Laplace's *Résultat Remarquable*[4] and its Ramifications

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1 Introduction

One of the most important and widely studied special functions in mathematics is the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad \operatorname{Re} z > 0,$$

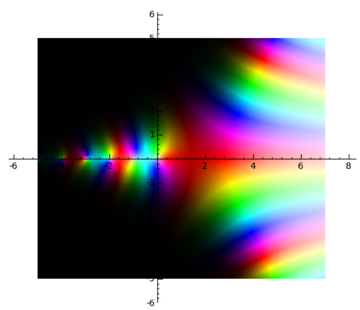

Figure 1: A domain-colored plot of $\Gamma(z)$, rendered with Sage

along with its meromorphic extension to the entire complex plane. However, in studying the gamma function, it is sometimes useful to consider its reciprocal. Like the gamma function itself, there are numerous definitions for this mathematical object. One common form is:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}, \qquad z \in \mathbb{C},$$

where $\gamma \approx .5772$ denotes the Euler-Mascheroni constant. For the derivation, see [3] p. 363. But occasionally more useful is Pierre-Simon Laplace's 1812 integral formula

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{a+it}}{(a+it)^z}\, dt = \frac{1}{\Gamma(z)}.$$

In [7], Wladimir de Azevedo Pribitkin explores Laplace's integral and applies it to the proofs of several properties of the gamma function. This paper will highlight some of the major results of Pribitkin's exposition, culminating in the derivations of the Maass and Lipschitz summation formulas.

2 Laplace's Integral

We will begin by defining Laplace's Integral and recording a few of its essential properties.

Definition 2.1. We define **Laplace's Integral** as

$$L_a(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{a+it}}{(a+it)^z} \, dt,$$

where a, Re $z > 0$.

We will use the convention $z^\alpha = e^{\alpha \log z}$ where we take the principal branch of the logarithm, with $-\pi \leq \arg z < \pi$. Almost immediately clear from the definition is that $L_a(z)$ converges absolutely and uniformly for Re $z \geq 1 + \epsilon$ and Im $z \leq C$ for any C. Let $z = x + iy$. We estimate the integrand:

$$\begin{aligned}
\left| \frac{e^{a+it}}{(a+it)^z} \right| &= |\exp(a + it - z \log(a+it))| \\
&= |\exp(a - x \log |a+it| + y \arg(a+it))| \\
&\leq e^{a+\pi C} |a+it|^{-x}.
\end{aligned}$$

For $t \in [-a, a]$, the integrand is bounded from above by $2e^{a+\pi C}|a|^{-x}$, and is hence proper. Elsewhere, it is bounded by $2e^{a+\pi C}|t|^{-x}$. Since $\int_{-\infty}^{a} |t|^{-x}$ and $\int_{a}^{\infty} |t|^{-x}$ converge, $L_a(z)$ converges absolutely and uniformly by the Weierstrass M-Test. Since C was arbitary, we have absolute and uniform convergence for all compact subsets of $x > 1$. We leave it to the reader to show that $L_a(z)$ also converges uniformly on compact subsets of $x > 0$, and hence defines an analytic function on this half-plane.

Next we will demonstrate the functional equation

$$L_a(z) = z L_a(z+1). \tag{1}$$

As in the proof of the analogous property for the gamma function, we will integrate by parts:

$$\begin{aligned}
L_a(z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{a+it}}{(a+it)^z} \, dt \\
&= \frac{1}{2\pi} \left(\frac{e^{a+it}}{(a+it)^{z+1}} \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{-z e^{a+it}}{(a+it)^{z+1}} \right) \\
&= z L_a(z+1).
\end{aligned}$$

We can use this recurrence relation to perform an analytic continuation (See [3], pp. 158 - 162) of $L_a(z)$ to arbitrary z. As a result, we see that $L_a(z)$ can be extended to be an entire function.

Thirdly, we will show that $L_a(z)$ is in fact independent of a. We accomplish this by formally differentiating with respect to a:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial a}\frac{e^{a+it}}{(a+it)^z}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(a+it)^z e^{a+it} - e^{a+it}z(a+it)^{z-1}}{(a+it)^{2z}}\,dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{a+it}}{(a+it)^z} + \frac{ze^{a+it}}{(a+it)^{z+1}}\,dt$$
$$= L_a(z) - zL_a(z+1).$$

Since these integrals converge uniformly, and by (1) their difference is zero, we find that $L_a(z) = L(z)$ is independent of our choice of positive a. For convenience in calculations, we will generally choose $a = 1$ in the remainder of this paper.

