \varepsilon)$. We therefore need only prove that v is continuous at $t = 0$. To show this continuity we simply need to show that the limits in the approach toward zero exist on both sides, and that both sides of the limit agree. Notice that the velocities are constant when not in a collision, and so the limits both exist. Now by the reflection law, we have that $v(0^+) = v(0^-) + 2\langle v(0^-), n(q)\rangle n(q)$. Since we have $\langle n(q(0)), T_*(0)\rangle$, we have that $\langle v(0^+), T_0(0)\rangle = \langle v(0^-), T_*(0)\rangle$, and so the claim is true.

We are now ready to show the unidirectionality of $\tilde{Q}$. First note that the invariance of N implies that if $v_0(\bar{t}) = 0$ for some billiard orbit and $\bar{t} \in \mathbb{R}$, then v_* is identically zero along the orbit. Now let us suppose that $\tilde{Q}$ does not have the unidirectionality property. We can then find a billiard orbit such that $v_*(t_1)v_*(t_2) < 0$ for some $t_1 < t_2$. Since v_* is continuous, there exists $t_1 < \bar{t} < t_2$ for which $v_0(\bar{t}) = 0$. By our previous observation, it follows that $v_0 \equiv 0$, which contradicts $v_*(t_1)v_*(t_2) < 0$, and so we have that $\tilde{Q}$ has the unidirectionality property. $\square$

We can then easily apply this result to the two-dimensional case.

Corollary 2. *A billiard in a track $Q \in \mathbb{R}^2$ has the unidirectionality property.*

Proof. Let us embed $Q \in \mathbb{R}^3$, or identify $\mathbb{R}^2$ with some plane $R \subset \mathbb{R}^3$, and denote $\tilde{Q}$ by the tubular neighborhood of γ in $\mathbb{R}^3$ with circular section Ω. Clearly, we have that $Q = \tilde{Q} \cap P$. Then let us choose th parameterization ζ of the circle $\partial\Omega$ so that $F(s)\zeta(0) \in P$. If γ is a straight segment, we define $N(s)$ to lie in the plane P. Now let $\tilde{n}$ be as in the proof of proposition 1. This corollary will be proved once it is shown that $n(s, 0) = \lambda N(s)$ for some $\lambda \neq 0$. Notice that from the equation for $\tilde{n}$ determined in 1 and from the fact that $\tau \equiv 0$, we immediately have that $-(1 - \zeta_1(0)\kappa(s))N(s)$. Notice that we have from assumptions that $|\zeta_1(0)\kappa(s)| < 1$, and so we have the proof. $\square$

Hyperbolicity

Track billiard is obviously a well defined dynamic system. We now look at a special type of track billiard, and show that its behavior is hyperbolic. These special tracks are composed of certain types of circular guides being kept a sufficient distance apart.

Definition. *A circular guide is of **type A** if $\alpha \geq \pi$ with no restriction upon r_1 or r_2, where α is the length of the central angle and r_1 and r_2 are the outer and inner radii, respectively. A circular guide is of **type B** if $r_2/r_1 < 1/2$ with no restriction upon α.*

In addition, let us scale the system of measure in such a way that $r_1 = 1$ and $r_2 = r < 1$.

Estimation of the Mapping

The value of D_xT_1 from the dynamics in the circular guide plays a large role in the hyperbolicity of this track. Notice that this is a well defined function for all $x \in D_1$, where D_1 is as defined in the section on billiard dynamics in circular guides, but we must partition D_1 into the several cases that can arise.

