Abel's Test, Uniform Version

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Abel's test has a uniform version. First we modify the statement of Abel's lemma

Lemma 1. *Suppose* $\sum b_k$ *converges. Let* $B_k = b_k + b_{k+1} + b_{k+2} + \dots$. *Then*

$$a_n b_n + \cdots + a_{n+k} b_{n+k} = B_n a_n + B_{n+1}(a_{n+1} - a_n) + \cdots + B_{n+k}(a_{n+k} - a_{n+k-1}) - B_{n+k+1} a_{n+k}. \quad (1)$$

Proof.

$$\begin{aligned} a_n b_n + \cdots + a_{n+k} b_{n+k} &= a_n(B_n - B_{n+1}) + \cdots + a_{n+k}(B_{n+k} - B_{n+k+1}) \\ &= B_n a_n + B_{n+1}(a_{n+1} - a_n) + \cdots + B_{n+k}(a_{n+k} - a_{n+k-1}) - B_{n+k+1} a_{n+k}. \end{aligned}$$

□

Theorem 1. *Suppose* $\sum_1^\infty b_n(x)$ *converges uniformly on* S *and that* $\{a_n(x)\}$ *is a monotone uniformly bounded sequence. Then* $\sum_1^\infty a_n(x) b_n(x)$ *converges uniformly on* S.

Proof. We show that $\sum_n^{n+k} a_j(x) b_j(x)$ is uniformly small if n is large enough. Using the notation of Lemma 1 let n be so large that $|B_n(x| \le \epsilon$ for all $x \in S$ and let $|a_n(x)| \le M$ for all $x \in S$. To be explicit, assume $a_n(x)$ is increasing. By (1)

$$\begin{aligned} |a_n b_n + \cdots + a_{n+k} b_{n+k}| &= |B_n a_n| + |B_{n+1}(a_{n+1} - a_n)| + \cdots + |B_{n+k}(a_{n+k} - a_{n+k-1})| + |B_{n+k+1} a_{n+k}| \\ &\le |B_n a_n| + |B_{n+1}(a_{n+1} - a_n)| + \cdots + |B_{n+k}(a_{n+k} - a_{n+k-1})| + |B_{n+k+1} a_{n+k}| \\ &\le \epsilon M + \epsilon(a_{n+k} - a_n) + \epsilon M \\ &\le 4M\epsilon. \end{aligned}$$

□

Corollary 1. ***(Abel's limit theorem)***

Suppose $\sum a_n$ *converges. Then* $\sum a_n x^n$ *converges uniformly on* $[0,1]$ *and hence to a continuous function* $f(x)$. *Consequently* $\lim_{x\to 1^-} \sum a_n x^n = \sum a_n$.

Dirichlet's test also has a uniform version.

Theorem 2. *Suppose* $\sum b_k(x)$ *is uniformly bounded on* S, *and that* $\{a_n(x)\}$ *tends monotonely and uniformly to* 0 *on* S. *Then* $\sum a_n(x) b_n(x)$ *converges uniformly.*

Proof. Let $B_n = b_1 + \cdots + b_n$. As in Lemma 1 check that when $n \ge 2$

$$a_n b_n + \cdots + a_{n+k} b_{n+k} = -B_{n-1} a_n + B_n(a_n - a_{n+1}) + \cdots + B_{n+k-1}(a_{n+k-1} - a_{n+k}) + B_{n+k} a_{n+k}.$$

By uniform boundedness $|B_n(x)| \le M$ for all n; and if $n \ge N$, $|a_n(x)| \le \epsilon$.

$$\begin{aligned} |a_n b_n + \cdots + a_{n+k} b_{n+k}| &\le |B_{n-1} a_n| + |B_n(a_n - a_{n+1}) + \cdots + B_{n+k-1}(a_{n+k-1} - a_{n+k})| + |B_{n+k} a_{n+k}| \\ &\le M\epsilon + M|a_n - a_{n+k}| + M\epsilon \\ &\le 4M\epsilon. \end{aligned}$$

□