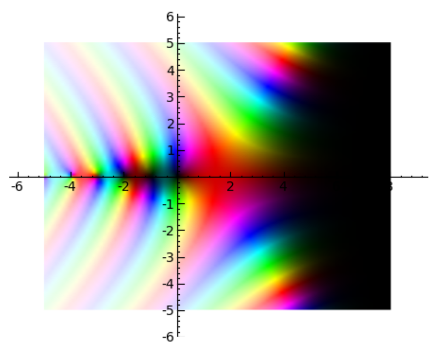

Figure 2: $L(z)$, rendered with Sage

Finally, we demonstrate the connection between $L(z)$ and the factorial function. To do so, we evaluate $L_1(2)$ via contour integration. Let C_R be the semicircular contour in the upper half-plane with radius $R > 1$. Then by the Residue Theorem, we have:

$$\int_{-R}^{R}\frac{e^{1+it}}{(1+it)^2}\,dt + \int_{C_R}\frac{e^{1+i\zeta}}{(1+i\zeta)^2}\,d\zeta = 2\pi i\ \mathrm{Res}\left[\frac{e^{1+i\zeta}}{(1+i\zeta)^2}, i\right]$$
$$= 2\pi i \lim_{\zeta\to i}\frac{d}{d\zeta}\left((\zeta - i)^2\frac{e^{1+i\zeta}}{(1+i\zeta)^2}\right)$$
$$= 2\pi.$$

By the ML-estimate, we have:

$$\left|\int_{C_R}\frac{e^{1+i\zeta}}{(1+i\zeta)^2}\,d\zeta\right| \le \frac{e}{R^2-1}\cdot \pi R,$$

which tends to 0 as $R \to \infty$. Since the integral defining $L(2)$ converges absolutely, we thus have $L(2) = 1$. By induction on our functional equation (1), we have:

$$L(n) = \frac{1}{(n-1)!} \tag{2}$$

for all $n \in \mathbb{Z}^+$. Our recurrence relation shows that we have simple zeroes at each nonpositive integer.

3 Laplace's Identity

The objective of this section is to prove the identity

$$\Gamma(z)L(z) = 1, \tag{3}$$

for all $z \in \mathbb{C}$, taking limits as appropriate for the nonpositive integers.

3.1 A Few Estimates

An easy result from complex power series is Liouville's Theorem, which states that a bounded entire function is constant. With a little more work, we can prove the following extension.

Theorem 3.1 (Extended Liouville Theorem)**.** *Let $P(z)$ be a polynomial of degree n. Suppose $f(z)$ is an entire function and $|f(z)| \leq P(|z|)$ for $|z| > \rho$. Then f is a polynomial of degree at most n.*

Proof. Suppose $P(|z|) = a_n|z|^n + a_{n-1}|z|^{n-1} + \cdots + a_1|z| + a_0$. Choose $R > \max\{1, \rho\}$. Then:

$$|P(|z|)| \leq |z|^n \sum_1^n |a_j| \leq R^n \sum_1^n |a_j|.$$

Hence, by the Cauchy estimates, we have

$$|f^{(m)}(0)| \leq \frac{m!}{R^m} R^n \sum_1^n |a_j|.$$

Suppose $m > n$. Then $|f^{(m)}(0)| \to 0$ as $R \to \infty$, so $f^{(m)}(0) = 0$ for all $m > n$. Hence, the power series expansion of $f(z)$ centered at the origin consists only of terms with degree less than or equal to n, so $f(z)$ is a polynomial of at most degree n. □

Example 1. *Suppose $f(z)$ is an entire function that satisfies $|f(z)| \leq |z|^4/\log|z|$ for $|z| > 1$. For $|z| > 1$, we have $\log|z| \leq |z|$, so $|f(z)| \leq |z|^3$. Then by the extended Liouville theorem, $f(z)$ is a polynomial of at most degree 3.*

A popular and somewhat surprising result characterizing the gamma function is the Bohr-Mollerup Theorem. The theorem states that a continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $F(1) = 1$, the functional relation $F(x+1) = xF(x)$ and the condition of log convexity is the gamma function. In 1939, Helmut Wielandt discovered a similar result for the gamma function in the complex plane. The following theorem is taken from [5].