Definition. *Let $\mathbf{E_1} = \{x \in D_1 : x$ is entering and $T_1^{n_1(x)}$ is leaving $\}$. The set E_1 can be partitioned as follows $E_1 = E_0 \cup E_+ \cup E_-$, where*

- $E_0 = \{x \in E_1 : n_1(x) = 0\}$, *or the set containing all those collisions that enter the guide, collide once, then leave,*
- $E_+ = \{x \in E_1 \setminus E_0 : \theta(x) \in (\bar{\theta}, \pi - \bar{\theta})\}$, *or the set for which the particle hits both the outer and inner guides, and*
- $E_- = \{x \in E_1 \setminus E_0 : \theta(x) \in (0, \bar{\theta}) \cup (\pi - \bar{\theta}, \pi)\}$, *or the set for which the particle only collides with the outer guide.*

In addition, for each $x \in E_1$, let us define the functions

$$\omega(x) = \alpha - 2n_1(x)\delta(x) \qquad\qquad \chi(x) = 2n_1(x)\delta'(x).$$

Quickly note that since $2n_1(x)\delta$ takes an entering collision out of the arc of radius α in steps of 2δ, we have that $0 \le \omega(x) < 2\delta(x)$.

Now, let us look at the behavior of these functions

Lemma 3. *If $x \in E_0 \cup E_-$, then $\chi(x) = 2n_1(x)$.*

Proof. Note that for $x \in E_0 \cup E_-$, we have that $\delta'(x) = 1$, and so

$$\chi(x) = 2n_1(x)\delta'(x) = 2n_1(x).$$

□

Lemma 4. *Consider a circular guide of type A. There exists a function $\chi_A = \chi_A(r, \alpha)$ non-increasing in r such that $\chi(x) \le \chi_A$ for all $x \in E_+$.*

Proof. Note that the symmetry of the guide allows us to only prove the lemma for $x \in E_+$ such that $\theta(x) \in (\bar{\theta}, \pi/2)$. For these value of x, we have that

$$\delta'(x) = 1 - \frac{\sin\theta}{\sqrt{r^2 - \cos^2\theta}} < 0.$$

And notice that $\delta'(x)$ is differentiable on this region as well, and so

$$\delta''(x) = \frac{\cos\theta}{(r^2 - \cos^2\theta)^{3/2}}(1 - r^2)$$

And note that this is positive for all $\theta(x) \in (\bar{\theta}, \pi/2)$. In addition, since $\delta'(x) \to \infty$ as $\theta \to \bar{\theta}^+$, there exists a $\vartheta \in (\bar{\theta}, \pi/2)$ such that $\chi(x) < -3$ for all $x \in E_+$ where $\theta(x) \in (\bar{\theta}, \vartheta]$.

Now let us look at the case for $x \in E_+$, and $\theta(x) \in (\vartheta, \pi/2)$. Notice that we can write $\delta(\theta) = -\delta'(\theta)\Delta\theta$, where $\Delta\theta$ is the length of the segment along the θ-axis who's endpoints are θ, and the intersection of the axis with the tangent of the graph of $\delta(\theta)$ evaluated at θ. Since we have that $\delta(\theta)$ is strictly convex, and since $\delta(\pi/2) = 0$, we have that $0 < \Delta\theta < \pi/2 - \theta$ for all $\theta \in (\vartheta, \pi/2)$. Therefore,

$$\frac{\delta'(\theta)}{\delta(\theta)} = -\frac{1}{\Delta\theta} < -\frac{2}{\pi - 2\theta}.$$

Notice that $\alpha - \omega(x) = 2n_1(x)\delta(x) > 0$, as $n_1(x) > 0$ and $\delta(x) > 0$. We therefore have that

$$\chi(x) = 2n_1(x)\delta'(x) = 2\frac{\omega(x)-\alpha}{2\delta(x)}\delta'(x) = (\alpha-\omega(x))\frac{\delta'(x)}{\delta(x)} < -2\frac{\alpha-\omega(x)}{\pi-2\theta(x)}$$
$$\chi(x) < -2\frac{\alpha-2\delta(x)}{\pi-2\theta(x)}\theta(x) < -2\frac{\alpha-2\theta}{\pi-2\theta} < -2 \qquad \text{for } \in (\vartheta, \pi/2)$$

If we then define $h(\alpha,\theta) = -2(\alpha - 2\delta(\theta))/(\pi - 2\theta(x))$, we have that $\partial_\alpha h < 0$ and $h < -2$. Thus, h is strictly decreasing in α, and so

$$\chi(x) < h(\alpha,\vartheta) < -2 \qquad \text{for } \theta(x) \in (\vartheta, \pi/2).$$

Now let us define $\chi_A = \max\{-3, h(\alpha,\vartheta)\}$, and note that χ_A is either constant with varying α or decreasing, and so we have a $\chi_A \geq \chi(x)$ for all $x \in E_+$ that is non-increasing in α.

