Harmonic Conjugate

Note Title 4/27/2009

Let u be harmonic on $D^* = \{0 < |z| < 1\}$.

Let $c = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial r} r\,d\theta$. Then

$u - c\log r$ has a harmonic conjugate.

First notice that $\int_0^{2\pi} \frac{\partial u}{\partial r} r\,d\theta = \int_{|z|=r} \frac{\partial u}{\partial n} ds$

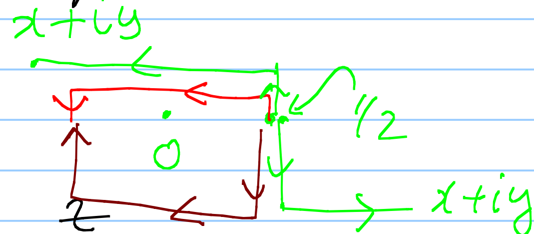
By Green's theorem this integral is independent of r. In fact it is the same for any rectangle surrounding 0. another formula is $\int \frac{\partial u}{\partial n} ds = \int -u_y dx + u_x dy$

Let $v = u - \frac{c}{2\pi}\log r$. a conjugate w of v satisfies

$$w_x = -u_y - \frac{c}{2\pi}\left(-\frac{y}{r^2}\right)$$

$$(*) \quad w_y = u_x - \frac{c}{2\pi}\left(\frac{x}{r^2}\right)$$

Define w as follows. For z not on the negative real axis define $w(z)$ by the integral from $z_0 = \frac{1}{2}$ to $x+iy$ by integrating from $\frac{1}{2}$ to $\frac{1}{2}+iy$ and then to $x+iy$



$$w(z) = \int_{z_0}^{z} -u_y\,dx + u_x\,dy - \frac{c}{2\pi}\int_{z_0}^{z} -\frac{y\,dx}{r^2} + \frac{x}{r^2}\,dy$$

For z on the negative real axis integrate up, over, and down. Because of the definition of C this is the same if we go down over and up. So $w(z)$ is well-defined. Now by a calculation made many times, $(*)$ is satisfied.