Theorem 3.2 (Wielandt's Theorem)**.** *Let $F(z)$ be an analytic function in the right half-plane $\mathbb{A} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ having the following properties:*

(a) $\quad F(z+1) = zF(z)$ *for all* $z \in \mathbb{A}$

(b) $\quad F(z)$ *is bounded in the strip* $S = \{z \in \mathbb{C} : 1 \leq \operatorname{Re} z < 2\}$

(c) $\quad F(1) = 1.$

Then $F(z) = \Gamma(z)$.

Proof. Let $f = F - \Gamma$, which is analytic in $\mathbb{A}$. Note that since $f(1) = 0$, we have $f(z) = (z-1)g(z)$ for $g(z)$ analytic at 1. From (a) and the recurrence relation for the gamma function, we can extend f to be analytic at 0:

$$f(0) \equiv \lim_{z \to 0} \frac{((z+1)-1)f(z+1)g(z+1)}{z} = g(1).$$

Therefore, by our recurrence formula, we can extend f to the entire function $\hat{f}$. Since $|\Gamma(z)| \leq |\Gamma(\operatorname{Re} z)|$, we have that Γ is bounded on S and hence f is bounded by (b). Let $S_0 = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z < 1\}$. For all $z \in S_0$ with $|\operatorname{Im} z| \leq 1$, it is clear that f is bounded. For points with $|\operatorname{Im} z| > 1$, the boundedness of f on S_0 follows from $f(z) = f(z+1)/z$ and the boundedness of f on S.

Consider the entire function $s(z) = \hat{f}(z)\hat{f}(1-z)$. Since $f(z)$ and $f(1-z)$ take on the same values in S_0, the function $s(z)$ is bounded in S_0. Since $\hat{f}(z+1) = z\hat{f}(z)$ and $\hat{f}(-z) = -\hat{f}(1-z)/z$, it follows that $s(z+1) = -s(z)$. Hence $s(z)$ is bounded on $\mathbb{C}$ and therfore constant by Liouville's theorem. Since $s(z) \equiv s(1) = f(1)\hat{f}(0) = 0$, we have $\hat{f}(z) \equiv 0$ so $F(z) = \Gamma(z)$ on $\mathbb{A}$. $\square$

Example 2. *Let*

$$F(z) = \pi^{-1/2} 2^{z-1} \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right).$$

Since the gamma function is analytic in the right half-plane, F is also analytic here. We also have

$$\begin{aligned} F(z+1) &= 2\pi^{-1/2} 2^{z-1} \Gamma\left(\frac{z+1}{2}\right) \Gamma\left(\frac{z+2}{2}\right) \\ &= \pi^{-1/2} 2^{z-1} \Gamma\left(\frac{z+1}{2}\right) z\Gamma\left(\frac{z}{2}\right) \\ &= zF(z). \end{aligned}$$

Since
$$|t^{z-1}| = t^{\operatorname{Re} z - 1}$$
we have that $|\Gamma(z)| \leq |\Gamma(\operatorname{Re} z)|$ *in the right half-plane. Hence,* $\Gamma(z)$ *is bounded on vertical strips* $\{z \in \mathbb{C} : a \leq \operatorname{Re} z \leq b\}$ *for* $0 < a < b$*. In addition, we have that* 2^{z-1} *is bounded on vertical strips, so* $F(z)$ *is bounded on* $\{z \in \mathbb{C} : 1 \leq \operatorname{Re} z < 2\}$*. Finally, it is easy to show directly that* $F(1) = 1$*. Therefore, by Wielandt's Theorem,* $F(z) = \Gamma(z)$*. By a simple change of variables, we have the Legendre Duplication Formula*

$$\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right).$$

3.2 The Reflection Formula

In this section, we will prove an analog of the Euler reflection formula for Laplace's Integral. We will use this formula to prove Laplace's Identiy, and as a corollary, give an easy proof of the reflection formula for the gamma function.

