Compactness

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Theorem 1. *Let S_k be a decreasing sequence of non-empty compact sets ($S_{k+1} \subset S_k$). Then $\cap S_k \neq \emptyset$.*

Proof. Let $x_k \in S_k$. Then $x_k \in S_1$ for all k. Hence there is a subsequence x_{k_j} that converges to a point $a \in S_1$. But ultimately all points of $\{x_{k_j} : j \geq N\}$ are in S_i for each fixed i. Since S_i is compact, $a \in S_i$. This is true for all i, so $a \in \cap S_i \neq \emptyset$. □

Corollary 1. *If S_j is a decreasing sequence of compact sets and $\cap_j S_j = \emptyset$ then $S_j = \emptyset$ for some j.*