□

Now let us prove a similar lemma for guides of type B.

Lemma 5. *Consider a guide of type B, and let $\chi_B = \chi_B(r) = 2(1-1/r) < -2$. Then $\chi(x) \leq 2n_1(x)(1-1/r) < \chi_B$ for all $x \in E_+$.*

Proof. Similar to the previous proof, notice that from the symmetry we need only prove this result for $\theta \in (\bar{\theta}, \pi/2)$. Note from the previous proof that $\delta'(x)$ is strictly increasing for $\theta \in (\bar{\theta}, \pi/2)$, and so we have that $\delta'(x) < \delta'(\pi/2) = 1 - 1/r$. Using this fact, we then have that

$$\chi(x) = 2n_1(x)\delta'(x) < 2n_1(x)\left(1 - \frac{1}{r}\right) \leq 2\left(1 - \frac{1}{r}\right) = \chi_B < -2.$$

□

Note that we will from now on consider only those guides of type A or B.

Focusing Times

Note that we defined M to be the phase space of the billiards system with track Q. Let u be some tangent vector at $x \operatorname{int} M$ in the tangent space $T_x M$, and let $s \mapsto \gamma(s) = (q(s), v(s)) \in \int M$ be a differentiable curve such that $y(0) = x$ and $y'(0) = u$. In addition, let us define the family of lines $s \mapsto \gamma_+(s)$ by setting $\gamma_+ = \{q(s) + tv(s) : t \in \mathbb{R}\}$. In the same manner, let us define $s \mapsto \gamma_-(s)$ the same as $\gamma_+(s)$ replacing γ with $-\gamma$: this second family can be interpreted as γ_+ reflected off the boundary. We then have from the results of [9] that all of the lines is γ_+ intersect at one point along the curve of $\gamma_+(0)$, and similarly for $\gamma_-(0)$.

Definition. *The **focal point** of u is defined as the point at which the family of lines γ_+ or γ_- defined as above intersect in linear approximation.*

We then have that if $x = (x,\theta)$ and $u = (ds, d\theta)$, that the distance from x and the focal points of u are given by

$$f_+(u) = \frac{\sin\theta}{\kappa(s) + m(u)} \qquad f_-(u) = \frac{\sin\theta}{\kappa(s) + m(u)},$$

where κ is the curvature of ∂Q at s and $m(u) = d\theta/ds$ [9]. Let us set the convention that the outer arc has positive curvature while the inner arc has negative curvature.

Definition. *The **forward** and **backward focusing times** of u are the distances $f_+(u)$ and $f_-(u)$, respectively.*

Notice that when we sum the inverses of the two focusing times, we achieve the *Mirror Formula*, the well known equation describing the focusing and dispersion of light with mirrors [7].

$$\frac{1}{f_+(u)} + \frac{1}{f_-(u)} = \frac{2\kappa(u)}{\sin\theta}.$$

Fractional Linear Transformation

The next step of the proof involves relating the focusing times of an infinitesimal family of billiard trajectories entering and leaving a circular guide. Let us define the set E as the set of all collisions $x \in M \setminus S_1^+$ entering a circular guide of Q. In addition, define the function $n(x) \geq 0$ the number of times a particle with initial state $x \in E$ hits the boundary of the circular guide before leaving it. Note that in this case, we redefine the definition of leaving the system to be that a collision $x \in E$ is leaving the guide if the last collision of the particle with either arc occurs at q. Note as well that the function $n(x)$ is similar to that of $n_1(x)$, except that $n(x)$ also counts the number of collisions with the inner arc.