Theorem 3.3.
$$L(z)L(1-z) = \frac{\sin \pi z}{\pi}.$$

Proof. Let $Q(z) = \pi L(z)L(1-z)/\sin \pi z$. Since the simple zeroes of $\sin \pi z$ coincide with those of $L(z)$, we see that $Q(z)$ is an entire function. Further, from our functional equation (1), we have $Q(z) = Q(z+1)$. Let $z = x + iy$. Then for $x \geq 2$ and $y \neq 0$ we have

$$\begin{aligned}
|L(z)| &\leq \frac{e}{2\pi} \int_{-\infty}^{\infty} \frac{e^{y \arg(1+it)}}{|1+it|^x} \, dt \\
&\leq \int_{-\infty}^{\infty} \frac{e^{y \arctan t}}{1+t^2} \, dt \\
&= \int_{-\pi/2}^{\pi/2} e^{yt} \, dt = \frac{1}{y}(e^{\pi y/2} - e^{-\pi y/2}).
\end{aligned}$$

Furthermore, for all x and for all $y \neq 0$

$$\begin{aligned}
\left| \frac{1}{\sin \pi z} \right| &= \left| \frac{2i}{e^{iz} - e^{-iz}} \right| \\
&\leq \frac{2}{e^{\pi|y|} - e^{-\pi|y|}}.
\end{aligned}$$

By repeated applications of (1), on the strip $2 \le x \le 3$ we have

$$\begin{aligned} Q(z) &= (1-z)(2-z)(3-z)(4-z)L(5-z)L(z)\frac{\pi}{\sin \pi z} \\ &= O(y^4)O(|y|^{-1}e^{\pi|y|/2})^2 O(e^{-\pi|y|}) \\ &= O(y^2), \end{aligned}$$

independent of x in this strip. Hence, there are positive constants A and B such that $|Q(z)| \le A|z|^2 + B$ on the strip. By our recurrence relation $Q(z) = Q(z+1)$, this estimate holds on $\mathbb{C}$. By the extended Liouville theorem, $Q(z)$ is a polynomial of at most degree 2. However, since $Q(1) = Q(2) = Q(3)$, the polynomial $Q(z) - Q(1)$ has at least three zeroes. By the fundamental theorem of algebra, $Q(z)$ must be constant. Since $Q(z) \equiv \lim_{z\to 0} \frac{\pi z}{\sin \pi z} = 1$, we have the desired reflection formula. □

3.3 A Simple Proof of the Identity

In [7], Pribitkin formally gives two rather different proofs of Laplace's Identity (3). Relegated to his closing remarks is an outline of the following proof. With the reflection formula and Wielandt's Theorem in hand, we can fill in the details.

Let $F(z) = 1/L(z) = \pi L(1-z)/\sin \pi z$ from the reflection formula, and as usual let $z = x + iy$. We note that where $\sin \pi z$ has simple zeroes, $L(1-z)$ does as well. Hence, the singularities of F are removable, so F is analytic for $x > 0$. Furthermore, we have

$$F(z+1) = \frac{\pi L(-z)}{\sin(\pi z + \pi)} = \frac{-z\pi L(1-z)}{-\sin \pi z} = zF(z).$$

Finally, we show that $F(z)$ is bounded in the strip $1 \le x < 2$. Making use of our estimates from Theorem 3.3, we have

$$\begin{aligned} |F(z)| &= \left|\frac{\pi(1-z)(2-z)(3-z)L(4-z)}{\sin \pi z}\right| \\ &\le \left|\frac{2\pi(1-z)(2-z)(3-z)(e^{\pi y/2} - e^{-\pi y/2})}{e^{\pi|y|} - e^{-\pi|y|}}\right| \\ &= O(y^2 e^{-\pi|y|/2}). \end{aligned}$$

Since the polynomial factor is dominated by the exponential as $|y| \to \infty$, we have that $F(z)$ is bounded in this strip. Finally, since $F(1) = 1/L(1) = 1/L(2) = 1$, we can apply the result of Wielandt's theorem to prove that $F(z) = \Gamma(z)$.

Since $\Gamma(z)$ has simple poles where $L(z)$ has simple zeroes, it follows that $\Gamma(z)L(z) \equiv 1$ on $\mathbb{C}$, taking limits at the nonpositive integers.

As promised, the following corollary is immediately clear from (3) and Theorem 3.3.

Corollary 3.4 (Euler's Reflection Formula)**.**

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

4 Some Summation Formulas

Pribitkin's primary application of Laplace's integral is to the derivation of the Maass and Lipschitz summation formulas. These formulas in turn have a wide range of consequences, especially in the theory of modular forms. To illustrate the power of this field, we present the following surprising result.

