LFT's that map $|z|=1$ to $|z|=1$

Note Title 5/5/2008

Suppose $\left|\dfrac{az+b}{cz+d}\right|^2 = 1$ when $|z|=1$.

Then $|a|^2+|b|^2+2\,\mathrm{Re}(\bar{b}az) = |c|^2+|d|^2+2\,\mathrm{Re}(\bar{d}cz)$.

This is identically true for all $|z|=1$. Hence

(1) $|a|^2+|b|^2=|c|^2+|d|^2$ and (2) $\bar{b}a=\bar{d}c$

also (3) $ad-bc\neq 0$

(i) Suppose $a=0$. Then $bc\neq 0$ and $\bar{d}c=0$, so $d=0$,
and by (1) $|b|=|c|$, so $\dfrac{az+b}{cz+d}=\dfrac{\lambda}{z}$, where $|\lambda|=\left|\dfrac{b}{c}\right|=1$.

(ii) Suppose $a\neq 0$. Then $\bar{b}=\bar{d}c/a$ and using (1)
$(|a|^2-|d|^2)(|a|^2-|c|^2)=0$. If $|a|=|c|$, $a=\lambda c$,
$|\lambda|=1$ and by (2) $b=\lambda d$. This contradicts (3). Thus $|a|\neq|c|$
and $|a|=|d|$; so $d=\bar{\lambda}\bar{a}$, $|\lambda|=1$. By (2) $c=\bar{\lambda}\bar{b}$,
and $\dfrac{az+b}{cz+d}=\dfrac{b+az}{\bar{a}\bar{\lambda}+\bar{b}\bar{\lambda}z}=\lambda\left(\dfrac{b+az}{\bar{a}+\bar{b}z}\right)=\mu\left(\dfrac{\alpha-z}{1-\bar{\alpha}z}\right)$, where

$\mu=\dfrac{-\lambda a}{\bar{a}}$, $\alpha=-b/a$, and $|\alpha|\neq 1$, since $|b|=|c|$
and $|a|\neq|c|$.

Conclusion: $\dfrac{az+b}{cz+d}=\lambda\dfrac{\alpha-z}{1-\bar{\alpha}z}$, $|\lambda|=1$, $|\alpha|\neq 1$.

(Modification of an argument in Copson, "Theory of Functions of a Complex Variable")