Let us consider $x \in E$ and $0 \neq u \in T_xM$. Let us define the map from the extended real line to itself given by $f_-(u) \mapsto f_+(D_xT^{n(1)}u)$ as $F_x(u)$. With a simple application of the Mirror Formula as defined above, we can deduce that F_x is a linear transformation, and moreover,

$$F_x(f) = \frac{a(x)f + b(x)}{c(x)f + d(x)},$$

where $a(x), b(x), c(x)$, and $d(x)$ satisfy the inequality $a(x)d(x) - b(x)c(x) < 0$. This inequality implies that $dF_x/df < 0$, and so F_x has two fixed points, $f_1(x) \geq f_2(x)$ on the real line.

Lemma 6. *For all $x \in E$, we have that*

$$f < f_2(x) \text{ or } f > f_1(x) \Leftrightarrow f_2(x) < F_x(f) < f_(x).$$

Proof. Note that since $dF_x/df < 0$, we have that F is monotone, and since F is an fractional linear transformation, we have the above property. □

The values of these fixed points are rather important.

Definition. *The **focal length of a circular guide** is the number*

$$\tilde{f} = \sup_{x \in E} f_1(x).$$

It is necessary in the proof to show that this number is always bounded from above for circular guides of both type A and B.

Theorem 7. *Let $\tilde{\chi} = \chi_A$ for guides of type A, and $\tilde{\chi} = \chi_B$ for guides of type B. Then*

$$\tilde{f} \leq \frac{\tilde{\chi}}{\tilde{\chi} + 2}.$$

Proof. We first want to prove that this is true for E_1. In order to do so, we must find the fixed points of F_x for $x \in E_1$. Note that in this case, $n(x) = n_1(x)$. From the definitions of f_- and f_+, we have that

$$f = f_-(u) = \frac{\sin\theta(x)}{1 - m(u)} \qquad\qquad f_+(D_x T^{n_1(x)} u) = \frac{\sin\theta(x)}{1 + m\left(D_x T^{n_1(x)} u\right)}.$$

Now realize, that we have already computed $D_x T_1^{n_1(x)}$, and so we have that

$$m\left(D_x T^{n_1(x)} u\right) = \frac{m(u)}{1 + \chi(x) m(u)} \rightarrow f_+(D_x T^{n_1(x)} u) = \frac{(1+\chi(x))\sin\theta(x)}{1 + (1+\chi(x)) m(u)}.$$

Using the above equations, and the mirror formula, we can then easily see that

$$F_x(f) = \frac{(1+\chi(x))\sin(x) f - \chi(x)\sin\theta(x)}{(2+\chi(x)) f - (1+\chi(x))\sin\theta(x)}.$$

By by Lemmas 3, 4, and 5, we have that $\chi(x) \geq 0$ for $x \in E_0 \cup E_-$, and $\chi(x) \leq -2$ for $x \in E_+$. Notice that in any case,

$$F(\sin\theta(x)) = \sin\theta(x) \qquad F\left(\frac{\chi(x)\sin\theta(x)}{2+\chi(x)}\right) = \frac{\chi(x)\sin\theta(x)}{2+\chi(x)},$$

and so these are the two fixed points. However, we have that $\sin\theta(x) < \frac{\chi(x)\sin\theta(x)}{2+\chi(x)}$ for $x \in E_+$ while the opposite is true for $x \in E_0 \cup E_-$. We therefore have that the two fixed points are

$$f_1(x) = \begin{cases} \sin\theta(x), & x \in E_0 \cup E_- \\ \frac{\chi(x)\sin\theta(x)}{2+\chi(x)}, & x \in E_+ \end{cases} \qquad f_2(x) = \begin{cases} \frac{\chi(x)\sin\theta(x)}{2+\chi(x)}, & x \in E_0 \cup E_- \\ \sin\theta(x), & x \in E_+. \end{cases}$$

