Dirichlet Problem

Note Title 5/8/2008

Let $\phi(t) = \phi(e^{it})$ be continuous and piecewise smooth on $|z|=1$. Let $a_n = \hat{\phi}(n)$ be the Fourier coefficients of ϕ. Then $\sum_{-\infty}^{\infty} |a_n| < \infty$ and $\phi(t) = \sum_{-\infty}^{\infty} a_n e^{int}$ and the convergence is uniform.

Let $0 \le r \le 1$, and $z = re^{i\theta}$. Then

$u(z) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta}$ converges uniformly on $|z| \le 1$ to a continuous function with $u = \phi$ on $|z|=1$.

Let $f(z) = \sum_{0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$. f is analytic and hence harmonic (real and imaginary parts are harmonic). $g(z) = \sum_{n=1}^{\infty} \bar{a}_{-n} z^n = \sum_{1}^{\infty} \bar{a}_{-n} r^n e^{in\theta}$ is also analytic. So the real and imaginary parts of $\overline{g(z)} = \sum_{n=-\infty}^{-1} a_n r^{|n|} e^{in\theta}$ are also harmonic. So

$u(z) = f(z) + \overline{g(z)}$ is harmonic & $u = \phi$ on $|z|=1$.

We have solved the Dirichlet problem (existence and uniqueness) for continuous, piecewise smooth functions on $|z|=1$. But we want to do it in more generality.