Math 335 Sample Problems

One notebook sized page of notes (*one side*)will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover 4.5, 4.6, 4.7, 5.6, 5.7, 5.8, 6.1, and 6.2. The midterm is on Monday, January 30.

1. Let $f(x)$ satisfy $0 \leq f(x) \leq f(y)$ if $x \geq y$. Suppose $\int_1^\infty f(x)dx$ converges. Prove $\lim_{x\to+\infty} xf(x) = 0$.

2. Assume $a_n \geq 0$ for all $n \geq 1$. Prove that if $\sum_1^\infty a_n$ converges then $\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges. Give an example of a sequence $a_n \geq 0$ such that $\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges and $\sum_1^\infty a_n$ diverges.

3. Prove that if $\sum_1^\infty a_n$ converges then $\sum_1^\infty \frac{\sqrt{a_n}}{n}$ converges. (Assume $a_n \geq 0$.)

4. Let x_n be a convergent sequence and let $c = \lim_{n\to\infty} x_n$. Let p be a fixed positive integer and let $a_n = x_n - x_{n+p}$. Prove that $\sum a_n$ converges and
$$\sum_1^\infty a_n = x_1 + x_2 + \ldots x_p - pc.$$

5. Suppose $a_n > 0,\ b_n > 0$ for all $n > 1$. Suppose that $\sum_1^\infty b_n$ converges and that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Prove that $\sum_1^\infty a_n$ converges.

6. Let S be the set of all positive integers whose decimal representation does *not* contain 2. Prove that $\sum_{n \in S} \frac{1}{n}$ converges.

7. Prove that $\int_0^\infty \cos x^2 dx$ converges, but not absolutely.

8. Let $a = \lim_{n\to\infty} a_n$. Prove that $\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = a$.

9. Decide if the following integrals converge conditionally, converge absolutely, or diverge.

   (a)
$$\int_{-\infty}^{+\infty} x^2 e^{-|x|} dx$$

   (b)
$$\int_0^\pi \frac{dx}{(\cos x)^{\frac{2}{3}}}$$

   (c)
$$\int_1^\infty \frac{\sin(1/x)}{x} dx$$

10. Let f and g be integrable on $[a, b]$ for every $b > a$.

    (a) Prove that
$$\left(\int_a^b |fg|\right)^2 \leq \int_a^b f^2 \int_a^b g^2.$$

    (b) Prove that if $\int_a^\infty f^2$ and $\int_a^\infty g^2$ converge then $\int_a^\infty fg$ converges absolutely.

11. (a) Suppose $\sum_1^\infty a_n$ converges. Fix $p \in \mathbb{Z}^+$. Prove that $\lim_{n\to\infty}(a_n + a_{n+1} + \ldots a_{n+p}) = 0$.

    (b) Suppose $\lim_{n\to\infty}(a_n + a_{n+1} + \ldots a_{n+p}) = 0$ for every p. Does $\sum_1^\infty a_n$ converge?

12. Let C be the curve of intersection of $y+z=0$ and $x^2+y^2=a^2$ oriented in the counterclockwise direction when viewed from a point high on the z-axis. Use Stokes' theorem to compute the value of $\int_C (xz+1)dx + (yz+2x)dy$.

13. (a) Prove that $\int_C \frac{-ydx + xdy}{x^2+y^2}$ is not independent of path on $\mathbf{R}^2 - \mathbf{0}$.

    (b) Prove that $\int_C \frac{xdx + ydy}{x^2+y^2}$ is independent of path on $\mathbf{R}^2 - \mathbf{0}$. Find a function $f(x,y)$ on $\mathbf{R}^2 - \mathbf{0}$ so that $\nabla f = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$.

14. Let $a_n > 0$ and suppose $a_n \geq a_{n+1}$. Prove that $\sum_1^\infty a_n$ converges if and only if $\sum_0^\infty a_{3n}$ converges.

15. Suppose that $a_n > 0$ is a sequence of positive numbers and suppose that the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists. Then prove that $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \sqrt[n]{a_n}$$

16. You will need to know the definitions of the following terms and statements of the following theorems.

    (a) Convergence and divergence of a series
    (b) Comparison test
    (c) Integral test
    (d) Cauchy condensation test
    (e) Root test and ratio test
    (f) Stokes' theorem

(g) Potentials and independence of path
(h) Poincare's lemma
(i) Improper single and multiple integrals
(j) Integrals dependent on a parameter

17. There may be homework problems or example problems from the text on the midterm.