Now note that $\frac{\chi(x)}{2+\chi(x)} \leq 1$ for $x \in E_+$, and since the function $z \mapsto \frac{z}{z+2}$ is increasing for $z \in (-\infty, -2)$, we have

$$\begin{aligned} \sup_{x \in E_1} f_1(x) &\leq \sup_{x \in E_+} \frac{\chi(x)}{2+\chi(x)} \\ &\leq \frac{\sup_{x \in E_4} \chi(x)}{2 + \sup_{x \in E_+} \chi(x)} \\ &\leq \frac{\tilde{\chi}}{2+\tilde{\chi}}. \end{aligned}$$

We have therefore proved this for the case that $x \in E_1$. To prove the situation for the case $x \in E \setminus E_1$, we will attempt to reduce the case down to that of E_1. To do that, we look at a circular guide containing the original guide, with the same inner and outer radii, but a central angle $\beta > \alpha$. For the rest of the proof, we shall denote all symbols dealing with this larger guide by using the original symbol with the addition of the hat character. For example, $\hat{M}$ denotes the phase space of collisions for the larger guide. Note that $x \in M$ and $y \in \hat{M}$ are equal if x and y coincide as tangent vectors in $\mathbb{R}^2$. Now that we have determined this notation, we can continue to the proof.

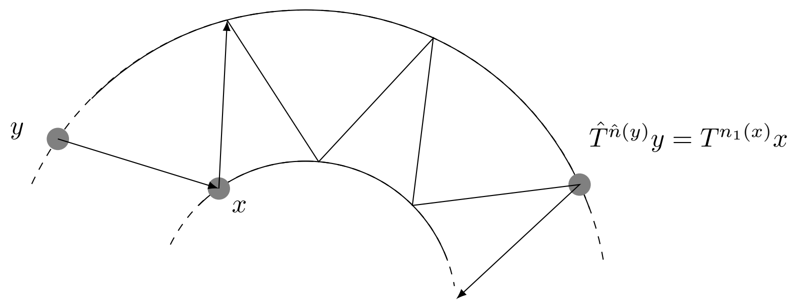

Figure 6: A picture describing the extension of a circular guide

Let us embed our original guide into a larger guide as described above, such that there exists a $y \in \hat{E}_1$ with $n(x) \le \hat{n}(y) \le n(x) + 2$ for which $\{x, \cdots, T^{n(x)}x\} \subset \{y, \cdots, \hat{T}^{\hat{n}(y)}y\}$. We have that the condition $n(x) \le \hat{n}(y) \le n(x)$ implies that $\hat{T}y = x$ or $\hat{T}^{\hat{n}_1(y)-1} = T^{n(x)}x$. We shall look only at the case $\hat{T}y = x$, as the other case proceeds along the same lines of proof. We can further split the case $\hat{T}y = x$, namely (i) $\hat{T}y = x$ and $\hat{T}^{\hat{n}_1(y)-1}y = T^{n(x)}x$ and (ii) $\hat{T}y = x$ and $\hat{T}^{\hat{n}_1(y)}y = T^{n(x)}x$. We shall again only study the second case, as the first case is similar to the second and is actually easier to prove. We therefore have by assumption that $\hat{T}y = x$ and $\hat{T}^{\hat{n}_1(y)-1}y = T^{n(x)}x$.

Let f_+ be the difference between $f_1(x)$ and the length of the segment connecting $\pi(x)$ and $\pi(y)$. Then we can find f_- using the mirror formula, noting that $\hat{\theta}(y) = \theta(x)$. We then have from the definition of F_x that $\hat{F}_y(f_-) = F_x(f_1)$, and so we have that

$$\hat{F}_y(f_-) = f_1(x).$$

We then want to show that $f_1(x) \le \hat{f}_1(y)$. As proof, let us assume that $f_1(x) > \hat{f}_1(y)$. Since we have that $y \in \hat{E}_1$ by assumption, we know from the first part of the proof that the fixed points of $\hat{F}_y$ have $\sin\hat{\theta}(y) = \hat{f}_2(y) < \hat{f}_1(y)$. In addition, since $\hat{\theta}(y) \in (\bar{\theta}, \pi - \bar{\theta})$, we have from the geometry of the system that $|\pi(x) - \pi(y)| < \sin\hat{\theta}(x)$.