Theorem 4.1 (Radamacher's Formula)**.** *A partition of a positive integer n is an expression of n as the sum of smaller positive integers. Let $p(n)$, called the* ***partition function****, be the number of partitions of n (where order does not matter). Then*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{\left(n-\frac{1}{24}\right)}}\right),$$

where

$$A_k(n) = \sum_{0\le m<k, \gcd(m,k)=1} e^{\pi i(s(m,k)-2nm/k)},$$

and

$$s(m,k) = \sum_{n \mod k} f(n/k)f(mn/k),$$

and finally

$$f(x) = \begin{cases} x - \lfloor x \rfloor - 1/2, & \text{if } x \in \mathbb{R}\setminus\mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases}$$

The summation formulas we will derive thus build a bridge between analysis and number theory. For a further discussion, see [6].

4.1 Confluent Hypergeometric Functions

The coefficients of Maass' Formula are most easily expressed in terms of the confluent hypergeometric function of the second kind. Here, we present a few of the basics.

Definition 4.2. We define the **hypergeometric equation** as

$$z(1-z)w'' + (c-(a+b+1)z)w' - abw = 0$$

for $a, b, c \in \mathbb{C}$.

If we replace z with z/b, the equation has singularities at $0, b$, and ∞. If we let $b \to \infty$, then ∞ becomes a confluence of two singularities. This gives rise to the confluent hypergeometric equation.

Definition 4.3. The **confluent hypergeometric equation** is given by

$$zw'' + (c - z)w' - aw = 0$$

for $a, c \in \mathbb{C}$.

The confluent hypergeometric functions are two linearly independent solutions of the confluent hypergeometric equation around $z = 0$.

Definition 4.4. Suppose $\operatorname{Re} c > \operatorname{Re} a > 0$. Then define the **confluent hypergeometric function of the first kind** as

$$\Phi(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 (1-t)^{c-a-1} t^{a-1} e^{zt} \, dt.$$

Example 3. *Here is a quick frustrating example from elementary calculus. For $x \in \mathbb{R}$*

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt = \frac{2x}{\sqrt{\pi}} \Phi(1/2, 3/2, x).$$

Definition 4.5. Suppose $\operatorname{Re} a > 0, c \in \mathbb{C}, \operatorname{Re} z > 0$. Then the **confluent hypergeometric function of the second kind** is

$$\Psi(a, c, z) = \frac{1}{\Gamma(a)} \int_0^\infty (t+1)^{c-a-1} t^{a-1} e^{-zt} \, dt.$$

Hypergeometric functions have diverse applications in physics and engineering, particularly in the study of waves. For instance, the confluent hypergeometric equation is related to the Bessel equation, which appears in the study of (coincidentally) Laplace's equation. Most importantly for this paper, we note that Ψ has an entire analytic continuation. In particular, $\Psi(0, b, z) = 1$. For more information see [1] and [2] .

4.2 Maass' Formula

A function that makes numerous appearances in the theory of automorphic forms is

$$f(z) = \sum_{l=-\infty}^{\infty} \frac{e^{-2\pi i \lambda (x+l)}}{(z+l)^s (\bar{z}+l)^w},$$

for $\lambda \in \mathbb{R}$ and $s, w, z = x + iy \in \mathbb{C}$. To ensure absolute convergence, we suppose $y \neq 0$ and $\operatorname{Re}(s + w) > 1$. Since $f(z)$ is periodic with period 1, we can express it as a Fourier series

$$f(z) = \sum_{n=-\infty}^{\infty} c_{n+\lambda}(y,s,w)e^{2\pi inx}, \tag{4}$$

where

$$c_{n+\lambda}(y,s,w) = \int_0^1 f(z)e^{-2\pi inx}\ dx.$$

Since our definition is symmetric about the imaginary axis, we can take $y > 0$. Evaluating the coefficients, and making use of the substitution $u = x + l$:

$$\begin{aligned} c_{n+\lambda}(y,s,w) &= \int_0^1 \sum_{l=-\infty}^{\infty} \frac{e^{-2\pi i\lambda(x+l)}}{(z+l)^s(\bar{z}+l)^w} e^{-2\pi inx}\ dx \\ &= \sum_{l=-\infty}^{\infty} \int_l^{l+1} \frac{e^{-2\pi i(n+\lambda)u}}{(u+yi)^s(u-yi)^w}\ du \\ &= (-i)^{s-w} \int_{-\infty}^{\infty} \frac{e^{-2\pi i(n+\lambda)u}}{(y-iu)^s(y+iu)^w}\ du. \end{aligned}$$

