Math 336 Makeup Exam, Summer, 2008

Name:____

One notebook-size page of notes is allowed (each side may be used).

1. Prove that $\Gamma(z)$ has a simple pole at -3 and compute its residue at this point.

2. Find the Green's function for the region $\{z : |z-1| < 3\}$, with pole at 2.

3. Let $u(x, y)$, $v(x, y)$ be continuously differentiable as functions of (x, y) in a domain Ω. Let $f(z) = u(z) + iv(z)$. Suppose that for every $z_0 \in \Omega$ there is an r_0 (depending on z_0) such that

$$\int_{|z-z_0|=r} f(z)dz = 0,$$

for all r with $r < r_0$. Prove that f is analytic in Ω. Hint: Show that f satisfies the Cauchy-Riemann equations in Ω.

4. Suppose that u is harmonic on all of $\mathbb{C}$ and $u(z) \geq 0$ for all $z \in \mathbb{C}$. Prove that u is constant. Is it true that a positive harmonic function on $\{(x, y) : x + y > 0\}$ is constant?

5. Let

$$g(x) = \begin{cases} 1, \text{ if } |x| < 1 \\ 0, \text{ if } |x| > 1; \end{cases}$$

$$h(x) = \begin{cases} |x|, \text{ if } |x| < 1 \\ 0, \text{ if } |x| > 1; \end{cases}$$

$$f(x) = g(x) - h(x).$$

Compute the Fourier transform (my definition in class) of g, h and hence of f.

6. Suppose $f_n(z)$ is a sequence of analytic functions on an open set Ω and that $|f_n(z)| < 1$ for all $z \in \Omega$. Suppose $\sum f_n(z)$ converges uniformly and absolutely on compact subsets of Ω. Prove that

$$f(z) = \prod(1 + f_n(z))$$

defines an analytic function and $f(z) \neq 0$ (f is never zero.)

7. Let $u(z)$ be harmonic in all of $\mathbf{C}$. Suppose $|u(z)| \leq c|z|^n$ for some positive constant c. Prove that u is the real part of a complex polynomial of degree n.