From this result, we then have that $f_+ < 0$, and so from the Mirror formula, we then have that $0 < f_- < (\sin\hat{\theta}(y))/2 < \hat{f}_2(y)$. By Lemma 6, we then have that $\hat{f}_2(y) < \hat{F}_y(f_-) < \hat{f}_1(y)$, and thus $\hat{f}_2(y) < f_1 < \hat{f}_1$. This inequality, however, contradicts our assumption that $f_1 > \hat{f}_1$, and so we have that $f_1(x) \le \hat{f}_1(y)$.

From the first part of the proof, we have that $\hat{f}_1(y)$ is bounded from above by $\tilde{\chi}(\beta)/(2 + \tilde{\chi}(\beta))$. Since we have that $\tilde{\chi}(\beta)$ is a non-increasing function in β we have that $\tilde{\alpha} \ge \tilde{\beta}$, an thus we conclude that $f_1(x) \le \tilde{\chi}(\alpha)/(2 + \tilde{\chi}(\alpha))$. This conclude the proof. □

We therefore have that the focal length of a circular guide is always bounded from above.

Cone Fields

In order to show that the track billiards are hyperbolic, we need to show that they satisfy some properties of cone fields.

Definition. *A* ***cone*** *in some 2-dimensional space V is a subset*

$$\{aX_1 + bX_2 : ab \ge 0\},$$

where X_1 and X_2 are two linear independent vectors of V. We have an equivalent statement in that a cone C is a closed interval of the projective space $\mathbb{P}(V)$, the space of lines in V. The interior of C is defined as $\operatorname{int} C = \{aX_1 + bX_2 : ab > 0\} \cup 0$.

Notice that since both the backwards and forwards focusing times are projective coordinates of $\mathbb{P}(T_x M)$, the set $C = \{u \in T_x M : f_-(u) \in I\}$ or equivalently $C = \{u \in T_x M : f_+(u) \in I\}$ is a cone in $T_x M$ for every closed interval, I.

Now let Λ be a subset of $\tilde{M}$ such that $\mu(\Lambda) > 0$, or that Λ is some subset of $\tilde{M}$ that has a probability of occurring. Let $T_\Lambda : \Lambda \to \Lambda$ be the first return map on Λ induced by the billiard map T, and let μ_Λ be the probability measure on Λ obtained by restricting μ to Λ. We have from previous results that the map T_Λ preserves μ_Λ.

Definition. *A **measurable cone field**, C, on Λ is a measurable map that associates to each $x \in \Lambda$ a cone $C(x) \subset T_x M$. The field C is **eventually strictly invariant** if for every $x \in \Lambda$, we have*

- $D_x T_\Lambda C(x) \subset C(T_\Lambda x)$ *and ,*
- $\exists$ *an integer $k(x) > 0$ such that* $D_x T_\Lambda^{k(x)} C(x) \subset \operatorname{int} C(T_\Lambda^{k(x)} x)$.

Now that we have these definitions of terms, let us define an invariant cone field for circular track billiards. Let $\tilde{E} = E \cap \tilde{M}$ be the set of collisions entering a circular guide. Let us define the a measurable cone field on $\tilde{E}$ as

$$C(x) = \{u \in T_x M : f_-(u) \geq \tilde{f(x)}\} \qquad \forall x \in \tilde{E},$$

where $\tilde{f}$ is the focal length of the circular guide containing $\pi(x)$. Let us note that since f_- as a function of x and $\tilde{f}$ are continuous on $\tilde{E}$, we have that the cone filed is also continuous on $\tilde{E}$.