Suppose $\operatorname{Re} s > 0$ and $\operatorname{Re} w > 1$. Then from the definitions of the gamma function and Laplace's integral, and by Fubini's theorem, we have

$$\begin{aligned} c_{n+\lambda}(y,s,w) &= (-i)^{s-w} \int_{-\infty}^{\infty} \left(\frac{1}{\Gamma(z)} \int_0^{\infty} v^{s-1}e^{-(y-iu)v}\ dv \right) \frac{e^{-2\pi i(n+\lambda)u}}{(y+iu)^w}\ du \\ &= \frac{(-i)^{s-w}}{\Gamma(s)} \int_0^{\infty} v^{s-1}e^{-yv} \int_{-\infty}^{\infty} \frac{e^{i[v-2\pi(n+\lambda)]u}}{(y+iu)^w}\ du\ dv \\ &= \frac{2\pi(-i)^{s-w}}{\Gamma(s)\Gamma(w)} e^{2\pi(n+\lambda)y} \int_{\max\{0,2\pi(n+\lambda)\}}^{\infty} v^{s-1}(v-2\pi(n+\lambda))^{w-1}e^{-2yv}\ dv. \end{aligned}$$

Hence we have

$$c_0(y,s,w) = \frac{2\pi(-i)^{s-w}\Gamma(s+w-1)}{\Gamma(s)\Gamma(w)(2y)^{s+w-1}}, \tag{5}$$

and

$$c_{n+\lambda}(y,s,w) = (2\pi)^{s+w}(-i)^{s+w}|n+\lambda|^{s+w-1}e^{-2\pi|n+\lambda|y}$$

$$\times \begin{cases} \frac{\Psi(w,s+w,4\pi(n+\lambda)y)}{\Gamma(s)}, & \text{if } n+\lambda > 0; \\ \frac{\Psi(s,s+w,-4\pi(n+\lambda)y)}{\Gamma(w)}, & \text{if } n+\lambda < 0. \end{cases} \tag{6}$$

Maass' formula is the Fourier series (4) given by coefficients (5) and (6).

4.3 Lipschitz's Formula

Although Maass' formula in its full generality has numerous important uses, we can easily derive from it a relatively simple summation formula with interesting consequences of its own. If we set $w = 0$ in Maass's formula (which is legal, since $\Psi(0, c, z) = 1$), then for $y > 0$ we obtain

$$\sum_{l=-\infty}^{\infty} \frac{e^{-2\pi i(z+l)\lambda}}{(z+l)^s} = \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n+\lambda>0} (n+\lambda)^{s-1} e^{2\pi i n z}, \tag{7}$$

for $\operatorname{Re} s > 1$.

Example 4. *Take $s = 2$ and $\lambda = 0$ in Lipschitz's formula. We have on the right-hand side*

$$\begin{aligned}
\frac{(-2\pi i)^2}{\Gamma(2)} \sum_{1}^{\infty} n e^{2\pi i n z} &= -4\pi^2 \sum_{0}^{\infty} \frac{d}{dz}\left(\frac{1}{2\pi i} e^{2\pi i n z}\right) \\
&= 2\pi i \cdot \frac{d}{dz}\left(\frac{1}{1 - e^{2\pi i z}}\right) \qquad \textit{since } |e^{2\pi i z}| < 1 \textit{ for } y > 0 \\
&= 2\pi i \frac{2\pi i e^{2\pi i z}}{(1 - e^{2\pi i z})^2} \\
&= -4\pi^2 \left(\frac{e^{\pi i z}}{1 - e^{2\pi i z}}\right)^2 \\
&= -4\pi^2 \left(-\frac{1}{2i} \sin \pi z\right)^2 \\
&= \frac{\pi^2}{\sin^2 \pi z}.
\end{aligned}$$

As a result, we have the identity

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}.$$

5 Conclusion

Pribitkin's examination emphasizes some of the truly powerful aspects of complex analysis. Just as the gamma function has a way of cropping up in various areas of mathematics, so too have we seen Laplace's integral. While the infinite product form for the reciprocal of the gamma function has its merits, Laplace's integral form has proven to be very convenient for making estimates. Further, its applications to the theory of modular forms reinforces the connection between complex analysis and number theory, much in the spirit of the prime number theorem. Although after two centuries, this result still has yet to catch on as a mainstream mathematical tool, its elegance and versatility still make it an exceptional piece of mathematics.

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