Hyperbolicity

Note that we finally reach the main conclusion in the paper. The bases for this proof rests on a well known result of Wojtkowski [9]. This result when applied to track billiards implies that if an eventually strictly invariant measurable cone field exists on Λ, then the mapping T_Λ is hyperbolic. In addition, from the results of [8], if the set $\cup_{x\in\mathbb{Z}} T^k \Lambda$ has full μ-measure, then the mapping T is hyperbolic as well. As such, to prove the hyperbolicity of track billiard maps, we need only prove that the cone field C define as above is eventually strictly invariant, and that $\cup_{k\in\mathbb{Z}} T^k \tilde{E}$ has full μ-measure.

As a means to this result, let Q be a track and assume that the guides of Q are ordered such that the i^{th} straight guide connects the i^{th} and $(i+1)^{th}$ circular guides. We also have that the $(n+1)^{th}$ circular guide coincides with the first circular guide, such that there are exactly n circular guides separated by n straight guides. In addition, let us assume that each circular guide is of type A or B.

Definition. *We say that a track Q as above satisfies **condition H** if the distance between any pair of consecutive circular guides of Q is greater than the focal length of the two circular guides, i.e.,*

$$l_i > \tilde{f}_i + \tilde{f}_{i+1} \qquad \textit{for each } i = 1, \cdots, n.$$

We are now prepared to give a proof of the main result of the paper.

Theorem 8. *Suppose that a track satisfies condition H. Then the billiard map T in Q is hyperbolic.*

Proof. As explained above, it is sufficient to prove this result we prove that the cone field C defined on Q as above is eventually strictly invariant, and that the set $\cup_{k\in\mathbb{Z}}T^k\tilde{E}$ has full measure.

Let $x \in \tilde{E}$ and let us consider $u \in C(x)$ with $u \neq 0$. From the definition of $C(x)$, we have that $f_-(u) > \tilde{f}(x) \geq f_1(x)$, and so by Lemma 6 we have that $f_+(D_xT^{n(x)}u) < f_1(x) \leq \tilde{f}(x)$. Let us now note that $T^{n(x)}$ is a collision leaving a circular guide, and that the piece of the orbit of x between x and $T_{\tilde{E}}x$ crosses a straight guide of length l. From condition H, we thus have that $l > \tilde{f}(x) + \tilde{f}(T_{\tilde{E}}x)$, so that

$$\begin{aligned} f_-(D_xT_{\tilde{E}}u) &= l - f_+(D_xT^{n(x)}u) \\ &\geq l - \tilde{f}(x) \\ &> \tilde{f}(T_{\tilde{E}}x). \end{aligned}$$

This then implies that $D_xT_{\tilde{E}}u \in \operatorname{int} C(T_{\tilde{E}}x)$, and thus we can conclude that C is eventually strictly invariant with $k(x) = 1$ for every $x \in \tilde{E}$. In addition, it is clear that $\cup_{k\in\mathbb{Z}}T^k\tilde{E} = \tilde{M}\setminus N$, where N is the invariant set $\{(q,v) \in M : \langle v, n(q)\rangle = 0\}$ defined in the proof on the unidirectionality of the track billiard map. Since we have $\mu(N) = 0$, we then have that $\cup_{k\in\mathbb{Z}}T^k\tilde{E}$ has full measure, which ends the proof. $\square$

Notice that an extremely simple example of such a system is one that looks like a track field with a long straight section. Notice that in this case, we have that the two arcs are both of type A, as $\alpha = \pi \geq \pi$, and so for sufficiently large straight sections, this system is hyperbolic.

Conclusion

The authors' article is interesting in that it describes a system that is completely determined, and yet exhibits chaotic behavior. Even more astounding is the simple character of the system that can lead to hyperbolic behavior. This paper is a perfect example of taking an extremely simple system and using it to explain and show complex ideas.

In addition, the study of such track billiards can help to provide information on the classical mechanics of tubular billiards, a system closely linked to that of track billiards except in $\mathbb{R}^3$. These tubular billiards have much applicability in that they can model several systems used in nanotechnology.

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